Ascending Auctions with Multidimensional Signals *

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Abstract

We study an ascending auction of an indivisible good in which agents observe multidimensional Gaussian signals about their valuation of the good. The equilibrium is solved using a two-step procedure. The first step is to project the signals into a one-dimensional *equilibrium statistic*. The second step is to solve for the equilibrium as if agents observed *only* the equilibrium statistic (and hence, as if agents observe one-dimensional signals). The solution method can also be applied to other trading mechanisms, including supply function competition.

We provide predictions of ascending auctions that arise only when agents observe multidimensional signals. The first prediction is that an ascending auction may have multiple symmetric equilibria. Each equilibrium induces a different allocative efficiency and different profits for the seller. The second prediction is that, with multidimensional signals, public signals can be detrimental for profits (even in symmetric environments). In fact, a precise enough public signal can induce profits arbitrarily close to 0. The third prediction is that public signals increase the surplus generated by the auction. These predictions already arise in a model with two-dimensional signals that combines a classic model of private values and a classic model of common values. Hence, the only difference between the model we study and the classic models of ascending auctions is the multidimensionality of the information structure.

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1 Introduction

1.1 Motivation and Model

In most auctions an agent's valuation of the object is determined by at least two independent shocks: an idiosyncratic shock and a common shock. If the valuation of an agent is determined by two independent shocks, it is natural that agents observe two signals; one signal about their idiosyncratic shock and a second signal about the common shock. For example, an agent's valuation of an oil field is determined by his idiosyncratic cost of extracting the oil and the size of the oil reservoir. In this example, it is natural that an agent observes his own cost of extracting the oil and also observes a noisy signal about the size of the oil reservoir. Similarly, an agent's valuation of timber, mineral rights, art, highway procurements, and others is also subject to multiple shocks.[1] Of course, in most of these examples it is also natural to think of additional shocks and additional signals. Hence, auction models in which agents observe more than one signal should be commonplace.

The ascending auction is one of the most widely used auction formats, yet, the properties of an ascending auction when agents observe multidimensional signals are unknown.[2] Even the possibility of constructing an equilibrium in a simple two-dimensional example that combines an idiosyncratic shock and a common shock is an open question. The objective of our paper is to characterize the equilibrium of an ascending auction when agents observe multidimensional signals, analyze its properties, and compare the predictions with the predictions in one-dimensional environments.

There is a fundamental difference between ascending auctions with one-dimensional signals and ascending auctions with multidimensional signals. This difference explains the gap in the literature and motivates our analysis. If agents observe one-dimensional signals, then for agent i observing the bid of agent j is informationally equivalent to observing the signal observed by agent j. In contrast, in environments with multidimensional signals, the drop-out time of an agent only reflects a summary statistic of all the signals this agent observed. Hence, the drop-out time of agent i is determined by the summary statistic that agent i learns from the drop-out time of agent j. Finding the informational content of the drop-out time of agent j is a

[1]In timber auctions, agents may differ in their harvesting cost and their estimate about the harvest quality (see Haile (2001) or Athey and Levin (2001)). In highway procurement auctions, bidders are exposed to idiosyncratic cost shocks and common cost shocks (see Somaini (2011) or Hong and Shum (2002)). In art auctions and real estate auctions, agents have a known taste shock and an unknown common shock that can represent the quality of the good or the future resale value.

[2]The same difficulty is pervasive to first-price auctions and second-price auctions. The ascending auction is a natural starting point, but we believe studying the other classic auction formats is also interesting.

difficult problem because this in turn depends on what agent j learns from the drop-out time of agent i. This problem is not present when agents observe one-dimensional signals. This is the fundamental difference between one-dimensional signals and multidimensional signals.

The model consists of N agents bidding for an indivisible good in an ascending auction. The utility of an agent if he wins the object is determined by a payoff shock, which may be correlated across agents. Each agent observes J signals about the realization of his own payoff shock. The joint distribution of signals and payoff shocks is Gaussian.

1.2 Results

Solving for a Nash equilibrium of an ascending auction when agents observe multidimensional signals will be equivalent to solving for the equilibrium in one-dimensional environments, but considering an endogenously determined one-dimensional signal. In order to make sense of the results in multidimensional environments, it is important to understand how the joint distribution of signals and payoff shocks determines the outcome in a one-dimensional environment. To this end, we characterize the surplus generated and the seller's profits for any distribution of signals, but keeping the distribution of payoff shocks unchanged.

In one-dimensional environments, we characterize the surplus generated and the seller's profits in terms of a orthogonal decomposition of the signals an agent observes. This allow us to separate the components of the joint distribution of signals and payoff shocks that determine the surplus generated and the split of the surplus between the seller and the buyers. The surplus generated is determined by the informativeness of the signals of all agents about the differences in agents' payoff shocks. The division of the surplus between the buyers and the seller is determined by the *informational interdependence*. The informational interdependence measures how much of the ex post valuation of agent i is explained by agent j's signal. A higher level of informational interdependence imply that the buyers keep a larger share of the total surplus generated.

We then study symmetric two-dimensional signals. The payoff shock is decomposed as the sum of two independent shocks; a common shock and an idiosyncratic shock. The idiosyncratic shock may be correlated across agents. The first signal agent i observes is perfectly informative about agent i's idiosyncratic shock. The second signal agent i observes is a noisy signal about the common shock. This model combines the two classic models in the auction literature: pure common values and pure private values.[3]

[3] If agents observed only their idiosyncratic shock, this would be a classic private value environment. If agents observed only the signal on the common shock, this would be a classic common value environment.

We characterize a class of equilibria using a two-step procedure. In the first step, we project the two signals of each agent into a one-dimensional *equilibrium statistic*. In the second step, we characterize the equilibria "as if" agents observed only the one-dimensional equilibrium statistic. Consequently, in the second step, the equilibrium characterization for one-dimensional signals is applied. To the best of our knowledge, we are the first paper that characterizes the equilibrium of an ascending auction that combines pure common values and pure private values. We later show that the same solution method can be applied to any Gaussian information structures (possibly asymmetric).[4]

In equilibrium agents behave "as if" they observed only the one-dimensional equilibrium statistic. Nevertheless, the mapping between signals and the equilibrium statistic is subtle. This leads to two differences between the analysis of auctions with one-dimensional signals and multidimensional signals. First, with multidimensional signals, there is no straightforward mapping between the distribution of signals and the surplus generated or the seller's profits in the auction. Second, any comparative statics will be affected by changes in the equilibrium statistic; hence comparative statics will different than in one-dimensional environments. To illustrate the differences between one-dimensional environments and multidimensional environments, we provide predictions of the ascending auction that arise only when agents observe multidimensional signals.

The first novel prediction is that the ascending auction may have multiple symmetric equilibria. One equilibrium resembles the equilibrium of a model of pure private values; the surplus generated and the profits earned by the seller are large. Another equilibrium resembles the equilibrium of a model of common values; the surplus generated and the profits earned by the seller are low. The multiplicity of equilibria shows that the outcome of the auction is determined in a non-trivial way by the weights agents place on their private signals. The following properties are necessary for the existence of multiple equilibria: (i) the idiosyncratic shocks are correlated, and (ii) the noise in the private signal an agent observes about the common shock is large.

The weight agents place on their own idiosyncratic shock in their bidding strategy may exhibit strategic complementarities, which explains the multiplicity of equilibria. The complementarity is caused by a negative spurious correlation between the idiosyncratic shock of agent i and the common shock. Conditional on the drop-out time of agent j, if agent i observes a high idiosyncratic shock he lowers his expectation of the common shock. Since a high idiosyncratic shock is associated with a low common shock, agent i places a low weight on his own idiosyncratic

[4]The information structure is the joint distribution of signals and payoff shocks.

shock. If agent i decreases the weight on his idiosyncratic shock, this causes an even bigger negative spurious correlation for agent j. Hence, there is a complementarity.

The second difference between one-dimensional signals and multidimensional signals is how public signals impact the equilibrium outcome. In one-dimensional environments, public signals do not change the total surplus, and public signals increase the profits. In contrast, in environments with multidimensional signals, a public signal may jointly increase the surplus and decrease the seller's profits.[5]

The intuition of how public signals impact the surplus generated is as follows. In one-dimensional environments, the equilibrium is efficient and hence public signals do not change the surplus. If agents observe multidimensional signal, then a public signal decreases the weight that agents place on their private signal about the common shock. This decreases the correlation between an agent's drop-out time and the noise in the agent's signal, which in turn increases the surplus. Interestingly, a public signal about the average idiosyncratic shock also increases the surplus. This is because the public signal reduces the negative spurious correlation between the idiosyncratic shock of agent i and the common shock.

The intuition of how public signals impact the seller's profits is as follows. A public signal about the average idiosyncratic shock has the effect of increasing the information an agent has about the differences in payoff shocks. This exacerbates the winner's curse from the common shock. This is because when agent i expects a lower idiosyncratic shock than agent j there are two effects. First, there is a direct effect that a lower expected idiosyncratic shock lowers agent i's drop-out time even in a private value environment. Second, agent i anticipates that to win the object the signal of agent j about the common shock must be low enough to compensate the differential in idiosyncratic shock (hence, not "just" lower than his own signal about the common shock). Hence, agent i faces an exacerbated "winner's curse". If the signal about the average idiosyncratic shock is precise enough, the seller's profits can be arbitrarily close to 0.

We show that the same solution method can be applied to a class of trading mechanisms that includes supply function competition with linear-quadratic payoffs.[6] The Nash equilibrium in these other mechanisms can be computed using the same two-step procedure. First, for each agent we find a one-dimensional equilibrium statistic of the signals each agent observes. Second, we compute the equilibrium as if agents observed *only* the one-dimensional equilibrium

[5] This can be interpreted as a failure of the linkage principle. We discus this in Section 5.5.

[6] Technically, we show that the analysis can be applied to every game that has an ex post equilibrium when the agents observe one-dimensional signal. There is a large class of mechanisms that have an ex post equilibrium when agents observe one-dimensional signals, including classic trading mechanisms (e.g. generalized VCG mechanisms), as well as mechanisms recently proposed by the literature (a detailed discussion is provided after Theorem 4).

statistic. The one-dimensional equilibrium statistic does not depend on the game, but only on the information structure. The extension to other mechanisms illustrates that the equilibrium statistic distills the essential elements of information aggregation that is present in mechanisms where an agent learns from other agents actions. This is discussed in Section 8.1.

1.3 Literature

The literature in auctions when agents observe one-dimensional signals is very extensive. A large part of this literature is based on the seminal contribution of Milgrom and Weber (1982), which we discuss later. To the best of our knowledge, there is no paper in the literature that characterizes the bidding strategies of agents in an ascending auctions with multidimensional signals, except for the case in which signals are independently distributed across agents. We now discuss the results that have been able to be obtained in an ascending auction when agents observe multidimensional signals.

Jackson (2009) provides a class of examples in which an equilibrium does not exist. The model studied therein is similar to our model — with a private and a common signal — except the distribution of signals and fundamentals is non-Gaussian (moreover, the signals have a finite support). The extent to which it is possible to construct equilibria with multidimensional non-Gaussian information structures is still an open question. We discuss Jackson (2009) in more detail in Section 8.

Wilson (1998) studies an ascending auction with log-normal random variables with a two-dimensional information structure. Wilson (1998) assumes that the random variables are drawn from a diffuse prior. Hence, the signals are not technically random variables, and the updating is not technically done by Bayes' rule. Relaxing the assumption of diffuse priors is not only a technical contribution, but it is also fundamental to derive the novel predictions in the ascending auction. This allows us to provide some insights on how the analysis can be extended to non-Gaussian information structures. Wilson (1998) can be seen as a particular limit of our model, which we explain in Section 4.

The closest connection in the literature to the equilibrium statistic we propose can be found in Dasgupta and Maskin (2000). They show that if agents' signals are independently distributed across agents, then there is a way to project the multidimensional signals into a one-dimensional statistic. Our equilibrium statistic coincides with the one in Dasgupta and Maskin (2000) if signals are independently distributed. In this case, the one-dimensional statistic corresponds to an agent's expectation of their own payoff shock conditional *only* on his private information. Hence, when signals are independently distributed across agents there is no "equilibrium component" to the equilibrium statistic. This equilibrium statistic has been used to study auctions (under the assumption of independent signals).[7] As mentioned, the predictions we find hinge critically on how the equilibrium statistic of one agent depends on the information the agent learns from the drop-out time of other agents.

The connection of our paper to the mechanism design literature and other trading mechanisms is discussed throughout the paper.

The paper is organized as follows. In Section 2 we provide the model. In Section 3 we study one-dimensional signals. In Section 4 we characterize the equilibrium with two-dimensional signals. In Section 5 we analyze the equilibrium with two-dimensional signals. In Section 6 we generalize the methodology to allow for multidimensional asymmetric signals. In Section 7 we generalize the methodology to other trading mechanisms. In Section 8 we conclude. All proofs are collected in the appendix.

2 Model

2.1 Payoff Structure

We study N agents bidding for an object in an ascending auction. The utility of agent i if he wins the object at price p is given by:

$$u_i(\theta_i, p) = \exp(\theta_i) - p, \tag{1}$$

where $\exp(\cdot)$ denotes the exponential function and $\theta_i \in \mathbb{R}$ is a payoff shock. If an agent does not win the object he gets a utility equal to 0. The reason we take the exponential function in (1) is to guarantee that the agent i's valuation of the good is always positive. Yet, this has no role in the equilibrium characterization. The characterization of the equilibrium goes through for any strictly increasing function of θ_i (see Section 7.1).

[7]Levin, Peck, and Ye (2007) and Goeree and Offerman (2003) study auctions in which agents observe independent signals, and hence the bidding strategy can be analyzed using the same one-dimensional statistic as in Dasgupta and Maskin (2000).

2.2 Information Structure

The only source of uncertainty is the realization of the payoff shocks $(\theta_1, ..., \theta_N)$. Each player observes J signals:

$$\mathbf{s}_i \triangleq (s_{i1}, ..., s_{iJ}),$$

where vectors are denoted in bold font. We assume that the random variables $(\theta_1, ..., \theta_N, \mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^{(J+1)N}$ are jointly normally distributed, and all random variables have a mean equal to 0. The assumption that the means are equal to 0 allow us to reduce the amount of notation, but the analysis goes through in a straightforward way if the means of the random variables are not 0.

We assume that the information structure is symmetric. That is, for all $i, \ell, k \in N$ the joint distribution of $(\theta_i, \theta_k, \mathbf{s}_i, \mathbf{s}_k)$ is the same as the joint distribution of $(\theta_\ell, \theta_k, \mathbf{s}_\ell, \mathbf{s}_k)$. We later generalize the analysis by studying asymmetric environments.

2.3 Ascending Auction

We study an ascending auction.[8] In an ascending auction an auctioneer rises the price continuously. At each moment in time, an agent can decide to drop out of the auction, in which case the agent does not pay anything and does not get the object. The last agent to drop out of the auction wins the object and pays the price at which the second to last agent dropped out of the auction. We restrict attention to ascending auctions in which agents are symmetric and agents use symmetric strategies. This allows us to simplify the notation and provide the main insights of the paper. We generalize the analysis in Section 6.

The strategy of player i is a set of functions $\{p_\ell^i\}_{\ell \in \{2,...,N\}}$, with

$$p_\ell^i : \mathbb{R}^J \times \mathbb{R}^{N-\ell} \to \mathbb{R}_+. \tag{2}$$

The function $p_\ell^i(\mathbf{s}_i, p_{\ell+1}, ..., p_N)$ is the drop-out time of agent i, when ℓ agents are left in the auction and the observed drop-out times are $p_N < ... < p_{\ell+1}$. The function $p_\ell^i(\mathbf{s}_i, p_{\ell+1}, ..., p_N)$ must satisfy:

$$p_\ell^i(\mathbf{s}_i, p_{\ell+1}, ..., p_N) \geq p_{\ell+1}.$$

That is, agent i cannot drop out of the auction at a price lower than the price at which another agent has already dropped out. Note that we are restricting attention to symmetric environments,

[8] We follow Krishna (2009) in the formal description of the ascending auction.

and hence it is not necessary to specify the identity of the agent that drops out of the auction, but only the price at which this agent dropped out.

The outcome of the ascending auction is described by the order at which each agent drops out and the price at which each agent drops out. We describe the order at which each agent drops out of the auction by a permutation π.[9] $\pi(i)$ is the number of agents left in the auction when agent i dropped out of the auction. The identity of the last agent to drop out of the auction is given by $\pi^{-1}(1)$. The price at which agents drop out of the auction is denoted by $p_1 > .... > p_N$. Hence, for any strategy profile the expected utility of agent i is:

$$\mathbb{E}[\mathbb{1}\left\{\pi^{-1}(1) = i\right\}(e^{\theta_i} - p_2)],$$

where $\mathbb{1}\{\cdot\}$ is the indicator function. We study the Nash equilibria of the ascending auction.

3 One-Dimensional Signals

We first study symmetric one-dimensional signals (s_i). We use the Gaussian structure of the signals to characterize how the information structure determines the surplus generated in the auction, and how this surplus is split between the buyers and the seller.

3.1 Examples of Information Structures

We first provide some examples of one-dimensional signals studied in the literature.

Example 1 (Example 2 in Dasgupta and Maskin (2000))**.** *Agents observe a noisy signal about their own payoff shock:*

$$s_i = \theta_i + \varepsilon_i.$$

Example 2 (Reny and Perry (2006))**.** *Agents observe a signal s_i and there is a common shock ω.[10] The signals and the shock have a correlation $corr(s_i, \omega)$. The payoff shock is given by:*

$$\theta_i = \omega + s_i.$$

[9] A permutation is a bijective function $\pi : N \to N$.

[10] We provide a linear-Gaussian version of the signals studied in Reny and Perry (2006). Beyond some technical differences, it is clear that the signals studied in this example have the same economic interpretation.

Example 3 (Common Values). *Agents observe a signal on the common component of the shock:*[11]

$$s_i = \frac{1}{N} \sum_{j \in N} \theta_j + \varepsilon_i.$$

Example 4 (Noise-Free Signals). *Agent i observes a signal equal to a linear combination of his own payoff shock and the average payoff shock across agents:*

$$s_i = \theta_i + (\mu - 1) \cdot \frac{1}{N} \sum_{j \in N} \theta_j. \tag{3}$$

Agent i can learn θ_i if he observes all signals $(s_1, ..., s_N)$. Unless $\mu = 1$, signal s_i alone is not sufficient to learn θ_i.

3.2 Characterization of Equilibrium with One-Dimensional Signals

Throughout Section 3, we assume that:

$$corr(s_i, \theta_i)^2 > corr(s_j, \theta_i)^2. \tag{4}$$

That is, the signal of agent i more informative than the signal of agent j about the payoff shock of agent i. This is a version of the single crossing condition.

We now characterize the equilibrium of the ascending auction. We relabel agents such that the realization of signals satisfy:

$$s_1 > ... > s_N.$$

As signals are noisy, we might have that the order over payoff shocks is not preserved. For example, we may have $\theta_{i+1} > \theta_i$ (even though by construction $s_{i+1} \leq s_i$). The expectation of θ_i assuming that signals $(s_1, ..., s_{i-1})$ are equal to s_i (that is, assuming that all signals higher than s_i are equal to s_i) is denoted by $\mathbb{E}[\theta_i | s_i, ..., s_i, ..., s_N]$.

[11] Agents do not have any information about $\Delta\theta_i$, so it is effectively a common value environment.

Proposition 1 (Equilibrium of Ascending Auction)**.**
The ascending auction has a Nash equilibrium in which agent i's drop-out time is given by:

$$p_i = \mathbb{E}[\exp(\theta_i)|s_i, ..., s_i, s_{i+1}..., s_N]. \tag{5}$$

In equilibrium, agent $i = 1$ gets the object and pays $p_2 = \mathbb{E}[\exp(\theta_2)|s_2, s_2, s_3, ..., s_N]$.

Proposition 1 shows that the player with the i^{th} highest signal drops out of the auction at his expected valuation, if the $i-1$ signals that are higher than s_i are equal to s_i. This is the classic equilibrium characterization found in Milgrom and Weber (1982). The equilibrium characterized in Proposition 1 is an ex post equilibrium. That is, even if every agent knew the realization of the signals of all other agents, the strategy profile described by (5) would still be an equilibrium (see the proof of Proposition 1). This is essentially the unique symmetric equilibrium.[12]

The surplus generated by the auction is equal to the expected valuation of the buyer that wins the object. Since we relabel agents such that signals satisfy $s_1 > ... > s_N$, it is easy to check that the expected surplus is equal to:

$$S(s_1, ..., s_N) \triangleq \mathbb{E}[\exp(\theta_1)|s_1, ..., s_N]. \tag{6}$$

To characterize the seller's profits, we define:

$$m \triangleq \frac{cov(s_i, \theta_i) + (N-1)cov(s_i, \theta_j)}{var(s_i) + (N-1)cov(s_i, s_j)} \cdot \frac{var(s_i) - cov(s_i, s_j)}{cov(s_i, \theta_i) - cov(s_i, \theta_j)}. \tag{7}$$

We call m the degree of informational interdependence. In order to interpret the degree of informational interdependence m, note that the expected value of θ_i conditional on all signals is given by:[13]

$$\mathbb{E}[\theta_i|s_1, ..., s_N] = \frac{var(s_i) - cov(s_i, s_j)}{cov(s_i, \theta_i) - cov(s_i, \theta_j)} \frac{(N-1) + m}{N} \left(s_i + \frac{m-1}{(N-1) + m} \sum_{j \neq i} s_j \right), \tag{8}$$

The ratio between the weight that agent i places on the signal of agent j, relative to the weight that agent i places on his own signal is increasing in m. That is, $(m-1)/(N-1+m)$ is strictly increasing in m.[14] For this reason, we call m the degree of informational interdependence. For

[12] Bikhchandani, Haile, and Riley (2002) show that there is a continuum of symmetric equilibria. Nevertheless, the allocation and equilibrium price is the same across equilibria. See Krishna (2009) for a textbook discussion.

[13] This is straightforward to compute from the formulation in (12), which we later provide.

[14] $(m-1)/(N-1+m)$ is discontinuous at $m = -(N-1)$. Yet, it is easy to check the (4) is equivalent to $m \geq 0$. Hence,

noise-free signals (see (3)), the parameter μ is equal to the degree of informational interdependence. This provides an alternative way of interpreting the informational interdependence. We now characterize the seller's profits in terms of the total surplus.

Theorem 1 (Profits)**.**
The seller's profits are equal to:

$$p_2 = exp\Bigg((\frac{1-m}{N} - 1)(\mathbb{E}[\theta_1|s_1,...,s_N] - \mathbb{E}[\theta_2|s_1,...,s_N]) \Bigg) \times S(s_1,...,s_N) \tag{9}$$

The characterization of the equilibrium strategies (Proposition 1) is well known in the literature, yet Theorem 1 provides a novel characterization of the seller's profits. Theorem 1 characterizes the seller's profits in terms of the parameter m, the first order statistic of the expected valuations, and the second order statistic of the expected valuations. The equilibrium is efficient (that is, $(\mathbb{E}[\theta_1|s_1,...,s_N] - \mathbb{E}[\theta_2|s_1,...,s_N]) > 0$), and hence, the seller's profits are decreasing in m.

The reason that the buyers' get a higher share of the surplus when m increases is that this increases the "winner's curse". Buyers adjust to the "winner's curse" by reducing their bids, which decreases the profits. If $m = 1$, then the expected value of agent i's payoff shock conditional on all signals is the same as the expectation conditional only on his private information (that is, $\mathbb{E}[\exp(\theta_i)|s_1,...,s_N] = \mathbb{E}[\exp(\theta_i)|s_i] \propto s_i$). This corresponds to the case in which agents have private values. In this case, the price paid is equal to the expected second highest valuation: $p_2 = \mathbb{E}[exp(\theta_2)|s_1,...,s_N]$. If $m > 1$, the price paid is higher than the expected second highest valuation. If $m < 1$, the price paid is lower than the expected second highest valuation.

3.3 Description of the Information Structure

In order to characterize how the ex ante expected surplus and the ex ante expected profits are determined by the information structure, it is convenient to first provide an explicit description of the joint distribution of signals and payoff shocks $(\theta_1,...,\theta_N,\mathbf{s}_1,...,\mathbf{s}_N)$. In order to provide a more compact description of the distribution of signals and payoff shocks, we first provide an orthogonal decomposition of signals and payoff shocks:

$$\bar{\theta} \triangleq \frac{1}{N}\sum_{i\in N}\theta_i \;\; ; \;\; \Delta\theta_i \triangleq \theta_i - \bar{\theta} \;\; ; \;\; \bar{\mathbf{s}} \triangleq \frac{1}{N}\sum_{i\in N}\mathbf{s}_i \;\; ; \;\; \mathbf{\Delta s}_i \triangleq \mathbf{s}_i - \bar{\mathbf{s}}. \tag{10}$$

$(m-1)/(N-1+m)$ is increasing in m on the relevant range.

Variables with an over-bar correspond to the average of the variable over all agents. Variables preceded by a Δ correspond to the difference between a variable and the average variable. We refer to variables that have an over-bar as the common component of a random variable and a variables preceded by a Δ as the orthogonal component of a random variable. For example, $\bar{\theta}$ is the common component of θ_i while $\Delta\theta_i$ is the orthogonal component of θ_i. Note that:

$$var(\bar{\theta}) = var(\theta_i)(\frac{1+(N-1)corr(\theta_i,\theta_j)}{N}) \text{ and } var(\Delta\theta_i) = var(\theta_i)\frac{(N-1)(1-corr(\theta_i,\theta_j))}{N}.$$

Similarly, for any random variable, the variances of the common and orthogonal components are determined by the correlation of the random variable across agents.

Since all random variables are symmetrically distributed, the joint distribution of $(\theta_i, \bar{\theta}, s_i, \bar{s})$ completely determines the joint distribution of signals and payoff shocks. Additionally, since all random variables are Gaussian with mean 0, the joint distribution of $(\theta_i, \bar{\theta}, s_i, \bar{s})$ is completely determined by its variance covariance matrix. Instead of describing the joint distribution of $(\theta_i, \bar{\theta}, s_i, \bar{s})$ we describe the joint distribution of $(\Delta\theta_i, \bar{\theta}, \Delta s_i, \bar{s})$, which is obviously equivalent.

Lemma 1 (Distribution of Equilibrium Statistic and Fundamentals)**.**
The variance covariance matrix of payoff shocks and signals $(\Delta\theta_i, \bar{\theta}, \Delta s_i, \bar{s})$ *is equal to:*

$$\begin{pmatrix} \sigma^2_{\Delta\theta_i} & 0 & \rho_{\Delta s_i,\Delta\theta_i}\sigma_{\Delta\theta_i}\sigma_{\Delta s_i} & 0 \\ 0 & \sigma^2_{\bar{\theta}} & 0 & \rho_{\bar{s},\bar{\theta}}\sigma_{\bar{\theta}}\sigma_{\bar{s}} \\ \rho_{\Delta s_i,\Delta\theta_i}\sigma_{\Delta\theta_i}\sigma_{\Delta s_i} & 0 & \sigma^2_{\Delta s_i} & 0 \\ 0 & \rho_{\bar{s},\bar{\theta}}\sigma_{\bar{\theta}}\sigma_{\bar{s}} & 0 & \sigma^2_{\bar{s}} \end{pmatrix}. \tag{11}$$

From (11) the role of the orthogonal decomposition in (10) becomes transparent. The orthogonal component of a random variable is always independent of the common component of a random variable. This will simplify the analysis and allows us to provide more compact expressions of the results.

We can provide an alternative formulation of the informational interdependence m. The expected value of θ_i conditional on all signals is given by:[15]

$$\mathbb{E}[\theta_i|s_1,...,s_N] = \mathbb{E}[\Delta\theta_i|\Delta s_i] + \mathbb{E}[\bar{\theta}|\bar{s}] = \frac{cov(\Delta\theta_i,\Delta s_i)}{var(\Delta s_i)}\Delta s_i + \frac{cov(\bar{\theta},\bar{s})}{var(\bar{s})}\bar{s}. \tag{12}$$

[15]It is straightforward to calculate (12). First, by symmetry we have that $\mathbb{E}[\theta_i|s_1,...,s_N] = \mathbb{E}[\theta_i|s_i,\bar{s}]$. Second, note that $(s_i,\bar{s})$ is informationally equivalent to $(\Delta s_i,\bar{s})$. Finally, using that the common component of signals is independent to the orthogonal component of signals we get the expression for the expectation.

The informational interdependence is equal to the ratio of the regression coefficients in (8):

$$m = \frac{\frac{cov(\bar{\theta},\bar{s})}{var(\bar{s})}}{\frac{cov(\Delta\theta_i,\Delta s_i)}{var(\Delta s_i)}}.$$

That is, the informational interdependence is equal to the ratio between both regression coefficients in (12).

It is easy to check that (11) is determined by 6 parameters:

$$(var(\Delta\theta_i), var(\bar{\theta}), var(\Delta s_i), var(\bar{s}), corr(\Delta s_i, \Delta\theta_i), corr(\bar{s}, \bar{\theta})). \tag{13}$$

Instead of describing the variance covariance matrix by (13), it is equivalent to describe the variance covariance matrix by:

$$(var(\Delta\theta_i), var(\bar{\theta}), var(s_i), m, corr(\Delta s_i, \Delta\theta_i), corr(\bar{s}, \bar{\theta})). \tag{14}$$

$var(\Delta\theta_i)$ and $var(\bar{\theta})$ are primitives of the payoff structure. $var(s_i) = var(\bar{s}) + var(\Delta s_i)$ is equal to the variance of the signals, and hence, this is just a normalization. m is a measure of the amount of informational interdependence. $corr(\Delta s_i, \Delta\theta_i)$ is the informativeness of the orthogonal component of the signals about the orthogonal component of the payoff shock. $corr(\bar{s}, \bar{\theta})$ is the informativeness of the common component of the signals about the common component of the payoff shock. By changing $(m, corr(\Delta s_i, \Delta\theta_i), corr(\bar{s}, \bar{\theta}))$ we change the information structure, but keep the payoff environment fixed.

3.4 Comparative Statics

We now compute the comparative statics of the ex ante surplus and ex ante profits in terms of the parameters $(m, corr(\Delta s_i, \Delta\theta_i), corr(\bar{s}, \bar{\theta}))$. Each of these parameters has a different impact on the surplus generated and the seller's profits. Hence, this allow us to disentangle how the information structure impacts the surplus generated and the seller's profits. We provide the comparative statics of the ex ante expected surplus.

Proposition 2 (Comparative Statics: Surplus)**.**
The ex ante expected surplus $\mathbb{E}[S(s_1, ..., s_N)]$ *is strictly increasing in* $corr(\Delta s_i, \Delta\theta_i)$ *and constant in* $corr(\bar{s}, \bar{\theta})$ *and* m*.*

Proposition 2 shows that the expected surplus is determined only on the informativeness of the signals observed by all agents $(s_1, ..., s_N)$ about the orthogonal component of the payoff shocks $(\Delta\theta_1, ..., \Delta\theta_N)$. This is natural as the surplus increases by assigning the object efficiently across agents. We now provide the comparative static on the expected profits.

Corollary 1 (Comparative Statics: Profits)**.**
The expected profits ($\mathbb{E}[p_2]$) *are decreasing in* m *and constant in* $corr(\bar{s}, \bar{\theta})$*.*

Corollary 1 shows that the profits are decreasing in the degree of informational interdependence m. This can be seen from (9), plus the fact that the distribution of $(\mathbb{E}[\theta_1|s_1, ..., s_N] - \mathbb{E}[\theta_2|s_1, ..., s_N])$ does not depend on m or $corr(\bar{s}, \bar{\theta})$.

Corollary 1 does not provide the comparative static of the seller's profits with respect to $corr(\Delta s_i, \Delta\theta_i)$. This is because the seller's profits can be increasing or decreasing with respect to $corr(\Delta s_i, \Delta\theta_i)$. The expected surplus is increasing in $corr(\Delta s_i, \Delta\theta_i)$, and hence, it is natural to believe the profits would also be increasing in $corr(\Delta s_i, \Delta\theta_i)$. Yet, $corr(\Delta s_i, \Delta\theta_i)$ increases the expected value of $(\mathbb{E}[\theta_1|s_1, ..., s_N] - \mathbb{E}[\theta_2|s_1, ..., s_N])$. If m is big enough, then the seller's profits are decreasing in $corr(\Delta s_i, \Delta\theta_i)$. The intuition is that more information about the idiosyncratic shock may have an excess effect on the "winner's curse".

In the limit $m \to \infty$, if $(\mathbb{E}[\theta_1|s_1, ..., s_N] - \mathbb{E}[\theta_2|s_1, ..., s_N]) > 0$, then the profits converge to 0 ($p_2 \to 0$). The pure common values model can be considered by taking the limit $corr(\Delta\theta_i, \Delta s_i) \to 0$ (see Example 3). In this limit two things happen; (i) $m \to \infty$, and (ii) $(\mathbb{E}[\theta_1|s_1, ..., s_N] - \mathbb{E}[\theta_2|s_1, ..., s_N]) \to 0$. It is easy to check that in this limit the profits do not converge to 0.

Finally, we note that the buyers' rents plus the seller's profits is equal to the total surplus. The buyers' rents are given by:

$$V(s_1, ..., s_N) \triangleq \mathbb{E}[exp(\theta_1) - p_2|s_1, ..., s_N]. \tag{15}$$

It is easy to check that the buyers' rents are increasing in m and constant in $corr(\bar{s}, \bar{\theta})$.[16]

[16]We believe the buyers' rents are increasing in $corr(\Delta s_i, \Delta\theta_i)$. Nevertheless, we have no been able to prove this.

4 Two-Dimensional Signals: Characterization of Equilibrium

We now study a class of symmetric two-dimensional signals. Although the information structure we study in this section is very stylized, the same methodology can be extended to general Gaussian information structures. The two-dimensional signals allow us to provide sharper intuitions on how the equilibrium is determined. Moreover, the two-dimensional signals is already sufficient to provide predictions of an ascending auction that arise only when agents observe multidimensional signals.

4.1 Information Structure

The payoff shock of agent i is equal to the sum of two independent random variables:

$$\theta_i = \iota_i + \bar{\varphi}. \tag{16}$$

where ι_i has correlation $corr(\iota_i, \iota_j) \geq 0$ across agents, while $\bar{\varphi}$ is common to all agents. Agent i observes two signals:

$$s_{i1} = \iota_i \ ; \ s_{i2} = \bar{\varphi} + \varepsilon_i, \tag{17}$$

where ε_i is a noise term independent across agents and independent of all other random variables in the model. All random variables are jointly normally distributed.

To fix an interpretation of the model, consider the auction of an oil field. Agent i is a firm that has technology ι_i. The size of the oil reservoir is determined by a shock $\bar{\varphi}$. The total amount of oil that firm i, with technology ι_i, can extract from an oil reserve $\bar{\varphi}$ is equal to $exp(\iota_i) \cdot exp(\bar{\varphi}) = exp(\theta_i)$. This is equal to the value of the oil field to firm i. A firm knows his own technology, but a firm only observe a noisy signal about the size of the oil reservoir

The model studied by Wilson (1998) corresponds to the limits $var(\bar{\varphi})$,$var(\bar{\iota}) \to \infty$ (formally, Wilson (1998) assumes diffuse priors). After we charcaterize the equilibrium, we explain why taking these limits simplify the analysis. If $N = 2$ and $corr(\iota_i, \iota_j) = 0$, then our model is a particular case of Pesendorfer and Swinkels (2000). Yet, they study the properties of the equilibrium as $N \to \infty$.[17]

[17] Pesendorfer and Swinkels (2000) study a sealed-bid auction. Nevertheless, both models coincide if $N = 2$. Pesendorfer and Swinkels (2000) do not characterize the equilibrium. In fact, they do not show that an equilibrium exists. Instead, they study the properties that any equilibrium must have (in case it exists) in the limit in which the market becomes large.

4.2 Definition of Equilibrium Statistic

We first define an equilibrium statistic, and then explain the intuition and how it is used in the equilibrium characterization.

Definition 1 (Equilibrium Statistic)**.**
A linear combination of signals $\zeta_i \triangleq s_{i1} + \beta s_{i2}$ *is an equilibrium statistic if:*

$$\mathbb{E}[\theta_i | s_{i1}, s_{i2}, \zeta_1, ..., \zeta_N] = \mathbb{E}[\theta_i | \zeta_1, ..., \zeta_N] \tag{18}$$

An equilibrium statistic is a linear combination of signals in which the weights on this signals satisfy statistical condition (18). The expected value of θ_i conditional on (s_{i1}, s_{i2}) and the equilibrium statistic of other agents $\{\zeta_j\}_{j \neq i}$ is equal to the expected value of θ_i conditional on all equilibrium statistics $\{\zeta_j\}_{i \in N}$. Although (18) is defined purely in terms of the information structure — without reference to the game or the solution concept — it is transparent to see that there is an equilibrium notion involved. If agent i knows the equilibrium statistic of other agents and agent i only needs to compute his expected payoff shock, then the equilibrium statistic of agent i is a sufficient statistic to compute his own payoff shock. Throughout the paper, we use ζ_i to denote an equilibrium statistic

4.3 Equilibrium Characterization

We show that for every equilibrium statistic there exists a Nash equilibrium in which each agent i behaves as if he observed *only* his equilibrium statistic ζ_i. Similar to the analysis of one-dimensional signals, we assume that agents are ordered as follows:

$$\zeta_1 > ... > \zeta_N. \tag{19}$$

If there are multiple equilibrium statistics, then there will be one Nash equilibrium for each equilibrium statistic. Different equilibrium statistics induces a different order (as in (19)), so the Nash equilibrium is described in terms of the order induced by each equilibrium statistic.

Theorem 2 (Symmetric Equilibrium with Multidimensional Signals)**.**
For every equilibrium statistic $\{\zeta_i\}_{i\in N}$ *there exists a Nash equilibrium in which agent i's drop-out time is given by:*

$$p_i = \mathbb{E}[\exp(\theta_i)|\zeta_i, ..., \zeta_i, ..., \zeta_N],$$

In equilibrium, agent 1 *gets the object and pays a price equal to* $p_2 = \mathbb{E}[\exp(\theta_2)|\zeta_2, \zeta_2, ..., \zeta_N]$.

Theorem 2 shows that there exists a class of equilibria in which agents project their signals into a one-dimensional statistic using the equilibrium statistic $\zeta_i = s_{i1} + \beta s_{i2}$. In equilibrium agents behave as if they observed *only* ζ_i, which is a one-dimensional object.

Theorem 2 implies that the analysis in Section 3 remains valid, with the modification that we need to replace s_i with ζ_i. It is easy to check that $corr(\Delta\zeta_i, \Delta\theta_i)$ is given by:

$$corr(\Delta\zeta_i, \Delta\theta_i) = \frac{var(\Delta\iota_i)}{var(\Delta\iota_i) + \beta^2 var(\Delta\varepsilon_i)}. \tag{20}$$

Hence, $corr(\Delta\zeta_i, \Delta\theta_i)$ is decreasing in β. Using Proposition 2, it is easy to check that the surplus generated by the auction is decreasing in β. This is because the inefficiencies in the allocation of the object come from the correlation between the drop-out time of agent i and the noise term (ε_i). If agent i observes only ζ_i, then m is given by:

$$m = \frac{var(\bar{\iota}) + \beta \cdot var(\bar{\varphi})}{var(\bar{\iota}) + \beta^2(var(\bar{\varphi}) + var(\bar{\varepsilon}))} \frac{var(\Delta\iota_i) + \beta^2 \cdot var(\Delta\varepsilon_i)}{var(\Delta\iota_i)}. \tag{21}$$

It is easy to check that m may be decreasing in β. Hence, β may impact in a non-monotonic way the split of the surplus between the buyers and the seller.

4.4 Intuition of the Proof

We provide the proof of the result for $N = 2$. We check the following two conditions: (i) agent 1 never regrets winning the object after agent 2 drops out of the auction; and (ii) agent 2 never regrets dropping out of the auction instead of waiting until agent 1 drops out of the auction. Formally, these two conditions are:

$$\mathbb{E}[\exp(\theta_1)|s_{11}, s_{12}, \zeta_2] - \mathbb{E}[\exp(\theta_2)|\zeta_2, \zeta_2] \geq 0 \quad \text{and} \quad \mathbb{E}[\exp(\theta_2)|s_{21}, s_{22}, \zeta_1] - \mathbb{E}[\exp(\theta_1)|\zeta_1, \zeta_1] \leq 0. \tag{22}$$

The first inequality states that agent 1 prefers winning the object at price $p_2 = \mathbb{E}[exp(\theta_2)|\zeta_2, \zeta_2]$ than not winning the object. The second inequality states that agent 2 prefers not to win the object at price $\tilde{p}_2 = \mathbb{E}[exp(\theta_1)|\zeta_1, \zeta_1]$, which is the price at which agent 1 would drop out of the auction.

To prove (22) is satisfied, we first note that:

$$\mathbb{E}[\exp(\theta_1)|\zeta_1, \zeta_2] - \mathbb{E}[\exp(\theta_2)|\zeta_2, \zeta_2] \geq 0 \text{ and } \mathbb{E}[\exp(\theta_2)|\zeta_1, \zeta_2] - \mathbb{E}[\exp(\theta_1)|\zeta_1, \zeta_1] \leq 0. \quad (23)$$

Checking (23) is equivalent to checking the equilibrium conditions with one-dimensional signals (see Proposition 1). Since the equilibrium statistic satisfies (4), both of these conditions are satisfied. Using (18), we note that:

$$\mathbb{E}[\exp(\theta_1)|\zeta_1, \zeta_2] = \mathbb{E}[\exp(\theta_1)|s_{11}, s_{12}, \zeta_2] \text{ and } \mathbb{E}[\exp(\theta_2)|\zeta_1, \zeta_2] = \mathbb{E}[\exp(\theta_2)|s_{21}, s_{22}, \zeta_1].$$

Note that in (18) the expectations are taken without the exponential function. Yet, as all random variables are Gaussian, the variance is constant. Hence, the distribution of θ_1 conditional on $(s_{11}, s_{12}, \zeta_2)$ is the the same as the distribution of θ_1 conditional on (ζ_1, ζ_2). Hence, if (18) is satisfied, then (18) is also satisfied for any function of θ_i. Hence, (22) is satisfied.

4.5 Characterization of the Equilibrium Statistic

The parameter β is the key element of the equilibrium characterization that determines the properties of the equilibrium outcome. We characterize the set of equilibrium statistics.

Proposition 3 (Equilibrium Statistic)**.**
A linear combination of signals $\zeta_i = s_{i1} + \beta s_{i2}$ is an equilibrium statistic if and only if β is a root of the cubic polynomial $a \cdot \beta^3 + b \cdot \beta^2 + c \cdot \beta + d$, with:

$$\begin{array}{ccc} a = \frac{(var(\Delta\iota_i)+var(\bar{\iota}))(var(\bar{\varepsilon})+var(\bar{\varphi}))}{var(\Delta\iota_i)var(\bar{\iota})var(\bar{\varphi})} & ; & b = \frac{-1}{var(\Delta\iota_i)} \ ; \\ c = \frac{var(\Delta\varepsilon_i)+var(\bar{\varepsilon})+var(\bar{\varphi})}{var(\Delta\varepsilon_i)var(\bar{\varphi})} & ; & d = \frac{-1}{var(\Delta\varepsilon_i)}. \end{array} \quad (24)$$

Proposition 3 shows that β is determined by a cubic equation. It is transparent to see that generically (24) has 1 or 3 solutions. Moreover, it is easy to check that all roots are positive.[18] This completes the characterization of the equilibrium with two-dimensional signals. Proposition

[18] This implies that, if agents observe only ζ_i, then the one-dimensional information structure satisfies (4).

3 shows how to project the signals into the equilibrium statistic, and Theorem 2 shows that agents behave "as if" they observe only the equilibrium statistic. Finally, we can explain why the assumption of diffuse priors simplify the analysis in Wilson (1998). In the limits $var(\bar{\varphi})$,$var(\bar{\iota}) \to \infty$, the unique solution to (24) is $\beta = 1$. In this limit, the equilibrium statistic of an agent is equal to his expected payoff shock conditional only on his private information ($\zeta_i = \mathbb{E}[\theta_i | s_{i1}, s_{i2}]$).

4.6 Discussion: Spurious Correlations and Complementarities

We now provide an intuition on how the information structure determines β. To understand how β is determined, note that for any equilibrium statistic there exists constants $\{m_j\}_{j \neq i} \in \mathbb{R}^{N-1}$ and constant $(c_1, c_2) \in \mathbb{R}^2$ such that:

$$\mathbb{E}[\theta_i | s_{i1}, s_{i2}, \zeta_1, ..., \zeta_N] = c_1 \cdot s_{i1} + c_2 \cdot s_{i2} + \sum_{j \neq i} m_j \zeta_j. \tag{25}$$

By construction $\beta = c_2/c_1$. That is, β is the relative weight that agent i places in both of his signals. The important thing is to take into account the additional information that agent i learns from the drop-out time of other agents. This is taken into account by $(\zeta_1, ..., \zeta_N)$. To explain how c_1 and c_2 are determined, consider $N = 2$ and note that:

$$\mathbb{E}[\theta_i | s_{i1}, s_{i2}, \zeta_j] = \iota_i + \mathbb{E}[\bar{\varphi} | s_{i1}, s_{i2}, \zeta_j]. \tag{26}$$

Note that s_{i1} is used to compute the expectation of $\bar{\varphi}$ (that is, $\mathbb{E}[\bar{\varphi} | s_{i1}, s_{i2}, \zeta_j] \neq \mathbb{E}[\bar{\varphi} | s_{i2}, \zeta_j]$). This is because ζ_j is correlated with ι_i (and hence s_{i1}).

It is easy to check that the value of c_2 is given by:[19]

$$c_2 = \frac{\partial}{\partial s_{i2}} \mathbb{E}[\bar{\varphi} | s_{i1}, s_{i2}, \zeta_j] < \frac{\partial}{\partial s_{i2}} \mathbb{E}[\bar{\varphi} | s_{i2}]. \tag{27}$$

That is, agent i learns about $\bar{\varphi}$ from the drop-out time of agent j. Hence, the weight agent i places on s_{i2} is less than if he did not learn anything from the drop-out time of agent j. We call this the learning effect.

[19] Since all random variables are Gaussian, all expectations are linear. Hence, the derivative with respect to a random variable represents the coefficient next to this variable in the expectation.

On the other hand, if $corr(\iota_i, \iota_j) > 0$, the value of c_1 is given by:[20]

$$c_1 = 1 + \frac{\partial}{s_{i1}} \mathbb{E}[\bar{\varphi}|s_{i1}, s_{i2}, \zeta_j] < 1. \tag{28}$$

That is, agent i places less weight on ι_i than if he did not learn anything from the drop-out time of agent j (that is, $\partial \mathbb{E}[\bar{\varphi}|s_{i1}, s_{i2}, \zeta_j]/\partial s_{i1} < 0$). This is because, conditional on the drop-out time of agent j, there is a negative spurious correlation between ι_i and $\bar{\varphi}$:[21]

$$corr(\bar{\varphi} - \mathbb{E}[\bar{\varphi}|\zeta_j], \iota_i) < 0. \tag{29}$$

This is because the drop-out time of agent j is determined by $\zeta_j = \iota_j + \beta \cdot s_{j2}$. Hence, if agent i observes a high shock ι_i, then agent i also expects a high shock ι_j. For a given drop-out time of agent j, a higher realization of ι_i makes agent i more pessimistic about $\bar{\varphi}$. Hence, agent i places a smaller weight on ι_i ($c_1 < 1$). This is the spurious correlation effect.

As previously explained, the value of β is determined by c_1 and c_2. Hence, the value of β will be determined by the two explained effects: (i) the learning effect (see (27)), and (ii) the spurious correlation (see (28)). An important insight is that the weight agents place on their idiosyncratic shock (c_1) may exhibit strategic complementarities.

To explain the complementarity, consider linear a combinations of signals:

$$z_j \triangleq \iota_j + b_j \cdot s_{j2}.$$

We use z_j and b_j instead of ζ_j and β_j only to clarify that we are taking linear combination of signals that may not be an equilibrium statistic. Of course, (26) can be written the same way by replacing ζ_j with z_j. If $corr(\iota_i, \iota_j) > 0$ and b_j is small enough, then:

$$\frac{\partial}{\partial b_j} \frac{\partial}{\partial s_{i1}} \mathbb{E}[\bar{\varphi}|s_{i1}, s_{i2}, z_j] < 0. \tag{30}$$

If agent j places smaller weight on s_{j1} (or, alternatively, b_j increases), then the spurious correlation for agent i increases (that is, the absolute value of $\partial \mathbb{E}[\bar{\varphi}|s_{i1}, s_{i2}, z_j]/\partial s_{i1}$ increases). This implies that agent i will place a smaller weight on s_{i1}. This in turn makes the spurious correlation

[20] To check (28), note that the expectation can be re-written as follows: $\mathbb{E}[\theta_i|s_{i1}, s_{i2}, \zeta_j] = \iota_i + \mathbb{E}[\bar{\varphi}|s_{i2}, \tilde{\zeta}_j]$, with $\tilde{\zeta}_j \triangleq (\iota_j - corr(\iota_i, \iota_j) \cdot \iota_i) + \beta \cdot s_{j2}$. By construction $\tilde{\zeta}_j$ is independent of ι_i, and hence the expectation of $\bar{\varphi}$ does not depend on ι_i. Now note that: $\frac{\partial}{s_{i1}} \mathbb{E}[\bar{\varphi}|s_{i1}, s_{i2}, \zeta_j] = -corr(\iota_i, \iota_j) \frac{\partial}{\tilde{\zeta}_j} \mathbb{E}[\bar{\varphi}|s_{i2}, \tilde{\zeta}_j] < 0$.

[21] The use of the word spurious is consistent with the use in the econometrics literature. Formally, the correlation arises because there is a missing variable, which is ι_j.

of agent j bigger, which decreases the weight on s_{j1} (or alternatively, b_j increases). Hence, there is a complementarity.

5 Two-Dimensional Signal: Novel Predictions

In the class of equilibria characterized by Theorem 2, agents behave as if they observed a one-dimensional signal equal to the equilibrium statistic. The only difference with one-dimensional environments is that, in multidimensional environments, the behavior of an agent is determined by the equilibrium statistic. Yet, the equilibrium statistic is an endogenous object. This has two implication for the analysis of auctions with multidimensional signals: (i) there is no straightforward way to map the information structure into the surplus and the profits generated by the auction, and (ii) comparative statics are partially determined by changes in the equilibrium statistic. To illustrate these two differences, in this section we provide three predictions of ascending auctions that arise only when agent observe multidimensional signals. Throughout this section, the information structure is as in Section 4.

5.1 Multiplicity of Equilibria

An ascending auction with multidimensional signals may have multiple symmetric equilibria. The multiplicity of equilibria provides a sharp illustration that the wieght agents place on their signal to determine their drop-out time exhibits a complementarity (see (30)). Moreover, it illustrates that in environments with multidimensional signal the surplus generated and the seller's profits are endogenously determined in a non-trivial way. The multiplicity of equilibria comes directly from the fact that (24) may have multiple solutions.

Proposition 4 (Multiplicity of Equilibria)**.**
The auction has a unique (multiple) equilibrium within the class of equilibria studied in Theorem 2 if: $18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 < 0 (> 0)$ *(with* $a, b, c,$ *and* d *defined in* (24)*).*[22]

Proposition 4 is a characterization of the environments in which the polynomial (24) has multiple roots. We use Proposition 4 to derive corollaries that are easier to interpret. This provides an intuition on the environments in which the use of privates signals exhibits the largest complementarities (see (30)). We show that the following conditions are necessary for the multiplicity

[22] The case $18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 = 0$ must be considered independently. If $18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 = 0$, then there is a unique equilibrium if and only if $b = 3ac$.

of equilibria: (i) the idiosyncratic shocks are positively correlated; and (ii) the private signal an agent observes about the common shock is large, but not too large.

***(i)* Correlated idiosyncratic shocks.** Multiple equilibria arise only if idiosyncratic shocks are positively correlated ($corr(\iota_i, \iota_j) > 0$).

Corollary 2 (Correlation in Idiosyncratic Shocks)**.**
If the idiosyncratic shocks are independently distributed ($corr(\iota_i, \iota_j) = 0$), then the ascending auction has a unique equilibrium (within the class of equilibria studied in Theorem 2).

Corollary 2 shows that correlated idiosyncratic shocks is a necessary condition to find multiple equilibria in the ascending auction. If the idiosyncratic shocks are independently distributed, then there is no spurious correlation between ι_i and $\bar{\varphi}$, and hence no complementarity in the weight agents place on ι_i (see (28)).

***(ii)* Intermediate size of the noise terms.** We show that for large enough $var(\varepsilon_i)$ or small enough $var(\varepsilon_i)$ there is a unique equilibrium.

Corollary 3 (Uniqueness of Equilibrium)**.**
If either $var(\varepsilon_i) \to 0$ or $var(\varepsilon_i) \to \infty$, then there exists a unique equilibrium (within the class of equilibria studied in Theorem 2).

Corollary 3 shows that if the noise term is large enough or small enough, then there always exist a unique equilibrium. If $var(\varepsilon_i) \to 0$, then agents have complete information. This limit corresponds to a private value environment, and hence there is a unique equilibrium. If $var(\varepsilon_i) \to \infty$, then agent i ignore s_{i2}. This limit also corresponds to a private value environment, and hence there is a unique equilibrium.

Corollary 3 shows that multiple equilibria only arise for an intermediate level of noise, but it does not provide any intuition on the magnitudes. Multiple equilibria arise when signal s_{i2} is noisy enough such that agent i does not learn "too much" about $\bar{\varphi}$ from s_{i2}. Additionally, signal s_{i2} must be precise enough such that the collection of all signals $(s_{12}, ..., s_{N2})$ is informative about $\bar{\varphi}$. In a loose sense, this can be stated as follows:

$$corr(s_{i2}, \bar{\varphi}) \approx 0 \text{ and } corr(\bar{s}_2, \bar{\varphi}) \approx 1, \tag{31}$$

where $\bar{s}_2 = \sum_{i \in N} s_{i2}/N = \bar{\varphi} + \bar{\varepsilon}$. If (31) is satisfied, when agent i observes only s_{i2} he cannot make a precise prediction of $\bar{\varphi}$. Yet, if agent i observed $(s_{12}, ..., s_{N2})$, then he would be able to make a precise prediction of $\bar{\varphi}$.

As the number of agents increases, the possibility that (31) is satisfied for very noisy signals increases. This is because $var(\bar{\varepsilon}) = var(\varepsilon_i)/N$ becomes smaller with N. Hence, $corr(\bar{s}_2, \bar{\varphi})$ converges to 1 as $N \to \infty$. Hence, by taking the limits $N \to \infty$ and $var(\varepsilon_i) \to \infty$ at the right rates, $corr(\bar{s}_2, \bar{\varphi})$ converges to 1 and $corr(s_{i2}, \bar{\varphi})$ converges to 0. This happens when N grows faster than $var(\varepsilon_i)$. This is the same as imposing $var(\bar{\varepsilon}) = 0$, and then taking the limit $var(\Delta\varepsilon_i) \to \infty$.

Corollary 4 (Multiplicity of Equilibria)**.**
If $var(\varepsilon_i) \to \infty,\ N \to \infty$ (with N diverging faster than $var(\varepsilon_i)$), then there are multiple equilibria if and only if:

$$var(\bar{\varphi}) \geq 4var(\Delta\iota_i)(1 + \frac{var(\Delta\iota_i)}{var(\bar{\iota})}). \tag{32}$$

Corollary 4 shows that it is possible to have multiple equilibria, even in the limit $var(\Delta\varepsilon_i) \to \infty$, as long as N grows faster. This shows that the model has an interesting discontinuity as we approach large markets ($N \to \infty$). In the limit $N \to \infty$ and $var(\varepsilon_i) \to \infty$, the model does not approach a model of private values. This is because if the number of agents grows faster than $var(\varepsilon_i)$, then the collection of the signals observed by all the agents is still informative about $\bar{\varphi}$.

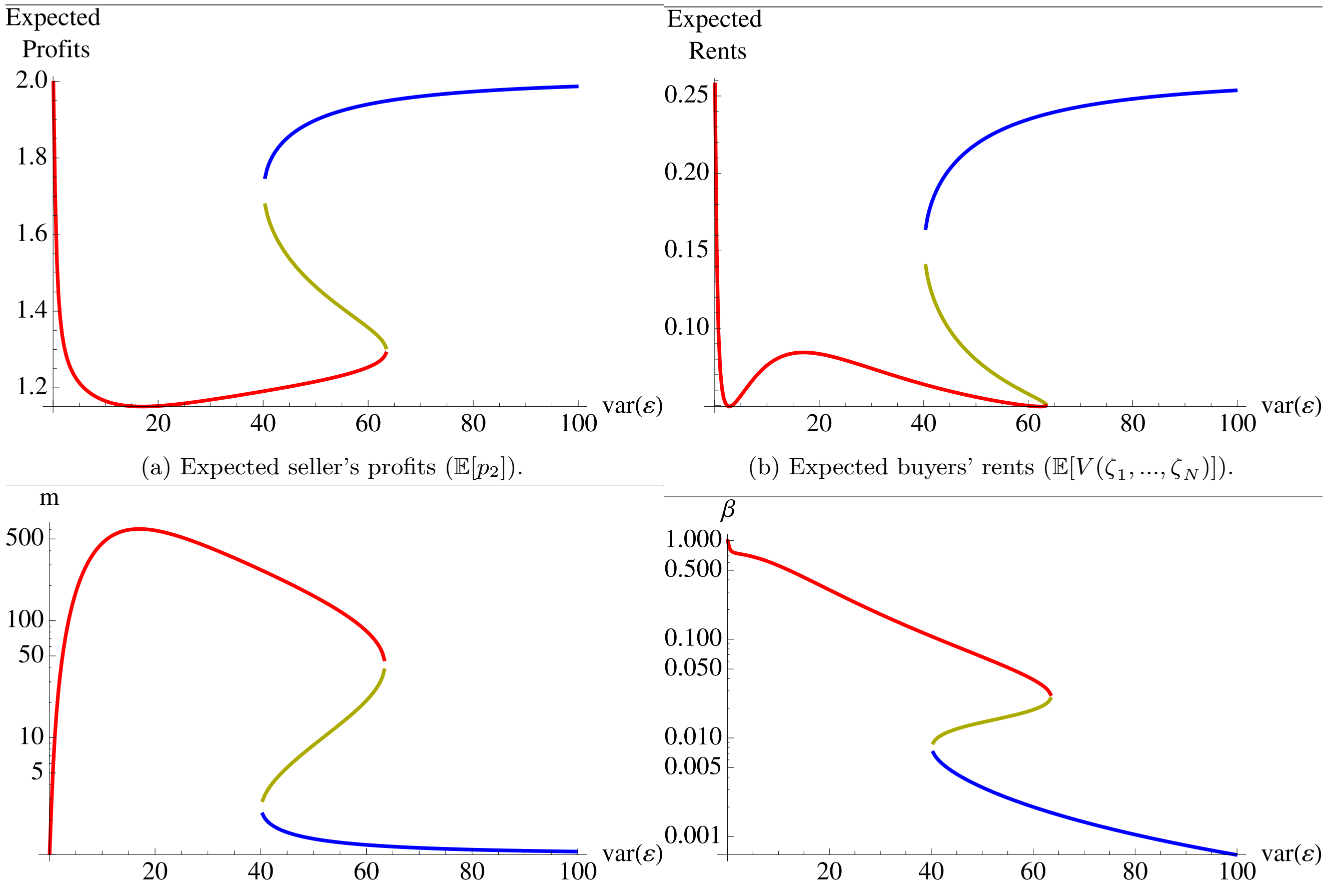
Finally, we show that having an environment with a large number of agents is not necessary to have multiple equilibria.

Corollary 5 (Multiplicity with Any Number of Agents)**.**
For any number of agents N, there exists $var(\bar{\varphi})$, $var(\iota_i)$, $corr(\iota_i, \iota_j)$, and $var(\varepsilon_i)$ such that the ascending auction has multiple equilibria.

Corollary 5 shows that there may be multiple equilibria, regardless of the number of agents.

5.2 Discussion: Outcomes of the Multiple Equilibria

In Figure 1 we plot the outcome of the ascending auction for a parametrized example. In Figure 1d we plot the value of β for different values of $var(\varepsilon_i)$ (note that this is in log-scale). The value of β gives the relative weight agents place on their signals. In Figure 1c we plot the value of m induced by the equilibrium statistic. In Figure 1a and Figure 1b we plot the seller's profits and the buyers' rents. It is easy to check from the scale of the plots that the buyers' rents are small, and hence the total surplus is qualitatively similar to the seller's profits. The different colors in the plot corresponds to the different roots of (24).



(a) Expected seller's profits ($\mathbb{E}[p_2]$).

(b) Expected buyers' rents ($\mathbb{E}[V(\zeta_1, ..., \zeta_N)]$).

(c) Parameter m in log-scale.

(d) β in log-scale.

Figure 1: Outcome of ascending auction for $var(\bar{\varphi}) = (5/2)^2$, $var(\iota_i) = (0.6)^2$, $corr(\iota_i, \iota_j) = 3/4$ and $N = 50$.

We begin by studying the equilibrium in blue. There is an equilibrium in which agents "almost" ignore their signals s_{i2} ($\beta \approx 0$). Agent i by looking at the drop-out time of agent j learns ι_j. Hence, agent i predicts θ_i only using his private information. Yet, as $var(\varepsilon_i)$ is large, the agent places a small weight on s_{i2}. Hence, in equilibrium the weight on s_{i1} is much larger than s_{i2}. This is the equilibrium plotted in blue in Figure 1.

If we restrict attention to the equilibrium in blue, the seller's profits and the buyers' rents increase with $var(\varepsilon_i)$. This is because as $var(\varepsilon_i)$ increases, agents place smaller weight on s_{i2} (β decreases). This leads to a higher surplus, which also generates a higher seller's profits and higher buyers' rents.

We now study the equilibrium in red in Figure 1. Suppose that all agents place a weight on s_{i2} that is non-negligible with respect to the weight on s_{i1}. This creates a negative spurious correlation between ι_i and $\bar{\varphi}$ (see Section 4.6). Hence, agent i reduces the weight that he places on s_{i1}. Hence, agent i places a small weight on both signals.

If we restrict attention to the equilibrium in red, the seller's profits are quasi-convex in $var(\varepsilon_i)$. For a large enough $var(\varepsilon_i)$, the intuition is similar to the equilibrium in blue. That is, as $var(\varepsilon_i)$ increases, the weight on s_{i2} decreases sufficiently fast such that the total surplus increases. Nevertheless, for small values of $var(\varepsilon_i)$, the rate at which the weight on s_{i2} decreases is not fast enough to compensate for the fact that ε_i has a higher variance. This implies that the total surplus decreases because the correlation between the drop-out time of agents and the noise term ε_i increases.

It is interesting to note that the buyers' rents have an unusual shape with a hump. This is because as $var(\varepsilon_i)$ increases two things happen. The first thing that happens is that the total surplus decreases. The second thing that happens is that the level of informational interdependence (m) increases. As the level of informational interdependence increases, buyers get a higher share of the total surplus. Hence, there are values of $var(\varepsilon_i)$ for which the total surplus decreases but the buyers' rents increases. This leads to the hump-shape of the buyers' rents in Figure 1b.

5.3 Impact of Public Signals on Surplus

We now study the impact of public information on the equilibrium outcome. We assume agents have access to two public signals (in addition to the signals in (17)):

$$\bar{s}_3 = \bar{\iota} + \bar{e}_3 \text{ and } \bar{s}_4 = \bar{\varphi} + \bar{e}_4, \tag{33}$$

where $\bar{\varepsilon}_3$ and $\bar{\varepsilon}_4$ are independent of all random variables defined so far. Hence, agent i observes the signals $(s_{i1}, s_{i2}, \bar{s}_3, \bar{s}_4)$. The signal $\bar{s}_4$ is additional information about the common shock ($\bar{\varphi}$), and hence this can be interpreted as disclosing additional information about the good. On the other hand, $\bar{s}_3$ is a signal on the average idiosyncratic shock of agents. This can be interpreted as allowing each agent to observe the other agents that are in the auction to get an estimate of their idiosyncratic shock.[23] Alternatively, public signal $\bar{s}_3$ has the same effect as reducing the correlation of idiosyncratic shocks across agents.[24]

We study how the public signals impact the total surplus.

[23] All the results go through in the same way if instead of having a public signal $\bar{s}_3 = \bar{\iota} + \bar{e}_3$ each agent i observes $N-1$ private signals on the payoff shocks of agents $j \neq i$. That is, if agent i observes signals $s_{ij3} = \iota_j + \varepsilon_{ij3}$ for all $j \neq i$.

[24] Analyzing a model with public signal $\bar{s}_3$, is the same as analyzing a model without public signal, but in which the idiosyncratic shock of each agent is equal to $\iota_i' = \iota_i - \mathbb{E}[\iota_i|\bar{s}_3]$. Under this alternative information structure $corr(\iota_i', \iota_j') < corr(\iota_i, \iota_j)$.

Proposition 5 (Comparative Statics of Public Signals: Surplus)**.**
If the ascending auction has a unique equilibrium, then the total surplus is decreasing in $var(\bar{e}_3)$ *and* $var(\bar{e}_4)$*. In the limit:*

$$\lim_{var(\varepsilon_3)\to 0} S(\zeta_1,...,\zeta_N) = \lim_{var(\varepsilon_4)\to 0} S(\zeta_1,...,\zeta_N) = \max_{i\in N} \quad \mathbb{E}[exp(\theta_i)|\{(s_{j1},s_{j2})\}_{j\in N}].$$

Proposition 5 shows that the surplus increases with the precision of the public signals. In the limit in which one of the public signals is arbitrarily precise, the equilibrium approaches the first best. Note that for any value of $var(\varepsilon_i)$ the ascending auction would implement the efficient outcome if agents "ignored" signal s_{i2}. Hence, a precise enough public signal reduces the weight that agents place on s_{i2} all the way to 0. Proposition 5 requires that the ascending auction has a unique equilibrium. If the ascending auction has three equilibria then the result holds for two of the equilibria, while the comparative static is reversed for the equilibrium "in the middle".

The intuition on why the surplus is decreasing in $var(\bar{e}_4)$ is simple. As the public information about $\bar{\varphi}$ is more precise, an agent needs to place less weight on their private signal s_{i2} to predict $\bar{\varphi}$. This implies that the correlation between the drop-out time of an agent and the realization of the noise term ε_i decreases. Hence, the surplus increases.

The mechanism by which $var(\bar{e}_3)$ impacts efficiency is more subtle. Signal $\bar{s}_3$ does not change an agent's expectation of his own payoff shock conditional only on his private information:

$$\mathbb{E}[\theta_i|s_{i1},s_{i2},\bar{s}_3] = \mathbb{E}[\theta_i|s_{i1},s_{i2}].$$

On the other hand, $\bar{s}_3$ allows an agent to extract more information from the drop-out time of another agent. That is, agent i uses the drop-out time of agent j as a signal about $\bar{\varphi}$. Since the drop-out time of agent j is determined by $\iota_j + \beta \cdot s_{2j}$, public signal $\bar{s}_3$ makes the information from the drop-out time of agent j a more precise signal about $\bar{\varphi}$. Nevertheless, the reason that a precise enough signal $\bar{s}_3$ induces the efficient outcome is not driven by a direct learning channel.

Suppose agent i could learn $(s_{12},...,s_{N2})$ from the drop-out of other agents. In this case agent i's expectation would be equal to:

$$\mathbb{E}[\theta_i|s_{i1},s_{12},...,s_{N2}] = \iota_i + \sum_{j\in N} \frac{\sigma^2_{\bar{\varphi}}}{\sigma^2_{\bar{\varphi}} + \sigma^2_{\varepsilon_i}/N}\frac{s_{j2}}{N} \tag{34}$$

In this case the equilibrium statistic would be equal to:

$$\hat{\zeta}_i = \iota_i + \frac{\sigma_{\bar{\varphi}}^2}{\sigma_{\bar{\varphi}}^2 + \sigma_{\varepsilon_i}^2/N} \frac{s_{i2}}{N}. \tag{35}$$

Yet, Proposition 5 shows that in the limit $var(\bar{\varepsilon}_3) \to 0$ the equilibrium is efficient. Hence, in the limit, the weight on s_{i2} is strictly less than: $\frac{\sigma_{\bar{\varphi}}^2}{\sigma_{\bar{\varphi}}^2+\sigma_{\bar{\varepsilon}_i}^2/N}\frac{1}{N}$. Hence, even if agent i could observe $(s_{12}, ..., s_{N2})$, this could still not explain why the weight on s_{i2} is so low.

If agents observe public signal, it is as if the the idiosyncratic shock of each agent is equal to:

$$\iota_i' \triangleq \iota_i - \mathbb{E}[\iota_i | \bar{s}_3].$$

Hence, the public signal has the same effect as reducing the correlation between idiosyncratic shocks.[24] This in turn reduces spurious correlation between ι_i and $\bar{\varphi}$ (see (28)). In the limit $var(\bar{\varepsilon}_3) \to 0$, the effective correlation between idiosyncratic shock is negative $(corr(\iota_i', \iota_j') < 0)$, and hence there is a positive spurious correlation. Hence, the weight on signal s_{i1} is excessively high. In relative terms this make β arbitrarily low.

5.4 Discussion: Efficiency of the Ascending Auction

In symmetric environments with one-dimensional signals an ascending auctions is efficient. Hence, it is straightforward that public signals do not change the total surplus. This is also true for most classic mechanisms (e.g. first-price auction). Hence, it is immaterial to compare different mechanisms according to the surplus they generate. If agents observe multidimensional signals, then the ascending auction is not efficient. Hence, it is natural to ask whether an ascending auction is the most efficient mechanism in this environment. More broadly, it is natural to compare the surplus generated by the ascending auction with the surplus generated by other mechanisms. We briefly discuss how the ascending auction compares with some direct mechanisms.[28]

In our model it is possible to construct a direct mechanism that implements the efficient allocation under Bayesian incentive compatibility. This is because the signals observed by agents are correlated across agents. Hence, it is possible to use the results in Cremer and McLean (1988). Yet, as it is well known in the literature, there are several reasons these mechanisms are not practical.

In our environment it is possible to construct a direct mechanism that has an ex post equilibrium.[25] [26] In any ex post equilibrium, the allocation of the good is determined by $(\hat{\zeta}_1, ..., \hat{\zeta}_N)$ (see (35)).[27] In fact, there exists a direct mechanism in which the agent that observes the highest one-dimensional statistic $\hat{\zeta}_i$ gets the object. This would not be an efficient mechanism. We can compare this mechanism with an ascending auction.

It is possible to check that allocating the good according to the one-dimensional statistic $\hat{\zeta}_i$ may be more efficient than the ascending auction. This is true under the parametrization in Figure 1 in the absence of any public signal. If a precise enough public signal is disclosed $\bar{s}_3$, then the allocation implemented by the ascending auction would be efficient. Nevertheless, the allocation of the direct mechanism that has an ex post equilibrium would continue to be determined by the one-dimensional statistic $\hat{\zeta}_i$. Hence, with a precise enough public signal the ascending auction would be more efficient than the direct mechanism.

The informal comparison is sufficient to illustrate a simple point: different mechanisms generate different surplus. Moreover, the rank may depend on the information structure. Of course, one expects that in multidimensional environments the efficiency of an ascending auction and a first-price auction also differ.[28] This leaves the open question on how to rank different mechanisms according to their surplus. Answering this question is outside the scope of this paper, but we expect the answer to depend on the details of the information structure.

5.5 Impact of Public Information on Profits

We continue to study the impact of public signals, as in (33), but we now study the impact of the public signals on the seller's profits.

Proposition 6 (Comparative Statics of Public Signals: Profits)**.**
If one of the public signals becomes arbitrarily precise:

$$\lim_{var(\varepsilon_3)\to 0} p_2 = 0 \quad \textit{and} \quad \lim_{var(\varepsilon_4)\to 0} p_2 = \max_{i\in N}{}^{(2)} \; \mathbb{E}[exp(\theta_i)|\{(s_{j1}, s_{j2})\}_{j\in N}],$$

where $\max^{(2)}\{\cdot\}$ *denotes the second order statistic (that is, the second maximum).*

[25]In an ex post equilibrium the action of an agent remains optimal, even if the agent knew the realization of the signals of all other agents. Hence, it has a robustness property that is shared with the ascending auction. This makes it a suitable benchmark.

[26]The impossibility theorem in Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006) does not apply to our model because we study an environment in which signals enter linearly in the expectation of a payoff shock (see (8)). That is, this is a separable type-space.

[27]See Jehiel, Meyer-ter Vehn, and Moldovanu (2008) for more results on implementation of ex post equilibrium in separable environments.

[28] It would be natural to compare the ascending auction with a first-price auction. Yet, solving a first-price auction in multidimensional environment is a difficult problem. This is material for future work.

Proposition 6 shows that, as the public signal about $\bar{\varphi}$ becomes arbitrarily precise ($var(\varepsilon_4) \to 0$), the outcome approaches the equilibrium under complete information. The intuition is that in the limit, agent ignore their private signal s_{i2}, and hence the only private signal they observe is $s_{i1} = \iota_i$. Hence, in this limit, the equilibrium is as if agents observed one-dimensional signals.

The impact of the public signal about $\bar{\iota}$ on profits is different. As $\bar{\iota}$ becomes common knowledge ($var(\bar{\varepsilon}_3) \to 0$), the equilibrium profits becomes arbitrarily close to 0.[29] In the limit $var(\bar{\varepsilon}_3) \to 0$, it is as if $var(\bar{\iota}) \to 0$. This is because agents learn $\bar{\iota}$ from the public signal, and hence they can "filter it out" from s_{i1}. Looking at (21), we can see that in the limit $var(\bar{\iota}) \to 0$ and $\beta \to 0$, we have that $m \to \infty$. Looking at (9) it becomes clear why the profits converge to 0 in the limit $m \to \infty$. Note that, it is the joint effect of a low average idiosyncratic shock ($var(\bar{\iota}) \approx 0$) and the fact that the equilibrium statistic endogenously places a small weight on s_{12} ($\beta \approx 0$), which reduces the profits to 0.

To illustrate the intuition, consider the case $N = 2$. As $var(\bar{\varepsilon}_3) \to 0$, the equilibrium statistic of agent i is almost equal to:

$$\zeta_i \approx \Delta\iota_i + \epsilon \cdot s_{i2} = \Delta\iota_i + \epsilon \cdot (\bar{\varphi} + \varepsilon_i). \tag{36}$$

In the limit $\epsilon \to 0$, the signals of agents are almost perfectly *negatively* correlated (remember that $\Delta\iota_1 + \Delta\iota_2 = 0$). If agent i observes a negative realization of his signal, he expects agent j to observe a positive realization of his signal. Hence, he expects to loose the auction. The only circumstance under which both agents can have a negative realization of their signals is that the realization of $\bar{\zeta}$ is negative and "far" from 0. Yet, as $\epsilon \to 0$ this implies that $\bar{\varphi}$ is "very" negative. That is, when an agent observes a negative realization of his signal, he knows that he observed the highest signal only if s_{2j} is very negative (and hence, $\bar{\varphi}$ is probably very negative). Hence, he will win the object only if his payoff shock ($\theta_i = \iota_i + \bar{\varphi}$) is very low. Hence, the bid of an agent who observes a negative signal will converge to 0. Hence, the agent that wins the object will pay almost 0.

5.6 Discussion: Failure of Linkage Principle

The intuition from proposition 6 is that a public signal about $\bar{\iota}$ exacerbates the "winner's curse" from $\bar{\varphi}$. If agent i expect a lower idiosyncratic shock than agent j, then agent i anticipates that

[29] Note that Proposition 6 makes no assumption on the variance of the shock ε_i or $\Delta\theta_i$. Hence, if $\varepsilon_i \approx 0$ and $var(\bar{\varepsilon}_4) \to 0$ then shocks are arbitrary close to common knowledge. Clearly, the result also holds for any distribution of payoff shocks.

to win the object the signal of agent j about the common shock must be not only lower than his own signal, but low enough to compensate the differential in idiosyncratic shock. Hence, agent i faces an exacerbated "winner's curse".

The same intuition can be applied to more general environments. Consider a model in which we modify s_{i1} as follows:

$$s'_{i1} = \iota_i + \varepsilon'_{i1}, \tag{37}$$

where ε'_{i1} is a noise term independent of all other random variables in the model and independent across agents. To make the argument simpler, assume that the variance of ε'_{i1} is small. Going back to the oil field example, (37) is interpreted as assuming agents observe only a noisy signal about their own cost of extracting oil. Yet, the costs may be correlated across agents. Public signal $\bar{s}_3$ can be interpreted as information about the oil field that gives agents better information about their costs.

If agents observe *only* s'_{i1}, this would constitute a classic one-dimensional environment with interdependent values (it is equivalent to Example 1). If agents observe only s'_{i1}, then the profits would be strictly increasing in the precision of $\bar{s}_3$ (decreasing in $var(\bar{\varepsilon}_3)$).

Lemma 2 (Impact of Public Signal: One-Dimensional Information Structure)**.**
If agents observe only signal s'_{i1} (see (37)), then the seller's profits are increasing in the precision of $\bar{s}_3$ (decreasing in $var(\bar{\varepsilon}_3)$).

Lemma 2 show that a public signal increases profits when agents observe one-dimensional signals. This is because it decreases the "winner's curse" from $\bar{\iota}$. Although Lemma 2 is stated in terms of a very stylized one-dimensional signal, Milgrom and Weber (1982) shows the result holds across a wide class of one-dimensional signals.

On the other hand, if agents observe s'_{i1} and s_{i2}, then the profits would continue to be decreasing in the precision of $\bar{s}_3$ (increasing in $var(\bar{\varepsilon}_3)$).[30] The key insight from Section 5.5 is that public signals have two effects. First, it decreases the winner's curse from one of the signals agents observe. This increases the profits. The second effect of public signals is that it exacerbates the "winner's curse" form another signal. This is because public signals provide agents with more information about the differences in the payoff shocks across agents. This increases the winner's curse from another common shock, which decreases the profits. In other words, a public signal decreases the informational interdependence from one signal, and increases the informational interdependence from another signal.

[30]This can be seen by Proposition 6 and a continuity argument.

The fact that profits can become arbitrarily close to 0 is reminiscent of the impact of ex ante asymmetries on profits. Klemperer (1998) shows that in an ascending auction with two players, ex ante differences in the payoff environment may lead to explosive behaviors. In contrast to the literature studying ascending auctions with asymmetric agents, our results are driven only by the information structure and not by assumptions on the payoff structure. Importantly, we show that it is always the case that public signals have two effects: decrease the “winner’s curse” from one signal and increase the “winner’s curse” from another signal.

Bergemann, Brooks, and Morris (2016) show that the profits in a first-price auction are bounded away from 0. Proposition 6 shows that the profits in an ascending auction are not bounded away from 0. Milgrom and Weber (1982) show that that in symmetric environments with one-dimensional signals, ascending auctions yields weakly greater profits than first-price auctions. The reason that the result in Milgrom and Weber (1982) does not apply in our model is because in the limit $var(\bar{\varepsilon}_3) \to 0$ the equilibrium statistic is negatively correlated across agents (see (36)). Hence, the affiliation property is not satisfied.

Proposition 6 can be interpreted as a failure of the linkage principle.[31] The linkage principle has been shown to fail in other environments.[32] In contrast to the previous literature we show that the linkage principle may fail in natural symmetric environments. This is only due to the multidimensionality of the information structure. Hence, our paper provides a new channel by which the linkage principle may fail.

6 General Multidimensional Signals

We extend the methodology in Section 4 to allow for asymmetric information structures. The idea remains the same as in Section 4. That is, we first compute an equilibrium statistic, and then compute the equilibrium as if agents observe only the equilibrium statistic. We later show that the analysis can be extended to a larger class of mechanisms.

6.1 Information Structure

We first study a model in which agents observe one-dimensional signals. In contrast to Section 3, we allow signals and payoff shocks to be asymmetrically distributed. We keep the information

[31] The linkage principle states that public signals increase profits and ascending auctions yield higher profits than first-price auctions (see Krishna (2009) for a textbook discussion).

[32] Perry and Reny (1999) show that the linkage principle may fail in multi-unit auctions. The linkage principle has also been shown to fail in environments in which the payoff structure is asymmetric (see Krishna (2009)) and in environments with independent and private values (see Thierry and Stefano (2003)).

structure the same as in Section 2.2, but allow for arbitrary information structures. That is, we allow for any distribution of signals and fundamentals $(\theta_1, ..., \theta_N, \mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^{(J+1)N}$ as long as the distribution is jointly Gaussian.

Example 5 (Asymmetric Agents)**.** *The agent i's payoff shock is decomposed as follows:*

$$\theta_i = \iota_i + \bar{\varphi},$$

where the idiosyncratic shocks $(\iota_1, ..., \iota_N)$ are jointly normally distributed. The idiosyncratic shocks are asymmetrically distributed. Agent i observes two signals:

$$s_{i1} = \iota_i \ ; \ s_{i2} = \bar{\varphi} + \varepsilon_i,$$

where ε_i is independent across agents and independent of all other random variables in the model, but $var(\varepsilon_i)$ is different across agents.

6.2 One-Dimensional Signals

We begin by studying one-dimensional signals. If agents observe one-dimensional signals, and the average crossing condition is satisfied, then the ascending auction has an ex post equilibrium that is efficient (see Krishna (2003)). The average crossing condition is defined as follows.

Definition 2 (Average Crossing Condition)**.**
The one-dimensional information structure $(s_1, ..., s_N, \theta_1, ..., \theta_N)$ satisfies the average crossing condition if for all $\mathcal{A} \subset \{1, ..., N\}$, and for all $i, j \in \mathcal{A}$ with $i \neq j$:

$$0 < \frac{\partial \mathbb{E}[\theta_i | s_1, ..., s_N]}{\partial s_j} \leq \frac{1}{|\mathcal{A}|} \sum_{k \in N} \frac{\partial \mathbb{E}[\theta_k | s_1, ..., s_N]}{\partial s_j}$$

The average crossing condition guarantees that the impact of agent i's signal on agent j's valuation is not to high. The comparison is done with respect to the average impact that agent i's signal has on any group of agents that contains i.

To characterize the equilibrium we assume that agents are ordered as follows:

$$\mathbb{E}[\theta_i | s_1, ..., s_N] > ... > \mathbb{E}[\theta_N | s_1, ..., s_N]. \tag{38}$$

That is, we assume that agents are ordered according to their expected valuation conditional on the signals of all agents. To characterize the equilibrium we define $\tilde{s}_1 \in \mathbb{R}$ as follows:

$$\begin{array}{llll} \tilde{s}_1 \triangleq & \arg\min_{s' \in \mathbb{R}} & & \mathbb{E}[\theta_1 | s', s_2, ..., s_N] \\ & \text{subject to} & \forall i \in N, & \mathbb{E}[\theta_1 | s', s_2, ..., s_N] \geq \mathbb{E}[\theta_i | s', s_2, ..., s_N] \end{array} \tag{39}$$

$\tilde{s}_1$ is the signal that yields the lowest expected payoff shock to agent 1, but keeping the expected payoff shock of agent 1 above the expected payoff shocks of other agents.

Proposition 7 (Equilibrium for One-Dimensional Signals)**.**
The ascending auction has a Nash equilibrium in which agent 1 *wins the object and pays a price:*

$$p_2 = \mathbb{E}[\theta_1 | \tilde{s}_1, s_2, ..., s_N]. \tag{40}$$

The ascending auction has an equilibrium in which the agent with the highest expected valuation wins the object. The price paid for the object is the expected valuation of the winner of the object, but evaluated at the minimum signal this agent could have observed and still win the object.

6.3 Equilibrium Statistic

We define an equilibrium statistic for general Gaussian information structures.

Definition 3 (Equilibrium Statistic)**.**
We say random variables $(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N) \in \mathbb{R}^N$ *are an equilibrium statistic of* $(\mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^{J \cdot N}$ *if:*

$$\forall i \in N, \quad \mathbb{E}[\theta_i | \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N] = \mathbb{E}[\theta_i | \mathbf{s}_i, \boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N]. \tag{41}$$

The definition of an equilibrium statistic is the natural extension of Definition 1, but allowing for general J-dimensional signals. Note that the weights on the signals of agent i ($\boldsymbol{\beta}_i$), may be different than the weights on the signals of agent j ($\boldsymbol{\beta}_j$). As before, in order to make the notation more compact, we often denote an equilibrium statistic by:

$$(\zeta_1, ..., \zeta_N) \triangleq (\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N). \tag{42}$$

The equilibrium statistic is the fundamental object that allows us to characterize the equilibrium in multidimensional environments.

In order to prove that an equilibrium statistic exists, we assume that every agent i observes an additional signal (label $J+1$):

$$s_{iJ+1} = \epsilon_i, \tag{43}$$

where ϵ_i is normally distributed with mean equal to 0 and variance equal to 1, independent across agents and independent of all other random variables in the model. That is, each agent observes an additional signal that is only noise. We call this the *augmented* information structure. We prove that an equilibrium statistic exists

Proposition 8 (Existence)**.**
If the variance covariance matrix var$(\mathbf{s}_1, ..., \mathbf{s}_N)$ has full rank, then the augmented information structure has an equilibrium statistic exists. If the information structure is symmetric, then there exists a symmetric equilibrium statistic.[33]

Proposition 8 guarantees the existence of equilibrium statistic for generic information structures.[34] The additional signal (43) is used in the proof only in the knife-edge cases in which agent i observes signals that contain no information about θ_i. Hence, in any natural example it is not necessary to consider the additional signal s_{iJ+1} to guarantee the existence of an equilibrium statistic. In the proof of Proposition 8 we explain in more detail under what circumstances the additional signal s_{iJ+1} could be needed.

Finding the set of equilibrium statistics $(\{\boldsymbol{\beta}_i\}_{i\in N})$ can be found by using a "guess-and-verify" method and checking that the coefficients satisfy (41). In symmetric environments, this reduces to finding the roots of a polynomial of order $2 \cdot J - 1$. In asymmetric environments, this reduces to solving a multilinear system of equations.

6.4 Equilibrium with Multidimensional Signals

It is possible to characterize a set of equilibria in multidimensional environments by computing the equilibrium "as if" agents observed only the equilibrium statistic. We assume that there exists an equilibrium statistic $(\zeta_1, ..., \zeta_N)$ that satisfies the average crossing condition. Agents are ordered as follows:

$$\mathbb{E}[\theta_i|\zeta_1, ..., \zeta_N] > ... > \mathbb{E}[\theta_N|\zeta_1, ..., \zeta_N]. \tag{44}$$

[33] For symmetric information structures it is possible to prove the existence of an equilibrium statistic without the need to consider the augmented information structure.

[34] The uniqueness of the equilibrium statistic is clearly not guaranteed (see Section 5.1).

That is, we assume that agents are ordered according to their expected valuation conditional on the equilibrium statistic of all agents. We define $\tilde{\zeta}_1 \in \mathbb{R}$ as follows:

$$\begin{aligned} \tilde{\zeta}_1 \triangleq \quad & \arg\min_{\zeta' \in \mathbb{R}} && \mathbb{E}[\theta_1 | \zeta', \zeta_2, ..., \zeta_N] \\ & \text{subject to} && \forall i \in N, \quad \mathbb{E}[\theta_1 | \zeta', \zeta_2, ..., \zeta_N] \geq \mathbb{E}[\theta_i | \zeta', \zeta_2, ..., \zeta_N] \end{aligned} \tag{45}$$

$\tilde{\zeta}_1$ is the analogous of $\tilde{s}_i$, but using the equilibrium statistic.

Theorem 3 (Equilibrium for Multidimensional Signals)**.**
The ascending auction has a Nash equilibrium in which agent 1 wins the object and pays a price equal to:

$$p_2 = \mathbb{E}[\theta_1 | \tilde{\zeta}_1, \zeta_2, ..., \zeta_N].$$

Theorem 3 characterizes a class of equilibria in which agents behave "as if" they observe only one-dimensional signals. Note that this characterization requires that the equilibrium statistic $(\zeta_1, ..., \zeta_N)$ satisfies the average crossing. This is because it is necessary to guarantee that the ascending auction has an ex post equilibrium when agents observe only their equilibrium statistic. It is simple to check in applications whether the information structure has an equilibrium statistic that satisfies the average crossing condition.

The same characterization can be applied if we consider an ascending auction with reentry (see the following section for a formal argument). Besides being a more realistic model in many applications, allowing for reentry relaxes the conditions under which an ex post equilibrium exists when agents observe one-dimensional signals (see Izmalkov (2001)). Hence, it is convenient to briefly discuss how Theorem 3 would change if we allowed for reentry. In an ascending auction with reentry, an ex post equilibrium exists for every symmetric one-dimensional Gaussian information structure.[35] Hence, an equilibrium exists for every multidimensional Gaussian signals. We have not been able to prove a general result for asymmetric information structures but, if we allow for reentry, in Example 5 an equilibrium exists.[36]

[35] In Section 3 we assumed $m \geq 0$ (this is the same as (4)). Nevertheless, the same characterization also yields an ex post equilibrium if $m \leq -(N-1)$, the proof is the same as the proof of Proposition 1 (the only difference is that this equilibrium is inefficient). If $m \in (-N-1), 0)$, then the characterization in Section 3 does not hold. Yet, in this case, an ascending auction with reentry yields an ex post efficient equilibrium, and the allocation and transfers are the same as in Proposition 1.

[36] We do not know if there exists an information structure as in Example 5 in which an equilibrium does not exists when reentry is not allowed.

7 Extension: Other Mechanisms

We now extend the methodology to accommodate for other mechanisms. We characterize a class of equilibria in which agents behave "as if" agents observe only their equilibrium statistic. Importantly, the definition of an equilibrium statistic does not change.

7.1 General Games

We consider a game with N players. Player $i \in N$ takes action $a_i \in A_i$, where A_i is assumed to be a metric space. The payoff of player $i \in \{1, ..., N\}$ depends on the realization of his payoff shock $\theta_i \in \mathbb{R}$ and the action taken by all players:

$$\mathbf{a} \triangleq (a_1, ..., a_N).$$

The payoff of player i is denote by $u_i(\theta_i, \mathbf{a})$. We denote by $(a_i', \mathbf{a}_{-i})$ the action profile:

$$(a_i', \mathbf{a}_{-i}) = (a_1, ..., a_{i-1}, a_i', a_{i+1}, ..., a_N).$$

We keep the information structure the same as in Section 6. The definition of an equilibrium statistic is the same as in Definition 3.

We distinguish between the payoff environment and the information structure. This is because we want to compute the set of equilibria for a fixed payoff environment, but under different information structures. The actions available to each agent and the utility functions are called the payoff environment and are denoted by P. The joint distribution of signals and fundamentals is the information structure and is denoted by $\mathcal{I}$. The game is defined by the payoff environment and the information structure $(P, \mathcal{I})$. Given an equilibrium statistic $(\zeta_1, ..., \zeta_N) \in \mathbb{R}^N$, the information structure in which agent i observes *only* ζ_i is called the reduced form information structure and is denoted by $\hat{\mathcal{I}}$.

In game $(P, \mathcal{I})$, a strategy profile for agent i is defined by a function $\alpha_i : \mathbb{R}^J \to A_i$. In game $(P, \hat{\mathcal{I}})$ a strategy for player i is a functions $\hat{\alpha}_i : \mathbb{R} \to A_i$. We denote by $(\boldsymbol{\alpha}(\mathbf{s}))$ the strategy profile given by:

$$(\boldsymbol{\alpha}(\mathbf{s})) \triangleq (\alpha_1(\mathbf{s}_1), ..., \alpha_N(\mathbf{s}_N)).$$

We denote by $(a_i', \boldsymbol{\alpha}_{-i}(\mathbf{s}_{-i}))$ the strategy profile in which all agents play according to $(\boldsymbol{\alpha}(\mathbf{s}))$ except for player i, and player i takes action a_i' for all realizations of the signals he observes.

That is,

$$(a_i', \boldsymbol{\alpha}_{-i}(\mathbf{s}_{-i})) \triangleq (\alpha_1(\mathbf{s}_1), ..., \alpha_{i-1}(\mathbf{s}_{i-1}), a_i', \alpha_{i+1}(\mathbf{s}_{i+1}), ..., \alpha_N(\mathbf{s}_N)).$$

7.2 Solution Concepts

In order to provide our results, it is convenient to work with stronger solution concepts than Nash equilibrium. This allow us to provide sharper results. We define posterior equilibrium.

Definition 4 (Posterior Equilibrium)**.**
A strategy profile $(\alpha_1, ..., \alpha_N)$ forms a posterior equilibrium if for all players $i \in N$, for all signals realizations $(\mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^J$, and for all actions $a_i' \in A_i$:

$$\mathbb{E}[u_i(\theta_i, \boldsymbol{\alpha}(\mathbf{s}))|\mathbf{s}_i, \boldsymbol{\alpha}(\mathbf{s})] \geq \mathbb{E}[u_i(\theta_i, (a_i', \boldsymbol{\alpha}_{-i}(\mathbf{s}_{-i})))|\mathbf{s}_i, \boldsymbol{\alpha}(\mathbf{s})]. \tag{46}$$

In a posterior equilibrium, the strategy of agent i remains optimal even if he knew the actions taken by all other agents. The definition of posterior equilibrium is due to Green and Laffont (1987). In contrast to a Nash equilibrium, the information set with respect to which the action needs to be optimal is augmented. The action taken by agent i remains optimal even if he knew the action taken by other agents. It is transparent to see that, if a strategy profile is an posterior equilibrium, then it is also a Nash equilibrium.

It is convenient to compare posterior equilibrium with ex post equilibrium. A strategy profile $(\alpha_1, ..., \alpha_N)$ forms an ex post equilibrium if for all players $i \in N$, for all signals realizations $(\mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^J$, and for all actions $a_i' \in A_i$:

$$\mathbb{E}[u_i(\theta_i, \boldsymbol{\alpha}(\mathbf{s}))|\mathbf{s}_1, ..., \mathbf{s}_N] \geq \mathbb{E}[u_i(\theta_i, (a_i', \boldsymbol{\alpha}_{-i}(\mathbf{s}_{-i})))|\mathbf{s}_1, ..., \mathbf{s}_N]. \tag{47}$$

In an ex post equilibrium, agent i's action is optimal even if he knew the realization of the signals of all other agents. The definition of ex post equilibrium is standard in the literature.[37]

The difference between posterior equilibria and ex-post equilibria is the amount of information with respect to which a strategies is optimal. That is, the difference lies in the conditioning variables in (47) and (46). The action taken by player i is less informative than the signal agent i observes. Hence, if an equilibrium is an ex-post equilibrium, then it is also a posterior equilibrium.

[37] Ex post incentive compatibility was discussed as "uniform incentive compatibility" by Holmstrom and Myerson (1983). Ex post equilibrium has been studied by many papers in different contexts. See Section 7.4 for a brief discussion.

7.3 General Characterization of Equilibria

We now show how to compute posterior equilibria in game $(P, \mathcal{I})$. We do this by providing an equivalence between ex post equilibria in game $(P, \hat{\mathcal{I}})$ and posterior equilibria in game $(P, \mathcal{I})$.

Theorem 4 (Equivalence)**.**
If $(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}_N \cdot \mathbf{s}_N) \in \mathbb{R}^N$ *is an equilibrium statistic and strategies profile* $\{\hat{\alpha}_i\}_{i \in N}$ *is an ex post equilibrium in game* $(P, \hat{\mathcal{I}})$*, then the following strategy profile* $\{\alpha_i\}_{i \in N}$ *is a posterior equilibrium in game* $(P, \mathcal{I})$ *:*

$$\alpha_i(\mathbf{s}_i) = \hat{\alpha}_i(\boldsymbol{\beta}_i \cdot \mathbf{s}_i). \tag{48}$$

Proposition 4 shows that equilibria can be computed using a two step procedure. The first step is to find the one-dimensional equilibrium statistic using (41). The second step is to compute a posterior equilibrium as if agents observed only the equilibrium statistic.

In order to characterize a posterior equilibrium when agents observe multidimensional signals, the mechanism must have an ex post equilibrium when agents observe only the equilibrium statistic. Yet, the equilibrium statistic is an endogenous object. So, to apply the methodology in our paper, it is necessary that the mechanism has an ex post equilibrium for a broad class of one-dimensional signals. For example, if one considers a direct mechanism that has an ex post equilibrium under a certain one-dimensional signal then, in general, this mechanism will no longer have an ex post equilibrium if we change the joint distribution of signals and fundamentals (but keep the mapping from messages to outcomes the same). Hence, the methodology can be applied to a large class of indirect mechanisms that have an ex post equilibrium, regardless of the precise description of the information structure. If a mechanism has an ex post equilibrium for some one-dimensional information structure but not for other one-dimensional information structures, then it is necessary to check whether the mechanism has an ex post equilibrium when agents observe only the equilibrium statistic.

7.4 Models that can be Solved with the Equilibrium Statistic

There is a large class of trading mechanisms that have an ex post equilibrium when agents observe one-dimensional signals. These mechanisms include classic trading mechanisms, as well as mechanisms proposed by recent papers. We briefly provide an overview of some of the mechanisms that have an ex post equilibria when agents observe one-dimensional signals.

There are classic trading mechanisms that have an ex post equilibrium when agents observe one-dimensional signals. For example, multi-unit ascending auctions (see for example, Ausubel (2004) or Perry and Reny (2005)) and generalized VCG mechanism (see for example, Dasgupta and Maskin (2000)). Additionally, supply function competition has an ex post equilibria when agents are symmetric (see for example Klemperer and Meyer (1989) or Vives (2011)).[38] Many recent papers study novel mechanisms that have an ex post equilibria when agents observe one-dimensional signals. Ausubel, Crampton, and Milgrom (2006) propose the Combinatorial Clock Auction that is meant to auction many related items. Sannikov and Skrzypacz (2014) study a variation of supply function equilibria in which each agents can condition on the quantity bought by other agents. Kojima and Yamashita (2014) study a variation of a double auction that improves upon the standard double auction along several dimensions. All the mechanisms previously mentioned have an ex post equilibria when agents observe one-dimensional signals.

It is worth mentioning some mechanisms that do not have an ex post equilibria when agents observe one-dimensional signals. Two classic examples are Cournot competition and first-price auction. In Cournot competition an agent tries to anticipate the quantity submitted by other agents, as these quantities will ultimately determine the equilibrium price. In a first-price auction agents try to anticipate the bid of other agents. It is interesting to compare this with supply function equilibria and English auction. In supply function equilibria an agent can condition the quantity they buy on the equilibrium price, and hence he does not need to anticipate the demands submitted by other agents. Nevertheless, agents learn from the equilibrium price. Analogously, in an English auction an agent can condition on the drop-out time of other agents, and hence an agent does not need to anticipate the bids of other agents.

8 Discussion

We conclude the paper with four discussions; (i) we explain how our paper fits a broader picture regarding problems of information aggregation, (ii) we explain how our paper relates to other papers studying multidimensional signals, (iii) we explain how the Gaussian signals simplify the analysis in our model, and (iv) we provide directions for future work.

[38]In symmetric environments the price aggregates all relevant information, and hence equilibria is privately revealing (see Vives (2011)). See Rostek and Weretka (2012) for a model with asymmetric agents in which there is no ex post equilibria even when agents observe one-dimensional signals.

8.1 Discussion: Information Aggregation with Multidimensional Signals

Although we have focused most of our analysis in an ascending auction, the paper addresses a broader problem of information aggregation. To understand the broader picture we describe in more detail a model of supply function competition, as in Vives (2011).

Consider N agents trading a divisible asset via supply function competition. The payoff of agent i is:

$$u(\theta_i, q_i, p) = \theta_i q_i - q_i \cdot p - \frac{1}{2}q_i^2,$$

where $q_i \in \mathbb{R}$ is the quantity bought by agent i of a divisible asset, $p \in \mathbb{R}$ is the price paid for the asset and $\theta_i \in \mathbb{R}$ is a shock to the marginal valuation of agent i. Each agent submits a demand function $x_i(p)$ to a Walrasian auctioneer. If the equilibrium price is equal to p^* then agent i buys a quantity $x_i(p^*)$ of the asset. The equilibrium price p^* is determined by the market clearing condition:

$$\sum_{i \in N} x_i(p^*) = 0.$$

The details on how to analyze this model can be found in Vives (2011). Here we just focus on discussing how our analysis distills the essential elements of information aggregation.

In this model agent i submits a supply functions that is determined by the signals he observes ($\mathbf{s}_i$). The equilibrium price, which is determined by the supply function submitted by all agents, aggregates some of the information privately observed by agents. The information in the price in turn feeds back into the behavior of agents. Hence, to find an equilibrium it is necessary to find the fixed point in which the information aggregated by the price is consistent with the behavior of agents.

Vives (2011) shows that, if agents observe one-dimensional signals then, the information in the price plus the signal privately observed by agent i is a sufficient statistic of the signal privately observe by all agents to predict θ_i. This is reminiscent of the equilibrium of the ascending auction with one-dimensional signals. If agents observe one-dimensional signals, then the drop-out time of an agent is equivalent to the information observed by this agent. Hence, observing the drop-out time of all agents is equivalent to observing the signals observed by all agents.

If agents observe multidimensional signals, the analysis becomes more complex. The reason is that observing the equilibrium price plus the signal privately observed by agent i is *not* a sufficient statistic of the signal privately observe by all agents to predict θ_i. In this case, it is best to think about the problem of information aggregation in two steps (see Figure 2 for a schematic diagram). First, the information of agent i must be aggregated into the demand of agent i. Second, the information in the demand of all agents gets aggregated by the price. Hence, when agents observe multidimensional signals, the demand of an agent cannot aggregate all the information an agent observes. Similarly, in an ascending auction, when agents observe multidimensional signals, the drop-out time of an agent cannot aggregate all the information an agent observes.

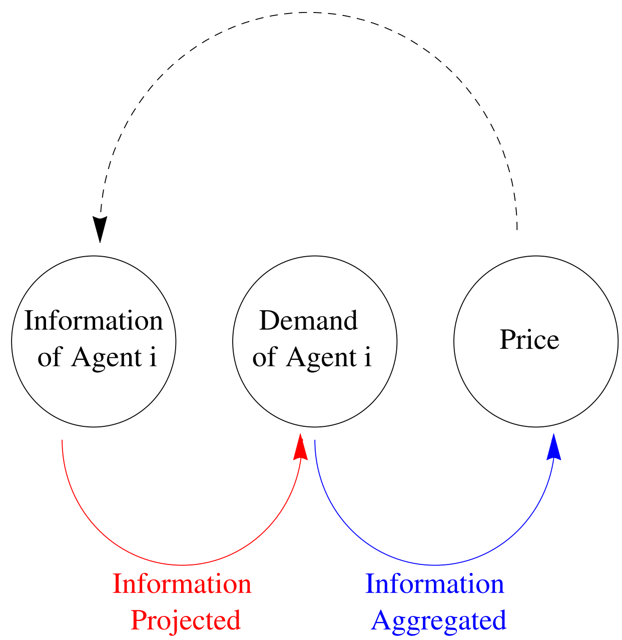


Figure 2: Diagram of Information Aggregation

The analysis of this paper concerns the first channel in Figure 2. That is, the channel of information aggregation by an agent's demand. The equilibrium statistic characterizes how the information is projected when the price perfectly reveals all the information contained in agents' demands. In supply function competition with linear-quadratic payoffs the price reveals the average signal observed by all agents. Since agents are symmetric and random variables are Gaussian knowing the average of the signals observed by all agents is a sufficient statistic of the signals observed by all agents. In an ascending auction, each agent observes the drop-out time of all other agents, and hence the information in the drop-out time of all other agents gets perfectly aggregate. In a second price auction, the price reveals the second highest expected valuation. Yet, this is not equivalent to observing the signals of all other agents (except for $N = 2$). Hence, the equilibrium statistic cannot be used to solve a second price auction when agents observe multidimensional signals. Of course, it is natural to expect that in a second price auction the "right projection" is "similar" than in an ascending auction.

In our model, the bid of an agent cannot aggregate all the information this agent observed due to the multidimensionality of the information structure. There is a recent literature in auctions that studies classic common values models with one-dimensional signals, but after the auction an agent needs to take an action that ultimately determines the value of the good (for example, see Axelson and Makarov (2016) or Atakan and Ekmekci (2014)). In this class of models there is a different mechanism that prevents the bid of an agent from fully revealing the signal this agent observed. In this class of models the final payoff of the good is not strictly monotonic in the realization of the signals observed by agents. This implies that in equilibrium, the bid of an agent is not necessarily informative about the signal observed by this agent. Hence, there is a natural sense in which this class of models exhibit a related problem of information aggregation.

8.2 Games and Mechanism Design with Multidimensional Signals

Studying an ascending auction with multidimensional signal has been elusive to the auction literature. This difficulty is pervasive in most of the game theory and mechanism design literature. We briefly provide an overview of how our paper connects to other papers studying multidimensional signals in other literatures.

Most of the results in the mechanism design literature have been negative results. Jehiel and Moldovanu (2001) shows that it is impossible to implement an efficient allocation with multidimensional independent signals. Jehiel, Meyer-ter-Vehn, Moldovanu, and Zame (2006) show that it is impossible to implement an ex post equilibrium with multidimensional signals. Neither of the results apply to our model because in our model signals are correlated across agents and the Gaussian structure implies this is a separable type-space. The fact that it is possible to construct an ex post equilibrium illustrates some properties of our construction and where the impossibility result is rooted.

As explained in Section 5.4, in our model it is possible to construct an ex post equilibrium in which the allocation of the object is determined by $\hat{\zeta}_i$ (see (35)). It is easy to check that the transfers associated to this mechanisms cannot be determined by knowing *only* $(\hat{\zeta}_1, ..., \hat{\zeta}_N)$. That is, the transfers in the ex post equilibrium are not determined by the same projection of signals as the projection that determines the allocation of the object. In contrast, in the ascending auction, the equilibrium statistic of all agents $(\zeta_1, ..., \zeta_N)$ determines who will win in the auction and how much the winner will pay. This difference provides an intuitive argument why constructing an ex post equilibrium is generically impossible, but solving for an ascending auction with non-Gaussian multidimensional signals may not suffer the same difficulty. We expect that finding the right projection in the ascending auction can be done also for non-Gaussian signals. As previously explained, the equilibrium we characterize is a posterior equilibrium. More broadly, we expect that studying mechanism design under posterior incentive compatibility constraints does not present the same impossibility results as ex post equilibrium.

The idea that information is muddled when agents observe multidimensional signals appears in signaling models (see, for example, Austen-Smith and Fryer Jr (2005), Esteban and Ray (2006), or Frankel and Kartik (2014)). This is reminiscent of the equilibrium in the ascending auction, in which the drop-out time of an agent muddles both of the signals he observes. In signaling models, agents that observe multidimensional signals perfectly know their own preferences. In contrast to signaling models, in our model, agents that observe the multidimensional signals do not have complete information over their preferences (the signals agents observe have a common value component). This implies that the way that the multidimensional signals of one agent is projected into a one-dimensional statistic depends on how other agents project their signals. Hence, there is a fixed point over how agents project their signals.[39]

The use of Gaussian signals has proven to be very useful in literatures studying linear best-response games.[40] Ganguli and Yang (2009) and Manzano and Vives (2011) study a rational expectations equilibrium in which agents observe two-dimensional signals. Amador and Weill (2010) studies a micro-founded macro model with informational externalities. Lambert, Ostrovsky, and Panov (2014) study a static version of a Kyle (1985) trading model. Our paper shares the common motivation of understanding a trading model with multidimensional signals. Nevertheless, a static version of a Kyle (1985) trading model does not have an ex post equilibrium when agents observe one-dimensional signals. Hence, the methodology developed in our paper is not useful to study a trading model as in Kyle (1985). Conversely, the methods developed in Lambert, Ostrovsky, and Panov (2014) are not useful to study the models we study in this paper.

[39] The same discussion applies to Deneckere and Severinov (2011), which studies optimal mechanism design in single agent problems under two-dimensional types.

[40] This includes among others, beauty-contest models (see Morris and Shin (2002)) and trading models with noise traders (see Grossman and Stiglitz (1980)).

8.3 On the Use of Gaussian Signals

The only assumption made throughout the paper that is not standard in the auction literature is the use of Gaussian signals. The objective of our model is not to provide a general characterization of the set of Nash equilibria when agents observe multidimensional signals. Instead, the objective of our paper to understand some of the differences that arise with respect to one-dimensional environments. In our opinion, the new predictions and results provided in our paper are unlikely to be overturned under non-Gaussian information structures. We now explain how the Gaussian signals simplifies the analysis, and why we believe the Gaussian signals is not the driving force of the new results we provide. This also serves as a guideline on how we believe the analysis should be modified to accommodate other distributions.

The main difficulty in constructing an equilibrium with multidimensional signals is that there is no straightforward way to order agents. Since there is no complete order over signals, the order in which agents drop out of the auction will be determined by some projection of the signals into a one-dimensional statistic. In fact, if a Nash equilibrium exists, then the drop-out time of agent i is in itself a projection of the signals agent i observes into a number. Hence, the difficulty is not in the realization that the equilibrium is determined by a projection of signals, but the difficulty is in characterizing this projection. The Gaussian information structure simplifies the analysis in two ways.

The first way in which the Gaussian information structures simplify the analysis is by guaranteeing that the projection of the signal into a one-dimensional statistic is independent of the realization of the signals. That is, the order over agents (i, j) does not depend on the time at which agent k drops out of the auction. In Gaussian environments, the equilibrium statistic determines an agent's behavior in the auction. Importantly, the equilibrium statistic does not depend on the history of the auction. This is not guaranteed to happen in non-Gaussian environments. In non-gaussian environments, it is possible that the projection of signals of agent i changes depending on the time at which agent k drops out.

The fact that the projection of signals of agent i is determined by the time at which agent k drops out of the auction is a non-trivial difficulty. Although we believe this is an important difficulty to overcome when considering non-Gaussian information structure, this difficulty is always absent if one considers two agents. This is because with two agents it is always the case that the history of the auction is empty until one of the two agents drops out of the auction. For this reason, we believe that a natural starting point to analyze non-Gaussian signals is to restrict attention to auctions with 2 agents.

The second way in which the Gaussian signals simplify the analysis is by inducing an order over agents that is constructed taking a linear combination of signals. This makes the analysis tractable, and allow us to derive analytic solutions. More complex indifference curves would make the analysis more complicated, and additional richness may arise. Nevertheless, we do not see a conceptual difference between the analysis of linear indifference curves and non-linear indifference curves.[41]

Jackson (2009) provides an example of a two-dimensional information structure for which it is not possible to find an equilibrium. We believe the finite support of the information structure is important in the construction of the example. The problem with finite support signals is that the drop-out time of an agent does not muddle both signals an agent observes. This is because there is only one combination of signals that is consistent with a given drop-out time.[42] This difference between finite-support information structures and continuous-support information structures has appeared in the rational expectations equilibrium. In a seminal contribution, Radner (1979) shows that with finite-support signals, in a competitive rational expectations equilibrium prices are fully revealing. In our environment, agents are not price takers. For this reason, the fully revealing nature of the bids conflict with the existence of an equilibrium. Hence, we believe that any extension of our analysis to non-Gaussian information structures must be done with continuous support information structures.

8.4 Directions for Future Work

The results in the paper can be naturally extended to other Gaussian information structures. Hence, there are plenty of open questions that can be answered with the same methodology. For example, it is natural to study a model in which the information structure is as in (17), but the signal about the common shock of different agents have different precision. It is also natural to study the case in which payoff shocks are asymmetrically distributed. Example 5 is an illustration of both of these extensions.

In the companion paper Heumann (2016), we use this methodology to study the limits of

[41] In a recent contribution, Breon-Drish (2015) extends classic models of rational expectations equilibrium with noise traders to allow for non-Gaussian distribution. These class of models share common features with ours. The model therein exhibits competitive agents that observe one-dimensional signals. Nevertheless, there are noise traders that play a similar role to the multidimensional signals in our model. We hope the techniques developed therein can be used in our model to extend the analysis to non-Gaussian information structures.

[42] It is possible to find an equilibrium statistic for the information structure studied in Jackson (2009). The equilibrium statistic does not muddle both sources of information. That is, the equilibrium statistic of agent i is informationally equivalent to both signals observed by agent i.

information aggregation in large markets. In particular, we study a continuum of traders trading a divisible asset via supply function competition. We study to what extent there is a limit to the amount of information that can be aggregate by prices. Agents observe multidimensional signals, but the supply function an agent submits only aggregates the information contained in the one-dimensional equilibrium statistic. Hence, there is a natural limit to the amount of information that can be aggregated by the price. In that paper we study the efficiency of the information revealed by the equilibrium statistic. We study whether the use of information by agents is optimal, and how can taxes increase or decrease the amount of information revealed by prices in equilibrium.

We believe it is also natural to apply our methodology to study other mechanisms. As we have explained in Section 7.3, the same methodology can be applied to many other trading mechanisms. Although the equilibrium statistic does not change, the characterization for one-dimensional signals does change. Since the impact of the distribution of a one-dimensional information structure is different for different mechanisms, the impact of multidimensional signals on these mechanisms will be different than in an ascending auction. Even within the class of linear model, like supply function competition (see Vives (2011)), our paper provides novel results on how to analyze multidimensional signals.

Finally, we believe that it is natural to compare the ascending auction with other mechanisms. With one-dimensional signals the ascending auction is efficient, but with multidimensional signals this is not the case. In Section 5.4 we provided a brief analysis of how the ascending auction compares with a direct mechanism that has an ex post equilibrium. An interesting comparison would be to compare the ascending auction with a first-price auction. It is natural to expect that the bidding in the first-price auction would also be determined by the projection of signals into a one-dimensional statistic, but a different one-dimensional statistic than the equilibrium statistic characterized in Section 4. Nevertheless, solving for a first-price auction is a difficult problem that is outside the scope of this paper.

9 Appendix A: Proofs

Proof of Theorem 1 First, note that for any pair of jointly normal random variables (x, y):

$$\mathbb{E}[exp(x)|y] = exp(\mathbb{E}[x|y] + \frac{1}{2}var(x|y)). \tag{49}$$

It is easy to check that:

$$var(\theta_i|s_1, s_2, ..., s_N) = (1 - corr(\Delta\theta_i, \Delta s_i)^2)var(\Delta\theta_i) + (1 - corr(\bar{\theta}, \bar{s})^2)var(\bar{\theta}).$$

Hence,

$$\begin{aligned} \mathbb{E}[exp(\theta_2)|s_2, s_2, ..., s_N] &= exp\Bigg(\mathbb{E}[\theta_2|s_2, s_2, ..., s_N] + \qquad (50) \\ &\frac{1}{2}((1 - corr(\Delta\theta_i, \Delta s_i)^2)var(\Delta\theta_i) + (1 - corr(\bar{\theta}, \bar{s})^2)var(\bar{\theta}))\Bigg) \qquad (51) \end{aligned}$$

It is important to note that, for any $i \in \{1, ..., N\}$, the errors of the prediction $\mathbb{E}[\theta_i|s_1, ..., s_N]$ are Gaussian. That is, $\theta_i - \mathbb{E}[\theta_i|s_1, ..., s_N]$ is a Gaussian random variables. Hence, we can use (49) to compute (50). Nevertheless, $\mathbb{E}[\theta_2|s_1, ..., s_N]$ is not a Gaussian random variable as this is the second maximum over N random variables. Hence, to compute $\mathbb{E}[\mathbb{E}[exp(\theta_2)|s_1, s_2, ..., s_N]]$ we cannot use (49).

Rewriting (12) explicitly for $i = 2$:

$$\mathbb{E}[\theta_2|s_1, ..., s_N] = \frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)}\Delta s_2 + \frac{cov(\bar{\theta}, \bar{s})}{var(\bar{s})}\bar{s}.$$

Replacing s_1 with s_2 in the previous expression, we get:

$$\begin{aligned} \mathbb{E}[\theta_2|s_2, s_2, ..., s_N] &= \frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)}\Delta s_1 + \frac{cov(\bar{\theta}, \bar{s})}{var(\bar{s})}\bar{s} \\ &\quad +(\frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)} - \frac{1}{N}\frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)} + \frac{1}{N}\frac{cov(\bar{\theta}, \bar{s})}{var(\bar{s})})(s_2 - s_1) \\ &= \frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)}\Delta s_1 + \frac{cov(\bar{\theta}, \bar{s})}{var(\bar{s})}\bar{s} \\ &\quad -(\frac{1-m}{N} - 1)\Bigg((\frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)}\Delta s_2 + \frac{cov(\bar{\theta}, \bar{s})}{var(\bar{s})}\bar{s}) \\ &\quad -(\frac{cov(\Delta\theta_i, \Delta s_i)}{var(\Delta s_i)}\Delta s_1 + \frac{cov(\bar{\theta}, \bar{s})}{var(\bar{s})}\bar{s})\Bigg) \\ &= \mathbb{E}[\theta_1|s_1, ..., s_N] + (\frac{1-m}{N} - 1)(\mathbb{E}[\theta_1|s_1, s_2, ..., s_N] - \mathbb{E}[\theta_2|s_1, s_2, ..., s_N]). \end{aligned}$$

Replacing into (50), we get:

$$\begin{aligned}\mathbb{E}[exp(\theta_2)|s_2, s_2, ..., s_N] &= exp\Bigg(\mathbb{E}[\theta_1|s_1, ..., s_N] \\ &+(\frac{1-m}{N} - 1)(\mathbb{E}[\theta_1|s_1, s_2, ..., s_N] - \mathbb{E}[\theta_2|s_1, s_2, ..., s_N]) \\ &+\frac{1}{2}((1 - corr(\Delta\theta_i, \Delta s_i)^2)var(\Delta\theta_i) + (1 - corr(\bar{\theta}, \bar{s})^2)var(\bar{\theta}))\Bigg).\end{aligned}$$

Note that:

$$\begin{aligned}S(s_1, ..., s_N) = \mathbb{E}[exp(\theta_1)|s_1, ..., s_N] &= exp\Bigg(\mathbb{E}[\theta_1|s_1, ..., s_N] \\ &+\frac{1}{2}((1 - corr(\Delta\theta_i, \Delta s_i)^2)var(\Delta\theta_i) + (1 - corr(\bar{\theta}, \bar{s})^2)var(\bar{\theta}))\Bigg)\end{aligned}$$

Hence,

$$\mathbb{E}[exp(\theta_2)|s_2, s_2, ..., s_N] = S(s_1, ..., s_N) \times exp\Bigg((\frac{1-m}{N} - 1)(\mathbb{E}[\theta_1|s_1, s_2, ..., s_N] - \mathbb{E}[\theta_2|s_1, s_2, ..., s_N])\Bigg).$$

Similarly, we can compete the buyers' rents:

$$\begin{aligned}V(s_1, ..., s_N) &= S(s_1, ..., s_N) - \mathbb{E}[exp(\theta_2)|s_2, s_2, ..., s_N] \\ &= S(s_1, ..., s_N) \times \Bigg(1 - exp\Bigg((\frac{1-m}{N} - 1)(\mathbb{E}[\theta_1|s_1, s_2, ..., s_N] - \mathbb{E}[\theta_2|s_1, s_2, ..., s_N])\Bigg)\Bigg).\end{aligned}$$

Hence, we prove the result. ■

Proof of Theorem 2 This is a direct corollary of Theorem 4 and the fact that the equilibrium characterized in Proposition 1 is an ex post equilibrium. ■

Proof of Theorem 3 This is a direct corollary of Theorem 4 and the fact that the equilibrium characterized in Proposition 7 is an ex post equilibrium. ■

Proof of Theorem 4. By the construction of the equilibrium statistic it is clear that for any equilibrium statistic, the joint distribution of the random variables $(\theta_1, ..., \theta_N, \mathbf{s}_1, ..., \mathbf{s}_N, \zeta_1, ..., \zeta_N)$ are jointly normally distributed. We first provide the main steps of the proof and then explain each step in detail. If $\hat{\alpha}_i : \mathbb{R} \to A_i$ is an ex post equilibrium of game $(P, \hat{\mathcal{I}})$, then:

$$\Rightarrow \quad \forall i \in N, \forall \boldsymbol{\zeta} \in \mathbb{R}^N,\ \forall a_i' \in A_i, \quad \mathbb{E}[u_i(\hat{\alpha}(\zeta_i), \hat{\alpha}(\zeta_{-i}), \theta_i)|\boldsymbol{\zeta}] \geq \mathbb{E}[u_i(a_i', \hat{\alpha}(\zeta_{-i}), \theta_i)|\boldsymbol{\zeta}] \tag{52}$$

$$\Rightarrow \quad \forall i \in N, \forall \boldsymbol{\zeta} \in \mathbb{R}^N,\ \forall \mathbf{s}_i \in \mathbb{R}^J,\ \forall a_i' \in A_i, \quad \mathbb{E}[u_i(\hat{\alpha}(\zeta_i), \hat{\alpha}(\zeta_{-i}), \theta_i)|\boldsymbol{\zeta}, \mathbf{s}_i] \geq \mathbb{E}[u_i(a_i', \hat{\alpha}(\zeta_{-i}), \theta_i)|\boldsymbol{\zeta}, \mathbf{s}_i] \tag{53}$$

$$\begin{aligned}\Rightarrow \quad \forall i \in N, \forall \boldsymbol{\zeta} \in \mathbb{R}^N,\ \forall \mathbf{s}_i \in \mathbb{R}^J,\ \forall a_i' \in A_i, \quad & \mathbb{E}[u_i(\hat{\alpha}(\zeta_i), \hat{\alpha}(\zeta_{-i}), \theta_i)|\mathbf{s}_i, \hat{\alpha}_1(\zeta_1), ..., \hat{\alpha}_N(\zeta_N)] \\ & \geq \mathbb{E}[u_i(a_i', \hat{\alpha}(\zeta_{-i}), \theta_i)|\mathbf{s}_i, \hat{\alpha}_1(\zeta_1), ..., \hat{\alpha}_N(\zeta_N)]\end{aligned} \tag{54}$$

$$\begin{aligned}\Rightarrow \quad \forall i \in N, \forall (\mathbf{s}_1, ..., \mathbf{s}_N) \in \mathbb{R}^{J \cdot N},\ \forall a_i' \in A_i, \quad & \mathbb{E}[u_i(\hat{\alpha}(\zeta_i), \hat{\alpha}(\zeta_{-i}), \theta_i)|\mathbf{s}_i, \hat{\alpha}_1(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1), ..., \hat{\alpha}_N(\boldsymbol{\beta}_N \cdot \mathbf{s}_N)] \\ & \geq \mathbb{E}[u_i(a_i', \hat{\alpha}(\zeta_{-i}), \theta_i)|\mathbf{s}_i, \hat{\alpha}_1(\boldsymbol{\beta}_1 \cdot \mathbf{s}_1), ..., \hat{\alpha}_N(\boldsymbol{\beta}_N \cdot \mathbf{s}_N)]\end{aligned} \tag{55}$$

$$\Rightarrow \alpha^* : \mathbb{R}^J \to M \text{ defined by } \alpha^*(\mathbf{s_i}) = \hat{\alpha}(\zeta_i) = \hat{\alpha}(\boldsymbol{\beta}_\mathbf{i} \cdot \mathbf{s}_i) \text{ is a posterior equilibrium of game } G \tag{56}$$

Step (52) This is by definition of ex post equilibria in game $(P, \hat{\mathcal{I}})$.

Step (53) First, note that the expectations are over random variable θ_i. Hence, we need to prove that:

$$\forall \zeta \in \mathbb{R}^N,\ \forall \mathbf{s}_i \in \mathbb{R}^J,\ \theta_i|_{\boldsymbol{\zeta}} = \theta_i|_{\boldsymbol{\zeta}, \mathbf{s}_i}.$$

That is, the distribution of θ_i conditional on $\boldsymbol{\zeta}$ is the same same as the conditional distribution of θ_i conditional on $\boldsymbol{\zeta}$ and $\mathbf{s}_i$. As the random variables are normally distributed, it suffices to prove that:

$$\forall \boldsymbol{\zeta} \in \mathbb{R}^N,\ \forall \mathbf{s}_i \in \mathbb{R}^J,\ \mathbb{E}[\theta_i|\boldsymbol{\zeta}] = \mathbb{E}[\theta_i|\boldsymbol{\zeta}, \mathbf{s}_i]; \tag{57}$$

$$\forall \boldsymbol{\zeta} \in \mathbb{R}^N,\ \forall \mathbf{s}_i \in \mathbb{R}^J, var(\theta_i|\boldsymbol{\zeta}) = var(\theta_i|\boldsymbol{\zeta}, \mathbf{s}_i) \tag{58}$$

(57) is true by the definition of an equilibrium statistic. (58) is true because the variables are jointly Gaussian and hence:

$$var(\theta_i|\boldsymbol{\zeta}) = var(\theta_i) - var(\mathbb{E}[\theta_i|\boldsymbol{\zeta}]) = var(\theta_i) - var(\mathbb{E}[\theta_i|\boldsymbol{\zeta}, \mathbf{s}_i]) = var(\theta_i|\boldsymbol{\zeta}, \mathbf{s}_i)$$

Step (54) Note that $\hat{\alpha}_i(\zeta_i)$ is measurable with respect to ζ_i. Hence,

$$\mathbb{E}[u_i(a_i', \hat{\alpha}(\zeta_{-i}), \theta_i)|\boldsymbol{\zeta}, \mathbf{s}_i] = \mathbb{E}[u_i(a_i', \hat{\alpha}(\zeta_{-i}), \theta_i)|\mathbf{s}_i, \boldsymbol{\zeta}, \hat{\alpha}_1(\zeta_1), ..., \hat{\alpha}_N(\zeta_N)]; \tag{59}$$

$$\mathbb{E}[u_i(\hat{\alpha}(\zeta_i),\hat{\alpha}(\zeta_{-i}),\theta_i)|\boldsymbol{\zeta},\mathbf{s}_i]=\mathbb{E}[u_i(\hat{\alpha}(\zeta_i),\hat{\alpha}(\zeta_{-i}),\theta_i)|\mathbf{s}_i,\boldsymbol{\zeta},\hat{\alpha}_1(\zeta_1),...,\hat{\alpha}_N(\zeta_N)]. \tag{60}$$

That is, we can add $\hat{\alpha}_i(\zeta_i)$ as conditioning variable. Hence, we can write (53) as follows:

$$\mathbb{E}[u_i(\hat{\alpha}(\zeta_i),\hat{\alpha}(\zeta_{-i}),\theta_i)|\mathbf{s}_i,\boldsymbol{\zeta},\hat{\alpha}_1(\zeta_1),...,\hat{\alpha}_N(\zeta_N)]\geq\mathbb{E}[u_i(a_i',\hat{\alpha}(\zeta_{-i}),\theta_i)|\mathbf{s}_i,\boldsymbol{\zeta},\hat{\alpha}_1(\zeta_1),...,\hat{\alpha}_N(\zeta_N)].$$

Taking expectation of the previous equation conditional on $(\mathbf{s}_i,\hat{\alpha}_1(\zeta_1),...,\hat{\alpha}_N(\zeta_N))$ and using the law of iterated expectations:

$$\mathbb{E}[u_i(\hat{\alpha}(\zeta_i),\hat{\alpha}(\zeta_{-i}),\theta_i)|\mathbf{s}_i,\hat{\alpha}_1(\zeta_1),...,\hat{\alpha}_N(\zeta_N)]\geq\mathbb{E}[u_i(a_i',\hat{\alpha}(\zeta_{-i}),\theta_i)|\mathbf{s}_i,\hat{\alpha}_1(\zeta_1),...,\hat{\alpha}_N(\zeta_N)].$$

Hence, we prove the step.

Step (55) This is using that $\boldsymbol{\beta}_i\cdot\mathbf{s}_i=\zeta_i$, hence the inequality obviously holds.

Step (56) Is just by the definition of posterior equilibria.

Hence, we prove the result.■

Proof of Proposition 1 It is standard in the literature to show that this is an ex post equilibrium (see Krishna (2009)). We repeat the argument for $N=2$. It is transparent that the general case can be extended the same way.

Using (8), we note that:

$$\mathbb{E}[\theta_1|s_1,s_2]-\mathbb{E}[\theta_2|s_2,s_2]=\frac{cov(\Delta\theta_i,\Delta s_i)}{var(\Delta s_i)}\frac{1}{N}\left((N-1)+\frac{\frac{cov(\bar{\theta},\bar{s})}{var(\bar{s})}}{\frac{cov(\Delta\theta_i,\Delta s_i)}{var(\Delta s_i)}}\right)(s_1-s_2);$$

$$\mathbb{E}[\theta_2|s_1,s_2]-\mathbb{E}[\theta_1|s_1,s_1]=\frac{cov(\Delta\theta_i,\Delta s_i)}{var(\Delta s_i)}\frac{1}{N}\left((N-1)+\frac{\frac{cov(\bar{\theta},\bar{s})}{var(\bar{s})}}{\frac{cov(\Delta\theta_i,\Delta s_i)}{var(\Delta s_i)}}\right)(s_2-s_1);$$

We have normalized signals such that:

$$\frac{cov(\Delta\theta_i,\Delta s_i)}{var(\Delta s_i)}\frac{1}{N}\left((N-1)+\frac{\frac{cov(\bar{\theta},\bar{s})}{var(\bar{s})}}{\frac{cov(\Delta\theta_i,\Delta s_i)}{var(\Delta s_i)}}\right)\geq 0,$$

and we have ordered agents such that $s_1>s_2$. Hence:

$$\mathbb{E}[\theta_1|s_1,s_2]-\mathbb{E}[\theta_2|s_2,s_2]\geq 0 \quad \text{and} \quad \mathbb{E}[\theta_2|s_1,s_2]-\mathbb{E}[\theta_1|s_1,s_1]\leq 0.$$

All random variables are Gaussian, and hence the variance does not depend on the realization of signals. This implies that:

$$\mathbb{E}[exp(\theta_1)|s_1, s_2] - \mathbb{E}[exp(\theta_2)|s_2, s_2] \geq 0 \quad \text{and} \quad \mathbb{E}[exp(\theta_2)|s_1, s_2] - \mathbb{E}[exp(\theta_1)|s_1, s_1] \leq 0.$$

The first inequality states that agent one prefers to win the object than to not win the object. The second inequality states that agent 2 prefers to not win the object, rather than wait and pay the price at which agent 1 would drop out. Hence, we prove the result.■

Proof of Proposition 2 We can write the expected surplus as follows:

$$\mathbb{E}[S(s_1, ..., s_N)] = \mathbb{E}[\mathbb{E}[\exp(\theta_1)|s_1, ..., s_N]] = \mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1 + \bar{\theta})|\bar{s}, \Delta s_1, ..., \Delta s_N]].$$

Since the common component of the variables are independent of the orthogonal component of the random variables, we have:

$$\mathbb{E}[S(s_1, ..., s_N)] = \mathbb{E}[\mathbb{E}[\exp(\bar{\theta})|\bar{s}]]\mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s_1, ..., \Delta s_N]].$$

Using the law of iterated expectations:

$$\mathbb{E}[\mathbb{E}[\exp(\bar{\theta})|\bar{s}]] = exp(\frac{1}{2}var(\bar{\theta})).$$

Hence,

$$\mathbb{E}[S(s_1, ..., s_N)] = exp(\frac{1}{2}var(\bar{\theta})) \times \mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s_1, ..., \Delta s_N]]. \tag{61}$$

Clearly the $\mathbb{E}[S(s_1, ..., s_N)]$ does not depend on m or $corr(\bar{s}, \bar{\theta})$. We need to prove that (61) is increasing in $corr(\Delta s_i, \Delta\theta_i)$. We prove this in Lemma 5, in the supplementary appendix. Hence, we prove the result.■

Proof of Lemma 2 This is standard in the literature (see Milgrom and Weber (1982)). In our environment it is to check that the public signal decreases the value of m, which increases the profits.■

Proof of Proposition 3 Using Proposition 9 for the statistic $\zeta_i = s_{i1} + \beta s_{i2}$ we get:

$$cov(\theta_i - \mathbb{E}[\Delta\theta_i|\Delta\zeta_i] - \mathbb{E}[\bar{\theta}|\bar{\zeta}], s_{i1}) = 0. \tag{62}$$

Writing the expectations explicitly:

$$\mathbb{E}[\Delta\theta_i|\Delta\zeta_i] = \mathbb{E}[\Delta\iota_i|\Delta\iota_i + \beta\Delta\varepsilon_i] = \frac{var(\Delta\iota_i)}{var(\Delta\iota_i) + \beta^2 var(\Delta\varepsilon_i)}(\Delta\iota_i + \beta\Delta\varepsilon_i);$$

$$\mathbb{E}[\bar{\theta}|\bar{\zeta}] = \mathbb{E}[\bar{\iota} + \bar{\varphi}|\bar{\iota} + \beta(\bar{\varphi} + \bar{\varepsilon})] = \frac{var(\bar{\iota}) + \beta var(\bar{\varphi})}{var(\bar{\iota}) + \beta^2(var(\bar{\varphi}) + var(\bar{\varepsilon}))}(\bar{\iota} + \beta(\bar{\varphi} + \bar{\varepsilon})).$$

Rewriting (62) we get:

$$var(\Delta\iota_i)(1 - \frac{var(\Delta\iota_i)}{var(\Delta\iota_i) + \beta^2 var(\Delta\varepsilon_i)}) + var(\bar{\iota})(1 - \frac{var(\bar{\iota}) + \beta var(\bar{\varphi})}{var(\bar{\iota}) + \beta^2(var(\bar{\varphi}) + var(\bar{\varepsilon}))}) = 0.$$

Rearranging and simplifying terms, we get:

$$\frac{-1}{var(\Delta\varepsilon_i)} + \frac{var(\Delta\varepsilon_i) + var(\bar{\varepsilon}) + var(\bar{\varphi})}{var(\Delta\varepsilon_i)var(\bar{\varphi})}\beta + \frac{-1}{var(\Delta\iota_i)}\beta^2 + \frac{(var(\Delta\iota_i) + var(\bar{\iota}))(var(\bar{\varepsilon}) + var(\bar{\varphi}))}{var(\Delta\iota_i)var(\bar{\iota})var(\bar{\varphi})}\beta^3 = 0$$

Hence, we prove the result. ■

Proof of Proposition 4 It is a standard property of cubic polynomials that they have a unique root if and only if their discriminant is greater than 0. For (24) this reduces to the condition in Proposition 4. Hence, we prove the result.■

Proof of Proposition 6 The proof is similar to the proof of Proposition 5. We use all the definitions and arguments therein, and extend them to show the results on profits.

(Part 1: $var(\varepsilon_4) \to 0$**)**. We first prove that:

$$\lim_{var(\varepsilon_4)\to 0} p_2 = \mathbb{E}[exp(\theta_2)|s_1, ..., s_N].$$

As we showed in Proposition 5, in the limit $var(\varepsilon_4) \to 0$ we have that $\beta \to 0$. Hence, agents behave as if they observe only $s'_{i1} = \iota'_i$. Hence, in the limit $var(\varepsilon_4) \to 0$ we have that $m \to 1$. The limit on m is easy to check as in the limit $var(\bar{\iota}') > 0$ and $var(\Delta\iota_i)$ are well defined, and hence the limit of m is well defined. Hence, in the limit agents behave as if they had private values. Hence:

$$\lim_{var(\varepsilon_4)\to 0} p_2 = \max{}^{(2)}\{\mathbb{E}[exp(\theta_1)|\mathbf{s}_1, ..., \mathbf{s}_N], ..., \mathbb{E}[exp(\theta_N)|\mathbf{s}_1, ..., \mathbf{s}_N]\}.$$

(Part 2: $var(\varepsilon_3) \to 0$). We now prove that:

$$\lim_{var(\varepsilon_3)\to 0} p_2 = 0.$$

The proof is more subtle than Part 1 because in the limit $var(\varepsilon_3) \to 0$ two things happen simultaneously. First, $\beta \to 0$ and second $var(\bar{\iota}') \to 0$. Hence, when we look at the limit of m we have that $cov(\bar{\zeta}, \bar{\theta}') \to 0$ and $var(\bar{\zeta}') \to 0$. Hence, the limit of m cannot be immediately calculated.

To calculate the limits we calculate the speed at which different terms converge to 0. We say $x(var(\bar{\iota}'))$ is of order $var(\bar{\iota}')^k$ if:

$$\lim_{var(\bar{\iota}')\to 0} \frac{x(var(\bar{\iota}'))}{var(\bar{\iota}')^\ell} = \begin{cases} \infty & \ell > k; \\ 0 & \ell < k. \end{cases}$$

We denote this by $x = O(var(\bar{\iota}')^k)$.

Now, note that, in the limit $var(\bar{\iota}') \to 0$, we must have that any root of $p(\beta)$ (defined in (63)) is of order $var(\bar{\iota}')^{1/3}$. If β is of an order bigger than this, then the polynomial $p(\beta)$ is greater than 0. If β is of an order smaller than $var(\bar{\iota}')^{1/3}$, then $p(\beta)$ is negative.

Hence, the equilibrium statistic in the limit $var(\varepsilon_3) \to 0$ satisfies that $\beta = O(var(\bar{\iota}')^{1/3})$. Hence, in the limit $var(\varepsilon_3) \to 0$,

$$var(\bar{\zeta}) = var(\bar{\iota}' + \beta(\bar{\varphi}' + \bar{\varepsilon})) = var(\bar{\iota}' + O(var(\bar{\iota}')^{1/3})(\bar{\varphi}' + \bar{\varepsilon})) = O(var(\bar{\iota}')^{2/3}).$$

On the other hand,

$$cov(\bar{\zeta}, \bar{\theta}') = cov(\bar{\iota}' + O(var(\bar{\iota}')^{1/3})\bar{\varphi}', \bar{\iota}' + \bar{\varphi}') = O(var(\bar{\iota}')^{1/3}).$$

Hence, in the limit $var(\varepsilon_3) \to 0$:

$$\frac{cov(\bar{\zeta}, \bar{\theta}')}{var(\bar{\zeta})} = O(var(\bar{\iota}')^{-1/3}) \to \infty.$$

This implies that in the limit in the limit $var(\varepsilon_3) \to 0$, $m \to \infty$. Looking at Theorem 1, this implies that in the limit $var(\varepsilon_3) \to 0$:

$$p_2 \to 0.$$

Hence, we prove the result. ■

Proof of Proposition 5 The analysis of the equilibrium with public signals is equivalent to redefining the variance of the common shocks. To formalize the argument define:

$$\bar{\iota}' \triangleq \bar{\iota} - \mathbb{E}[\bar{\iota}|s_3] \text{ and } \bar{\varphi}' \triangleq \bar{\varphi} - \mathbb{E}[\varphi|s_4].$$

Note that the equilibrium analysis is equivalent to a model in which the payoff shock of agents is:

$$\theta_i' = \Delta\iota_i + \bar{\iota}' + \bar{\varphi}',$$

and agents observe signals:

$$s_{i1}' = \Delta\iota_i + \bar{\iota}' \text{ and } s_{i2}' = \bar{\varphi}' + \varepsilon_i.$$

Hence, the model is equivalent to redefining $var(\bar{\iota})$ and $var(\bar{\varphi})$.

Define the polynomial:

$$p(\beta) \triangleq \frac{-1}{var(\Delta\varepsilon_i)} + \beta\frac{var(\Delta\varepsilon_i) + var(\bar{\varepsilon}) + var(\bar{\varphi})}{var(\Delta\varepsilon_i)var(\bar{\varphi})} + \beta^2\frac{-1}{var(\Delta\iota_i)} + \beta^3\frac{(var(\Delta\iota_i) + var(\bar{\iota}))(var(\bar{\varepsilon}) + var(\bar{\varphi}))}{var(\Delta\iota_i)var(\bar{\iota})var(\bar{\varphi})}. \tag{63}$$

It is easy to check that, $p(\beta)$ is decreasing $var(\bar{\iota})$ and $var(\bar{\varphi})$. If $p(\beta)$ has a unique root $p(\beta^*) = 0$, then $p(\beta^*)$ is increasing at β^*. Hence, if $p(\beta)$ has a unique root, then this root is increasing in $var(\bar{\iota})$ and $var(\bar{\varphi})$. This implies that the equilibrium statistic:

$$\zeta_i = s_{i1} + \beta s_{i2},$$

satisfies that $corr(\Delta\zeta_i, \Delta\theta_i)$ is decreasing in $var(\bar{\iota})$ and $var(\bar{\varphi})$. Looking at Proposition 2, this implies that the total surplus $\mathbb{E}[S(\zeta_1, ..., \zeta_N)]$ is decreasing in $var(\bar{\iota})$ and $var(\bar{\varphi})$.

For the limit, note that in the limit $var(\bar{\iota}) \to 0$ or $var(\bar{\varphi}) \to 0$ the polynomial $p(\beta)$ has a unique root $p(\beta^*) = 0$, and $\beta^* \to 0$. Hence, in the limit the equilibrium statistic satisfies $\zeta_i \to \iota_i'$. Hence, in the limit $corr(\Delta\zeta_i, \Delta\theta_i) \to 1$. Hence, in the limit the equilibrium approaches the first best. Hence, we prove the result. ■

Proof of Proposition 7 Krishna (2003) shows that the ascending auction has an efficient ex post equilibrium (see Theorem 2 therein). Second we prove that the price paid in the ex post efficient equilibrium is (40). Yet, this is standard in the literature (see for example Ausubel (1999)). Finally, Perry and Reny (1999) provides a revenue equivalence theorem for ex post equilibria. That is, if two mechanisms implement the same allocation as an ex post equilibrium, then the payments must be the same. Hence, (40) must also be the payment in the outcome of the ascending auction. Hence, we prove the result. ■

Proof of Proposition 8 The proof proceeds in two steps. We first define a perturbed payoff environment and show that an equilibrium statistic exists in this perturbed environment. In the second step we show that an equilibrium statistic exists in the original environment by taking a sequence of perturbed environments that converges to the original one.

(Step 1) Recall that the augmented information structure is equal to the original information structure, but adding a signal (labeled $J+1$) to each agent:

$$s_{iJ+1} = \epsilon_i, \tag{64}$$

where $\{\epsilon_i\}_{i\in N}$ have a variance equal to 1, independent across agents and independent of all other random variables in the model. $\mathbf{s}'_i$ denotes the set of all original signals observed by i plus s_{iJ+1}. That is. $\mathbf{s}'_i \triangleq (\mathbf{s}_i, s_{iJ+1})$.

Consider a payoff environment in which for all $i \in N$, the payoff shock of agent i is equal to:

$$\theta'_i \triangleq \theta_i + \frac{1}{k} s_{iJ+1},$$

where $k \in \mathbb{N}$. Note that we are adding a shock to θ_i, and each agent knows the realization of his additional shock.

We denote by $\mathcal{R} \subset \mathbb{R}^{J+1}$, the set of all vectors that have a $1/k$ in the component $J+1$ and have a norm smaller or equal than M. M is big enough to satisfy a condition specified later. Clearly, $\mathcal{R}$ is a convex and compact subset of $\mathbb{R}^{J+1}$.

We define function $v_i : \mathcal{R}^{N-1} \to \mathcal{R}$ as the weights agent i places on his own signals when he knows $(\boldsymbol{\beta}'_1 \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}'_{i-1} \cdot \mathbf{s}'_{i-1}, \boldsymbol{\beta}'_{i+1} \cdot \mathbf{s}'_{i+1}, ..., \boldsymbol{\beta}'_N \cdot \mathbf{s}'_N)$. That is, $v_i(\boldsymbol{\beta}'_1, ..., \boldsymbol{\beta}'_{i-1}, \boldsymbol{\beta}'_{i+1}, ..., \boldsymbol{\beta}'_N)$ is defined implicitly as follows:

$$\mathbb{E}[\theta'_i | \mathbf{s}'_i, \boldsymbol{\beta}'_1 \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}'_N \cdot \mathbf{s}'_N] \triangleq v_i(\boldsymbol{\beta}'_1, ..., \boldsymbol{\beta}'_{i-1}, \boldsymbol{\beta}'_{i+1}, ..., \boldsymbol{\beta}'_N) \cdot \mathbf{s}'_i + \sum_{j\neq i} m_j \cdot \boldsymbol{\beta}'_j \cdot \cdot \mathbf{s}'_j, \tag{65}$$

for some $m_j \in \mathbb{R}^{N-1}$. There are several things to note. (i) Since $\{s_{iJ+1}\}_{i\in N}$ is independent of all other random variables in the model, agent i always places a weight equal to $1/k$ on s_{iJ+1}. This (plus point (ii) below) guarantees that the range of $v_i(\boldsymbol{\beta}'_1, ..., \boldsymbol{\beta}'_{i-1}, \boldsymbol{\beta}'_{i+1}, ..., \boldsymbol{\beta}'_N)$ is equal to $\mathcal{R}$. (ii) We can bound the weights agent i places on his own signals. This is because the variance of the expectation is always smaller than the variance of the original random variable. That is, it is possible to find M large enough such that if $||\boldsymbol{\beta}_i|| > M$, then

$$var\left(\boldsymbol{\beta}_i \cdot \mathbf{s}'_i + \sum_{j \neq i} m_j \cdot \boldsymbol{\beta}'_j \cdot \cdot \mathbf{s}'_j\right) > var\left(\theta'_i\right),$$

for any $\{m_j\}_{j \neq i}$. This (plus point (i) above), guarantees that the range of $v_i(\boldsymbol{\beta}'_1, ..., \boldsymbol{\beta}'_{i-1}, \boldsymbol{\beta}'_{i+1}, ..., \boldsymbol{\beta}'_N)$ is equal to $\mathcal{R}$. (iii) Since the original signals have full rank, the expectation in (65) is uniquely defined. That is, this guarantees that $v_i(\boldsymbol{\beta}'_1, ..., \boldsymbol{\beta}'_{i-1}, \boldsymbol{\beta}'_{i+1}, ..., \boldsymbol{\beta}'_N)$ is a function (and not a correspondance). (iv) Since the variance covariance matrix of signals has full support, and the component $J+1$ is always equal to $1/k$, the function $v_i(\boldsymbol{\beta}'_1, ..., \boldsymbol{\beta}'_{i-1}, \boldsymbol{\beta}'_{i+1}, ..., \boldsymbol{\beta}'_N)$ is continuous in $\mathcal{R}^{N-1}$. That is, the expectation would be discontinuous at the point $(0, ..., 0)$, but we are not considering this vector in the domain of the function. This is because the domain of the function is $\mathcal{R}$, which has $1/k$ in the component $J+1$.

We now define function $\mathbf{v} : \mathcal{R}^N \to \mathcal{R}^N$ as folllows:

$$\mathbf{v}(\boldsymbol{\beta}'_1, ..., \boldsymbol{\beta}'_N) \triangleq (v_1(\boldsymbol{\beta}'_2, ..., \boldsymbol{\beta}'_N), ..., v_N(\boldsymbol{\beta}'_1, ..., \boldsymbol{\beta}'_{N-1})).$$

By the previous argument, $\mathbf{v}$ is a continuous function, with a domain equal to its range, and defined on a compact domain. By Brouwer's fixed-point theorem $\mathbf{v}$ has a fixed point. By construction, the fixed point $(\boldsymbol{\beta}^k_1, ..., \boldsymbol{\beta}^k_N)$ satisfies:

$$\mathbb{E}[\theta'_i | \mathbf{s}'_i, \boldsymbol{\beta}^k_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}^k_N \cdot \mathbf{s}_N] = \mathbb{E}[\theta'_i | \boldsymbol{\beta}^k_1 \cdot \mathbf{s}_1, ..., \boldsymbol{\beta}^k_N \cdot \mathbf{s}_N].$$

Hence, $(\boldsymbol{\beta}^k_1, ..., \boldsymbol{\beta}^k_N)$ is an equilibrium statistic under the perturbed payoff shock.

(Step 2) We denote by $(\boldsymbol{\beta}^k_1, ..., \boldsymbol{\beta}^k_N)$ the equilibrium statistic of the perturbed environment. We normalize the equilibrium statistic such that for all $i \in N$, $||\boldsymbol{\beta}^k_i|| = 1$ (remember the $J+1$ component is equal to $1/k$, so the norm is never 0). Since $(\boldsymbol{\beta}^k_1, ..., \boldsymbol{\beta}^k_N)$ is defined on a compact set, it has a convergent subsequence. By re-labeling the series, we can assume that $(\boldsymbol{\beta}^k_1, ..., \boldsymbol{\beta}^k_N)$ is a convergent sequence, and define the limit $(\boldsymbol{\beta}^\infty_1, ..., \boldsymbol{\beta}^\infty_N)$. Note that by taking the limit $k \to \infty$, we are not changing the joint distribution of signals and payoff shocks $(\theta_1, ..., \theta_N, \mathbf{s}_1, ..., \mathbf{s}_N)$. By changing k we only change the impact that ϵ_i has on θ'_i.

We have considered equilibrium statistics such that for all $i \in N$, $||\boldsymbol{\beta}^k_i|| = 1$, and the joint distribution of signals stays constant as we take the limit. Hence,

$$\lim_{k\to\infty} \mathbb{E}[\theta_i|\mathbf{s}'_i, \boldsymbol{\beta}_1^k \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}_N^k \cdot \mathbf{s}'_N] = \mathbb{E}[\theta_i|\mathbf{s}'_i, \boldsymbol{\beta}_1^\infty \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}_N^\infty \cdot \mathbf{s}'_N]; \tag{66}$$

$$\lim_{k\to\infty} \mathbb{E}[\theta_i|\boldsymbol{\beta}_1^k \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}_N^k \cdot \mathbf{s}'_N] = \mathbb{E}[\theta_i|\boldsymbol{\beta}_1^\infty \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}_N^\infty \cdot \mathbf{s}'_N]. \tag{67}$$

That is, the informational content of $\boldsymbol{\beta}_i^k \cdot \mathbf{s}'_i$ changes continuously in the limit $k \to \infty$. Hence, the expectations must change continuously. By construction of the equilibrium static:

$$\mathbb{E}[\theta_i + \frac{1}{k}\epsilon_i|\mathbf{s}'_i, \boldsymbol{\beta}_1^k \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}_N^k \cdot \mathbf{s}'_N] = \mathbb{E}[\theta_i + \frac{1}{k}\epsilon_i|\boldsymbol{\beta}_1^k \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}_N^k \cdot \mathbf{s}'_N],$$

then we must also have that:

$$\lim_{k\to\infty} var\Bigg(\mathbb{E}[\theta_i|\mathbf{s}'_i, \boldsymbol{\beta}_1^k \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}_N^k \cdot \mathbf{s}'_N] - \mathbb{E}[\theta_i|\boldsymbol{\beta}_1^k \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}_N^k \cdot \mathbf{s}'_N]\Bigg) = 0. \tag{68}$$

This implies that:

$$\mathbb{E}[\theta_i|\boldsymbol{\beta}_1^\infty \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}_N^\infty \cdot \mathbf{s}'_N] = \mathbb{E}[\theta_i|\mathbf{s}'_i, \boldsymbol{\beta}_1^\infty \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}_N^\infty \cdot \mathbf{s}'_N].$$

Hence, $(\boldsymbol{\beta}_1^\infty \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}_N^\infty \cdot \mathbf{s}'_N)$ is an equilibrium statistic of the environment in which agents observe an additional signal s_{iJ+1} that is pure noise.

Note that when taking the expectation $\mathbb{E}[\theta_i|\mathbf{s}'_i, \boldsymbol{\beta}_1^\infty \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}_N^\infty \cdot \mathbf{s}'_N]$, an agent always places 0 weight on s_{iJ+1} (as this is pure noise). Hence, if the equilibrium statistic $\boldsymbol{\beta}_i^\infty \cdot \mathbf{s}'_i$ places non-zero weight on s_{iJ+1}, this means that the expectation $\mathbb{E}[\theta_i|\boldsymbol{\beta}_i^\infty \cdot \mathbf{s}'_1, ..., \boldsymbol{\beta}_N^\infty \cdot \mathbf{s}'_N]$ places 0 weight on $\boldsymbol{\beta}_1^\infty \cdot \mathbf{s}'_i$. Hence, if the equilibrium statistic $\boldsymbol{\beta}_i^\infty \cdot \mathbf{s}'_i$ places non-zero weight on s_{iJ+1}, this means an agent ignore his own signals in equilibrium. In most natural applications this obviously is not satisfied, and hence the weight on s_{iJ+1} is 0.

Finally, it is easy to check that in any equilibrium statistic signal s_{iJ+1} will have 0 weight as this signal only contains noise that is independent of all other random variables in the model. Hence, we prove the result.■

(Symmetric Information Structures.) Clearly, if the information structure is symmetric, then we can repeat the argument using symmetric equilibrium statistic. That is, instead of considering $\boldsymbol{\beta}_{-i} \triangleq (\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_{i-1}, \boldsymbol{\beta}_{i+1}, ..., , \boldsymbol{\beta}_N)$, one needs to consider $\boldsymbol{\beta}_{-i}$ in which $\boldsymbol{\beta}_j = \boldsymbol{\beta}_\ell$, for all $j, \ell \neq i$.

Hence, we prove the result.■

Proof of Lemma 1 This is by the definition of variance covariance matrix and using Lemma 3 (in the appendix) to show some of the covariances are 0.

Additionally, the following covariances are equal to 0 by symmetry (see Lemma 3):

$$cov(\bar{\theta}, \Delta\theta_i) = cov(\bar{s}, \Delta\theta_i) = cov(\bar{\theta}, \Delta s_i) = cov(\bar{s}, \Delta s_i) = 0.$$

Hence, $(\Delta\theta_i, \bar{\theta}, \Delta s_i, \bar{s})$ is determined by 6 coefficients. Hence, the coefficients in (11) completely determine the variance covariance matrix of $(\Delta\theta_i, \bar{\theta}, \Delta s_i, \bar{s})$. Hence, we prove the result.■

Proof of Corollary 1 The seller's profits are given by:

$$\begin{aligned}
\mathbb{E}[\pi_2] &= \mathbb{E}\left[exp\left((\frac{1-m}{N} - 1)(\mathbb{E}[\theta_1|s_1, ..., s_N] - \mathbb{E}[\theta_2|s_1, ..., s_N])\right)\mathbb{E}[\exp(\theta_1)|s_1, ..., s_N]\right] \\
&= \mathbb{E}\left[exp\left((\frac{1-m}{N} - 1)(\mathbb{E}[\Delta\theta_1|\Delta s_1, ..., \Delta s_N] - \mathbb{E}[\theta_2|\Delta s_1, ..., \Delta s_N])\right)\right. \\
&\quad \left. \times\mathbb{E}[\exp(\bar{\theta} + \Delta\theta_1)|\bar{s}, \Delta s_1, ..., \Delta s_N]\right]
\end{aligned}$$

Since the expectation does not depend on m, it is clear that $\mathbb{E}[\pi_2]$ is decreasing in m (note that by construction $\mathbb{E}[\Delta\theta_1|\Delta s_1, ..., \Delta s_N] > \mathbb{E}[\theta_2|\Delta s_1, ..., \Delta s_N]$). Additionally, the realization of the common component of random variables is independent of the realization of the orthogonal component of random variables. Hence, we can use the law of iterated expectations to take expectations over the common component of the signals. We get:

$$\begin{aligned}
\mathbb{E}[\pi_2] &= \mathbb{E}\left[exp\left((\frac{1-m}{N} - 1)(\mathbb{E}[\Delta\theta_1|\Delta s_1, ..., \Delta s_N] - \mathbb{E}[\theta_2|\Delta s_1, ..., \Delta s_N])\right)\right. \\
&\quad \left. \times\mathbb{E}[\exp(\Delta\theta_1)|\bar{s}, \Delta s_1, ..., \Delta s_N]\right] \times \exp\left(\frac{1}{2}corr(\bar{s}, \bar{\theta})^2 var(\bar{\theta})\right).
\end{aligned}$$

Since all the terms depend only on the realization of the orthogonal component of signals, we have that the expectation does not depend on $corr(\bar{s}, \bar{\theta})$. The proof for the rents of the buyers is completely analogous, except for the fact that the rents are increasing in m. Hence, we prove the result.■

Proof of Corollary 2 Since $corr(\iota_i, \iota_j) = 0$, we have that:

$$var(\Delta\iota_i) = \frac{(N-1)var(\iota_i)}{N} \; ; \; var(\bar{\iota}) = \frac{var(\iota_i)}{N} \; ; \; var(\Delta\varepsilon_i) = \frac{(N-1)var(\varepsilon_i)}{N} \; ; \; var(\bar{\varepsilon}) = \frac{var(\varepsilon_i)}{N}.$$

Replacing this into $18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$ we get:

$$-\frac{n^4\left(4var(\iota_i)^4\left(var(\bar{\varphi}) + var(\varepsilon_i)^2\right)^3\left(nvar(\bar{\varphi}) + var(\varepsilon_i)^2\right) + 4var(\bar{\varphi})^4 var(\varepsilon_i)^4\right)}{(n-1)^4 var(\bar{\varphi})^4 var(\varepsilon_i)^6 var(\iota_i)^6}$$

$$+\frac{n^4\left(var(\bar{\varphi})^2 var(\varepsilon_i)^2 var(\iota_i)^2\left((9(2-3n)n+1)var(\bar{\varphi})^2 + 4(5-9n)var(\bar{\varphi})var(\varepsilon_i)^2 - 8var(\varepsilon_i)^4\right)\right)}{(n-1)^4 var(\bar{\varphi})^4 var(\varepsilon_i)^6 var(\iota_i)^6}.$$

It is easy to check all terms are negative, and hence the ascending auction has a unique equilibrium.■

Proof of Corollary 3 We prove the result using the characterization in Proposition 4.

In the limit $var(\varepsilon_i) \to 0$, we have that $var(\Delta\varepsilon_i) \to 0$. In this case, $d \to \infty$ and $c \to \infty$. Clearly the term that dominates is $-4ac$.

In the limit $var(\varepsilon_i) \to 0$, we have that $var(\bar{\varepsilon}) \to \infty$. In this case, $a \to \infty$. Clearly the term that dominates is $-27a^2d^2$.

Proof of Corollary 4 We prove the result using the characterization in Proposition 4. The limit $var(\varepsilon_i) \to \infty$, $N \to \infty$ (with N diverging faster than $var(\varepsilon_i)$) is the same as assuming $var(\bar{\varepsilon}) = 0$ and considering the limit:

$$\lim_{var(\Delta\varepsilon_i)\to\infty} 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 = \frac{-4var(\Delta\iota_i)^2 - 4var(\Delta\iota_i)var(\bar{\iota}) + var(\bar{\iota})var(\bar{\varphi})}{var(\Delta\iota_i)^2 var(\bar{\iota})var(\bar{\varphi})^3}.$$

By re-arranging terms we get the result.■

Proof of Corollary 5 Fix an information structure and payoff environment $(var(\Delta\iota_i), var(\bar{\iota}), var(\bar{\varphi}), var(\Delta$ for a given number of agents N such that the ascending auction has multiple equilibria. Now consider a number N', and construct the payoff environment and information structure $(var(\Delta\iota_i'), var(\bar{\iota}'), var(\bar{\varphi}$ such that:

$$var(\Delta\iota_i') = var(\Delta\iota_i) \; ; \; var(\bar{\iota}') = var(\bar{\iota});$$

$$\frac{var(\bar{\varphi})^2}{var(\bar{\varphi}) + var(\bar{\varepsilon})} = \frac{var(\bar{\varphi}')^2}{var(\bar{\varphi}') + var(\bar{\varepsilon}')}$$

$$\frac{var(\bar{\varphi})var(\Delta\varepsilon_i)}{(var(\bar{\varphi}) + var(\bar{\varepsilon}))^2} = \frac{var(\bar{\varphi}')var(\Delta\varepsilon_i')}{(var(\bar{\varphi}') + var(\bar{\varepsilon}'))^2}$$

It is possible to check that under this alternative information structure the number of equilibria remains the same. That is, the sign of $18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 > 0$ does not change. Hence, we prove the result. ■

10 Appendix B: Auxiliary Results used in Proofs

This appendix contains auxiliary results that are used in the proofs of the results in the paper.

Lemma 3 (Orthogonal Decomposition)**.**
If the information structure is symmetric then the random variables $(\Delta\theta_i, \mathbf{\Delta s}_i)$ *are independent of* $(\bar{\theta}, \bar{\mathbf{s}})$*.*

Lemma 4 (Symmetric Equilibrium Statistic)**.**
The random variables $(\boldsymbol{\beta} \cdot \mathbf{s}_1, ..., \boldsymbol{\beta} \cdot \mathbf{s}_N)$ *are an equilibrium statistic if and only if:*

$$\mathbb{E}[\theta_i | \boldsymbol{\beta} \cdot \mathbf{s}_i, \boldsymbol{\beta} \cdot \bar{\mathbf{s}}] = \mathbb{E}[\theta_i | \mathbf{s}_i, \boldsymbol{\beta} \cdot \bar{\mathbf{s}}]. \tag{69}$$

Proposition 9 (Characterization of One-Dimensional Statistic)**.**
The random variables $(\boldsymbol{\beta} \cdot s_1, ..., \boldsymbol{\beta} \cdot s_N) \in \mathbb{R}^N$ *forms an equilibrium statistic if and only if:*

$$\forall j \in J, \quad cov(\theta_i - \mathbb{E}[\Delta\theta_i | \boldsymbol{\beta} \cdot \mathbf{\Delta s}_i] - \mathbb{E}[\bar{\theta} | \boldsymbol{\beta} \cdot \bar{\mathbf{s}}], s_{ij}) = 0. \tag{70}$$

Lemma 5 (Monotonicity of Expectation)**.**
$\mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1) | \Delta s_1, ..., \Delta s_N]]$ *is increasing in* $corr(\Delta s_i, \Delta\theta_i)$*.*

11 Appendix C: Proofs of Auxiliary Results

Proof of Lemma 5 We now use a coupling argument to prove that, for all s_i', s_i such that $corr(\Delta s_i', \Delta\theta_i) > corr(\Delta s_i, \Delta\theta_i)$, then:

$$\mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1) | \Delta s_1', ..., \Delta s_N']] > \mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1) | \Delta s_1, ..., \Delta s_N]].$$

Since $corr(\Delta s_i', \Delta\theta_i) > corr(\Delta s_i, \Delta\theta_i)$, we assume that s_i' is strictly more informative than s_i in a Blackwell sense. That is, we assume that s_i can be written as follows:

$$s_i = s_i' + \varepsilon_{i2}, \tag{71}$$

where ε_{i2} is a noise term independent of $\Delta\theta_i$ and s_i'. Of course, for two arbitrary signals s_i', s_i (71) might not be satisfied. Nevertheless, this does not matter for the argument because to compute:

$$\mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s_1', ..., \Delta s_N']]$$

what matters is the joint distribution of $(\Delta s_i', \Delta\theta_i)$. Hence, if we prove it for signal s_i' that satisfies (71) we will have proven it for all signals.

For each $i \in N$, we define random variables:

$$\Delta\varphi_i \triangleq \mathbb{E}[\Delta\theta_i|\Delta s_1, ..., \Delta s_N] \text{ and } \Delta e_i \triangleq \Delta\theta_i - \Delta\varphi.$$

Clearly, $\Delta\varphi_i$ is independent of Δe_i. On the other hand, we can write:

$$\begin{aligned} \mathbb{E}[\Delta\theta_i|\Delta s_i'] &= \mathbb{E}[\Delta\theta_i|\Delta s_i', \Delta s_i] \\ &= \mathbb{E}[\Delta\theta_i|\Delta s_i] + \mathbb{E}[\Delta\theta_i - \mathbb{E}[\Delta\theta_i|\Delta s_i]|\Delta s_i', \Delta s_i] \\ &= \Delta\varphi_i + \mathbb{E}[\Delta e_i|\Delta s_i', \Delta s_i]. \end{aligned}$$

Define,

$$\Delta\varphi_i' \triangleq \mathbb{E}[\Delta\theta_i|\Delta s_i'] \text{ and } \Delta q_i' \triangleq \mathbb{E}[\Delta e_i|\Delta s_i', \Delta s_i] \text{ and } \Delta e_i' \triangleq \Delta\theta_i - \mathbb{E}[\Delta\theta_i|\Delta s_i'].$$

Note that:

$$\begin{aligned} \Delta e_i' &= \theta_i - \mathbb{E}[\Delta\theta_i|\Delta s_i'] = \theta_i - (\Delta\varphi_i + \Delta q_i') \\ &= = (\theta_i - \Delta\varphi_i) - \Delta q_i' = \Delta e_i - \Delta q_i' \end{aligned}$$

Clearly, $\Delta e_i'$ is independent of $\Delta\varphi_i'$. Additionally, note that:

$$\mathbb{E}[q_i'\Delta\varphi_i] = \mathbb{E}[\mathbb{E}[\Delta e_i|\Delta s_i, \Delta s_i']\Delta\varphi_i]] = \mathbb{E}[\mathbb{E}[\Delta\varphi_i\Delta e_i|\Delta s_i', \Delta s_i]] = \mathbb{E}[\Delta\varphi_i\Delta e_i] = 0.$$

The first equality is by definition of $\Delta q_i'$. The second equality is using that $\Delta\varphi_i$ is measurable with respect to Δs_i. The third equality is using the law of iterated expectations. The fourth equality is just using that the errors from the expectations are uncorrelated with the expectation.Hence, q_i' is independent of $\Delta\varphi_i$.

We denote by $\Delta\hat{e}$ a random variable that has the same variance as Δe_i, but is independent of all random variables. Similarly, for each random variable previously defined, we denote by a hat over the variable a typical variable that has the same distribution but is independent of all other random variables. For example, $\Delta\hat{q}'$ is a random variable that has the same variance as $\Delta q_i'$, but is independent of all random variables.

It is clear to see that:

$$\mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s_1, ..., \Delta s_N]] = \mathbb{E}[\exp(\max\{\Delta\varphi_1, ..., \Delta\varphi_N\}+\Delta e_k)]] = \mathbb{E}[\exp(\max\{\Delta\varphi_1, ..., \Delta\varphi_N\}+\Delta\hat{e})]], \tag{72}$$

with k satisfying that $\Delta\varphi_k = \max\{\Delta\varphi_1, ..., \Delta\varphi_N\}$. The first equation is obtained by explicitly writing down the expectation. The second equality comes from the fact that we can replace Δe_k with $\Delta\hat{e}$ because Δe_k is independent of $\Delta\varphi_k$ (hence, the distribution of the random variables does not change).

We can write an equation analogous to (72) for s_i' as follows:

$$\mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s_1', ..., \Delta s_N']] = \mathbb{E}[\exp(\max\{\Delta\varphi_1 + q_1', ..., \Delta\varphi_N + q_N'\} + \Delta\hat{e}')]]. \tag{73}$$

Clearly, we have that:

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s_1', ..., \Delta s_N']] &= \mathbb{E}[\exp(\max\{\Delta\varphi_1 + \Delta q_1', ..., \Delta\varphi_N + \Delta q_N'\} + \Delta\hat{e}')]] \\
&< \mathbb{E}[\exp(\max\{\Delta\varphi_1, ..., \Delta\varphi_N\} + \Delta\hat{q}' + \Delta\hat{e}')]] \\
&= \mathbb{E}[\mathbb{E}[\exp(\Delta\theta_1)|\Delta s_1', ..., \Delta s_N']].
\end{aligned}$$

The inequality comes from the fact that the right hand side corresponds to the left hand side, but ignoring the realization of the random variables $\Delta q_i'$, and replacing this with a typical realization $\hat{q}'$. Hence, we prove the result.$\blacksquare$.

Proof of Proposition 9 (Only If) Let $(\boldsymbol{\beta}\cdot s_1, ..., \boldsymbol{\beta}\cdot s_N) \in \mathbb{R}^N$ be an equilibrium statistic. Using Lemma 4 we have that:

$$\mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}] = \mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{s}].$$

By construction of the expectation, we have that:

$$cov(\theta_i - \mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}], \mathbf{s}_i) = 0.$$

Using the property of the equilibrium statistic:

$$\mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}] = \mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}] = \mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}].$$

Hence,

$$cov(\theta_i - \mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}], \mathbf{s}_i) = 0.$$

Using Lemma 3, we know that variables with a bar are orthogonal to variables preceded by a Δ and hence:

$$\mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}] = \mathbb{E}[\Delta\theta_i|\boldsymbol{\beta}\cdot\boldsymbol{\Delta}\mathbf{s}_i] + \mathbb{E}[\Delta\theta_i|\boldsymbol{\beta}\cdot\bar{\mathbf{s}}].$$

Hence,

$$cov(\theta_i - \mathbb{E}[\bar{\theta}|\boldsymbol{\beta}\cdot\bar{\mathbf{s}}] - \mathbb{E}[\Delta\theta_i|\boldsymbol{\beta}\cdot\boldsymbol{\Delta}\mathbf{s}_i], \mathbf{s}_i) = 0.$$

Hence we prove sufficiency.

(If) Let $\boldsymbol{\beta}$ be such that: $cov(\theta_i - \mathbb{E}[\bar{\theta}|\boldsymbol{\beta}\cdot\bar{\mathbf{s}}] - \mathbb{E}[\Delta\theta_i|\boldsymbol{\beta}\cdot\boldsymbol{\Delta}\mathbf{s}_i], \mathbf{s}_i) = 0$. Hence, we have that: $cov(\theta_i - \mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}], \mathbf{s}_i) = 0$. This implies that: $\mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}] = \mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}\cdot\boldsymbol{\Delta}\mathbf{s}_i, \boldsymbol{\beta}\cdot\bar{\mathbf{s}}]$. Hence, we prove necessity. Hence, we prove the result.■

Proof of Lemma 3 Consider normal random variables $\{y_i\}_{i\in N}, \{z_i\}_{i\in N}$ normally and symmetrically distributed. Define:

$$\bar{y} \triangleq \frac{1}{N}\sum_{i\in N} y_i \ ; \ \Delta y_i \triangleq y_i - \bar{y} \ ; \ \bar{z} \triangleq \frac{1}{N}\sum_{i\in N} z_i \ ; \ \Delta z_i \triangleq z_i - \bar{z}. \tag{74}$$

We prove that $cov(\bar{y}, \Delta z_i) = 0$. We first provide the steps, and then explain each step.

$$\begin{aligned}
cov(\bar{y}, \Delta z_i) &= cov(\frac{1}{N}\sum_{k\in N} y_k, z_i - \frac{1}{N}\sum_{j\in N} z_j) &(75)\\
&= \frac{1}{N}\sum_{k\in N} cov(y_k, z_i) - \frac{1}{N^2}\sum_{k\in N}\sum_{j\in N} cov(y_k, z_j) &(76)\\
&= \frac{1}{N}(cov(y_i, z_i) + \sum_{k\neq i} cov(y_k, z_i)) - \frac{1}{N^2}\sum_{k\in N}(cov(y_k, z_k) + \sum_{j\neq k} cov(y_k, z_j)) &(77)\\
&= \frac{1}{N}(cov(y_i, z_i) + (N-1)cov(y_k, z_i)) - \frac{1}{N}(cov(y_k, z_k) + (N-1)cov(y_k, z_j)) &(78)\\
&= 0 &(79)
\end{aligned}$$

The explanation of each step is as follows. (75) is by construction of $\bar{y}$ and Δz_i. (76) is by the collinearity of the covariance. (77) is expanding terms. (78) is using symmetry. More specifically,

$$\forall i, k, \ell, j \in N, \quad \text{with } j \neq i \text{ and } k \neq \ell, \quad cov(z_i, y_j) = cov(z_k, y_\ell);$$

$$\forall i \in N, \quad cov(z_i, y_i) = cov(z_k, y_k).$$

(79) is trivially by checking both terms are the same.

Note that, it is clear from the proof that $cov(\bar{y}, \Delta y_i) = 0$ must also be satisfied. Since all random variables are Gaussian, if they have 0 covariance, they must be independent. Hence, we prove the result. ■

Proof of Lemma 4 For any $\boldsymbol{\beta} \in \mathbb{R}^J$, there exists constants $m_{i\ell}$, c_{ij} and $\tilde{m}_{i\ell}$ such that the expectation can be written as follows:

$$\mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N] = \sum_{j\in N} m_{ij}\boldsymbol{\beta}\cdot\mathbf{s}_j;$$

$$\mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N] = \sum_{j\in J} c_{ij}s_{ij} + \sum_{j\neq i} \tilde{m}_{ij}\boldsymbol{\beta}\cdot\mathbf{s}_j.$$

By symmetry, we can find $m, m', \tilde{m}, \{c_j\}_{j\in J}$ such that:

$$\mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N] = m\sum_{j\neq i} \boldsymbol{\beta}\cdot\mathbf{s}_j + m'\cdot\boldsymbol{\beta}\cdot\mathbf{s}_i;$$

$$\mathbb{E}[\theta_i|\mathbf{s}_i, \boldsymbol{\beta}\cdot\mathbf{s}_1, ..., \boldsymbol{\beta}\cdot\mathbf{s}_N] = \sum_{j\in J} c_j s_{ij} + \sum_{j\neq i} \tilde{m}\cdot\boldsymbol{\beta}\cdot\mathbf{s}_j.$$

We can re-write both equations as follows:

$$\mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_1,...,\boldsymbol{\beta}\cdot\mathbf{s}_N] = Nm\bar{s} - m\cdot s_i + m'\cdot\boldsymbol{\beta}\cdot\mathbf{s}_i;$$

$$\mathbb{E}[\theta_i|\mathbf{s}_i,\boldsymbol{\beta}\cdot\mathbf{s}_1,...,\boldsymbol{\beta}\cdot\mathbf{s}_N] = \sum_{j\in J} c_j s_{ij} + N\tilde{m}\cdot\boldsymbol{\beta}\cdot\bar{\mathbf{s}} - \tilde{m}s_i.$$

Hence, we have that in symmetric environments $\mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_1,...,\boldsymbol{\beta}\cdot\mathbf{s}_N]$ is measurable with respect to $(\boldsymbol{\beta}\cdot\mathbf{s}_i,\boldsymbol{\beta}\cdot\bar{\mathbf{s}})$ and $\mathbb{E}[\theta_i|\mathbf{s}_i,\boldsymbol{\beta}\cdot\mathbf{s}_1,...,\boldsymbol{\beta}\cdot\mathbf{s}_N]$ is measurable with respect to $(\boldsymbol{\beta}\cdot\mathbf{s}_i,\boldsymbol{\beta}\cdot\bar{\mathbf{s}})$. That is:

$$\mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_1,...,\boldsymbol{\beta}\cdot\mathbf{s}_N] = \mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_i,\boldsymbol{\beta}\cdot\bar{\mathbf{s}}];$$

$$\mathbb{E}[\theta_i|\mathbf{s}_i,\boldsymbol{\beta}\cdot\mathbf{s}_1,...,\boldsymbol{\beta}\cdot\mathbf{s}_N] = \mathbb{E}[\theta_i|\mathbf{s}_i,\boldsymbol{\beta}\cdot\bar{\mathbf{s}}].$$

Hence, $(\boldsymbol{\beta}\cdot\mathbf{s}_1...,\boldsymbol{\beta}\cdot\mathbf{s}_N)$ is an equilibrium statistic if and only if $\mathbb{E}[\theta_i|\boldsymbol{\beta}\cdot\mathbf{s}_i,\boldsymbol{\beta}\cdot\bar{\mathbf{s}}] = \mathbb{E}[\theta_i|\mathbf{s}_i,\boldsymbol{\beta}\cdot\bar{\mathbf{s}}]$. Hence, we prove the result. ■

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