Collective Progress: Dynamics of Exit Waves[*]

Doruk Cetemen[†] Can Urgun[‡] Leeat Yariv[§]

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Abstract. We study a model of collective search by teams. Discoveries beget discoveries and correlated search results are governed by a Brownian path, with its variance—the search scope—jointly controlled by members. Agents individually choose when to cease search and implement their best discovery. We show the emergence of endogenous exit waves, whereby possibly heterogeneous agents cease search all at once. Search scope is constant and independent of search outcomes as long as no member has left; it declines after each exit wave. We also characterize the optimal team search and illustrate inefficiencies equilibrium search entails, in terms of both search efforts and duration.

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[†]Collegio Carlo Alberto; doruk.cetemen@carloalberto.org

[‡]Princeton University; curgun@princeton.edu

[§]Princeton University CEPR, and NBER; lyariv@princeton.edu

1 Introduction

"Teamwork makes the dream work." John Maxwell's maxim is particularly germane to technological developments: advances in motor vehicles, communication devices, and pharmaceuticals frequently take place as joint ventures. For example, the Partnership for a New Generation of Vehicles was formed in 1993 and included eight federal agencies and the United States Council for Automative Research, comprised of various car manufacturers (see Chalk, Patil, and Venkateswaran (1996)). The program persisted till 2001, when automakers requested its dismantling and proceeded to each produce new car models. Similarly, the Paradigm Project started within Apple in 1989 and aimed at developing a digital communication platform. General Magic was created a year later by a subset of tech wizards who had worked on the Paradigm Project and hoped to bring their knowledge to fruition.[1] While General Magic ultimately failed to produce a successful product, many of its members went on to establish successful companies using some of the expertise they developed jointly.[2] Such industry examples abound. They entail alliances that form at early stages of development and dissolve over time.

In this paper, we provide a framework for analyzing such dynamic explorations by teams. We characterize the features of the search: members' effort and ambition while searching in an alliance and their decision to terminate or narrow it. In particular, we show that while alliances introduce well-known free-riding motives in search efforts, they encourage members to search for longer durations. In equilibrium, exit waves, in which multiple members leave an alliance at once—such as the car manufacturers in the New Generation of Vehicles partnership—often occur, even when members are heterogeneous. Furthermore, while the precise timing of exit waves depends on the realized search path, their sequencing does not. We also identify the inefficiencies generated through equilibrium when individuals take part in gradually dissolving alliances. These inefficiencies appear in both search efforts and exit patterns.

Technological developments rarely occur in a vacuum and discoveries build on one another. We therefore consider environments in which search results are correlated over time and follow a Brownian path, as first modeled by Callander

[1] See https://www.wired.com/1994/04/general-magic/

[2] For instance, Bill Atkinson conceived and implemented HyperCard, while Pierre Omidyar founded eBay.

(2011). The scope of search, captured by the Brownian path' variance, is chosen at each moment by the searching alliance. Specifically, each of the members of a searching alliance incurs a strictly positive cost that depends on their own search scope. While jointly searching, they observe outcomes of a Brownian path with variance corresponding to the sum of its members' search scopes. Any member can terminate her search at any point. A member ceasing her search receives a lump sum payoff corresponding to the maximal value the search has produced till her departure. For example, when automakers left their 1993 Partnership, they each retained the ability to use the most promising technologies and know-how developed jointly. Naturally, some alliance members may choose to continue their search even after certain members have exited. Those members experience prolonged search costs, but benefit from any further breakthroughs, as reflected by search results that exceed the previously-observed maxima. As search progresses, members gradually terminate their search until it halts altogether.

Our first results characterize equilibrium search in Markov strategies, where the state variables corresponds to the current search results, current assembly and the attained maximum. We show that, in any team one works in, search scope is constant and independent of the search results as long as no member leaves. Individual search scopes increase when members depart, reflecting the more limited free-riding opportunities present. The optimal time at which members depart and alliances shrink is governed by a simple stopping boundary, often referred to as a *drawdown stopping boundary*. Such boundaries are defined by one number, the *drawdown size*. Whenever search results fall by more than the drawdown size relative to the currently maintained maximal observation, a subset of members ceases search.

Relative to an individual searching on her own, standard free-riding motives drive search scopes down in an alliance. This is a form of a discouragement effect, whereby members do not search as intensely when they expect others to bear some of the search costs. Nonetheless, the externalities of others make search more valuable in a team: relative to searching alone, a member can reap the benefits of some of her peers' efforts. There is therefore also an encouragement effect that leads team members to search for longer than they would have on their own.

Our second result characterizes members' patterns of exits. In general, those exhibiting high ratios of marginal to fixed costs leave earlier than those exhibiting low such ratios. We show that, even when individual costs are fully heterogenous, clustered exits, or exit waves, may occur in equilibrium. Importantly, while the precise timing of exit waves may depend on the realized path of discoveries, their sequencing—who leaves first, second, etc., and with whom—does not.

Our last set of results identifies inefficiencies in such processes. We characterize the socially optimal search scope and exit wave sequencing. The socially optimal search scope is also constant and independent of search results for as long as an alliance is searching. Naturally, the positive externalities induced by each member's investment in search scope imply that the socially optimal level is higher than that chosen in equilibrium. Furthermore, in contrast to equilibrium search scopes, as alliance members terminate their search, the optimal scope of those remaining declines. As in equilibrium, clustered exits may be optimal even when individuals incur fully heterogeneous costs. The sequence of optimal exit waves is deterministic and independent of the realized search path. Furthermore, optimal exits are governed by drawdown stopping boundaries, although the drawdown sizes corresponding to each active alliance differ from those determined in equilibrium—optimal drawdown sizes are larger, corresponding to longer search durations.

Finding the optimal sequence of exit waves is a challenging combinatorial problem. A social planner needs to consider all possible partitions of the original searching team and assess search outcomes from the corresponding exit wave sequences. Nonetheless, we show a simple method for identifying the optimal sequencing for one class of settings, where individual search costs are proportional to one another. Similar to equilibrium, the social planner terminates the search of those with the highest search costs first. This observation limits the sequences of exit waves to consider. We illustrate a simple procedure, akin to a greedy algorithm (see, e.g., Papadimitriou and Steiglitz (1998)) that yields the optimal exit wave sequence. In rough terms, the social planner can use a recursive procedure, first identifying the optimal last alliance to search. That is, the alliance that would generate the highest welfare when all members are constrained to stop jointly. Once that alliance is identified, the social planner can find the optimal penultimate alliance. And so on. The procedure allows us to highlight settings in which equilibrium exit waves differ substantially from those set optimally.

Throughout the paper, we consider a simple environment in which late exits are not penalized by the existence of potential "first-movers" who exited early and already shifted any jointly-discovered technologies into production. This simplifies our analysis, but may not fit some applications. We show that most of our qualitative results carry over when introducing such penalties. However, naturally, the pattern of exits is modified somewhat when explicit incentives to exit early are in place. As it turns out, penalties for late exits do not induce substantial preemption motives. For example, with a team of two, the agent who exits first in a setting without late-mover penalties does not change her behavior in response to their introduction. However, the second agent, who without penalties searches alone after the first agent departs, may exit whenever the first mover does if exiting later implies a substantial drop in payoffs.

2 Literature Review

Since Weitzman (1979), much of the search with recall literature has focused on individual agents' discovery process, where the set of options is independent of one another. Our consideration of a Brownian path of discoveries, capturing intertemporal correlations, is inspired by the setting studied by Callander (2011). He studies short-lived agents who decide whether to choose an optimal, previously explored, result or experiment on their own. Urgun and Yariv (2020) analyze an individual-search setting similar to the one here. They consider agents who choose the variance of a Brownian path while searching, as well as when to stop.

In recent years, substantial attention has been dedicated to the study of collective search and experimentation. Some of this literature focuses on learning spillovers between team members. For instance, the classic papers of Bolton and Harris (1999), Keller, Rady, and Cripps (2005) extend the classic two-armed bandit problem to a many-agent setting, where agents can learn from others. Information is a public good. Thus, there is a free-rider problem that discourages experimentation. Nonetheless, there may also be an encouragement effect through the prospect of others' future experimentation. See Hörner and Skrzypacz (2016) for a survey of the literature that followed.

Another strand of literature contemplates settings in which team members decide jointly when to stop. Albrecht, Anderson, and Vroman (2010) and Strulovici (2010) consider sequential search and experimentation, respectively, in which a committee votes on when to stop. They illustrate when collective dynamics may impede search or experimentation. Bonatti and Rantakari (2016) offer a model in which agents can exert effort on different projects but stop experimentation jointly. They show that, optimally, one agent advances her preferred project quickly and her opponent agrees to early projects in order to avoid effort costs. Polarized outcomes can then emerge. Deb, Kuvalekar, and Lipnowski (2020) study a related problem from a design perspective—for a given deadline at which a project has to be chosen, the principal commits to a selection rule. Titova (2019) studies a public-good setting in which a team decides jointly whether to implement a public good. Payoffs are revealed through a Pandora's box problem à la Weitzman (1979). In this setting, she shows that optimal information and projects are selected, but that free-riding may generate inefficient delays.[3]

There are also several papers that illustrate patterns reminiscent of the clustered exits we characterize, mostly in settings in which agents have private information. Caplin and Leahy (1994) study a three-period irreversible-investment game in which each firm receives private information on the aggregate state of the economy as well as observes others' prior decisions. Firms' actions reveal information and can generate a wave. Gul and Lundholm (1995) considers a two-agent model in which both try to predict the value of a project. Each agent receives private information on the project's value and decides when to issue a prediction, where delay entails a flow cost. The timing of decisions is then informative and clustered predictions occur in equilibrium. Rosenberg, Solan, and Vieille (2007) study a two-agent version of the standard real-options problem (see Dixit and Pindyck (1994)). Agents observe private signals about common returns to a risky project, as well as the actions of others. If one agent switches to a safe projects—namely, exercise an option—this can lead the other agent to immediately switch to the safe project as well. See also Murto and Välimäki (2011) and Anderson, Smith,

[3]Dynamic contribution games without experimentation or uncertainty have also been heavily studied, see for instance Admati and Perry (1991), Marx and Matthews (2000), Yildirim (2006), and the literature that followed. These papers show that free-riding can be mitigated if agents can contribute over time. In these models, payoff is realized only when the project's state reaches a prespecified threshold. This makes effort choices strategic complements over time. Hence, besides the usual free-riding there is also an encouragement effect. Cetemen, Hwang, and Kaya (2019) shows that even without the threshold structure, if the returns from the public good are uncertain, there may still be an encouragement effect. An agent contributing makes others more optimistic and more willing to contribute.

and Park (2017). In a static information-collection setting, Bardhi and Bobkova (2021) characterize optimal subsets, or mini-publics, to be activated. [4]

The techniques we use relate to the applied mathematics literature on optimal stopping, see Peskir and Shiryaev (2006) and Azéma and Yor (1979) for particularly relevant sources.

3 A Model of Collective Search

Consider a team of N agents—product developers, academic researchers, etc.—searching through a terrain of ideas in continuous time. Time is indexed by t and runs through $[0, \infty)$. Each seeks good outcomes and ultimately benefits from the maximal value they have found when they stop their search. Formally, we assume all agents are risk neutral. At each time t, agent $i = 1, 2, ..., N$ decides on the scope of her search $\sigma_{i,t}^A \in [\underline{\sigma}, \overline{\sigma}]$, where $A \subseteq \{1, ..., N\}$ is the alliance of agents still searching at time t. Let $\mathcal{A} = 2^{\{1,...,N\}}$ denote the set of all possible alliances. We assume $\bar{\sigma} \geq \underline{\sigma} > 0$. As we soon describe, the search scope naturally feeds into the breadth of search conducted by the alliance of active agents. Any search scope σ comes at a cost of $c_i(\sigma)$, and we assume that c_i is log-convex for $i = 1, 2, \ldots N$. The special case of $\underline{\sigma} = \bar{\sigma}$ corresponds to settings in which search scope is not controlled and agents' only choose when to stop search.

We model the progress of discoveries using a Weiner process, which allows us to capture the correlation of new developments over time, and the impact of search scope of those who engage in search.[5] Formally, for any time t, denote by B_t the standard Brownian motion and $B_0 = 0$, and let σ_t^A denote the controlled breadth of search, which will depend on the search scope of all members of the active alliance A as we soon describe. The observed project value at time t is denoted by X_t, where

[4] There is also a literature that tries to explain industry "shakeouts," corresponding to times at which firm numbers plummet, absent a decline in output. For example, Jovanovic and MacDonald (1994) suggests shakeouts result from exogenous technological shocks. Initially, firms enter new markets when they are profitable. Profits naturally decrease as more firms enter. When there is a technological shock, some firms become more productive than others, potentially leading to clustered exits.

[5] We view correlation as an important feature of discovery processes. Nonetheless, from a purely theoretical perspective, one could analyze an analogous model with independent samples. As it turns out, such a model is far less tractable. We elaborate on this point in our concluding discussion, see Section 6.3, as well as in the Online Appendix.

$X_0 = 0$ and the law of motion is given by:

$$dX_t = \sigma_t^A dB_t.$$

Whenever the alliance A of agents is searching, we assume $\sigma_t^A = \sum_{i \in A} \sigma_{i,t}^A$.[6]

The search scope can be interpreted in two ways. First, it can capture search breadth. Investment in development, through acquisition of instruments or expert time, often entails an increase in risk: it either leads to substantial leaps, or to more pronounced losses. Second, given our modeling of search values, the search scope can also be thought of as capturing search *speed*. Changing the search scope from 1 to σ at any small interval of time is tantamount to "speeding up" the process by a factor of σ^2. As we soon show, the returns of search with recall depends linearly on the search scope.

We assume the discovery process exhibits no drift: in applications, the mere passage of time rarely improves or worsens search outcomes over standard horizons of research and development. Naturally, one could consider a team that *controls* drift rather than search scopes, which would also translate to the returns of search with recall. The analysis would follow similar lines to those we describe, although with an important loss in tractability.[7] We view search scope as more natural for most market applications, where investments in innovation either affect the speed at which progress is made, or entail non-trivial risks.

3.1 Payoffs

Each agent is rewarded according to the maximal project value observed up to her stopping time. Let M_t denote the maximum value observed by time t :

$$M_t = \left(\max_{0 \le r \le t} X_r \vee M_0 \right),$$

where we assume that $M_0 = 0$.

For any aggregate fixed search scope σ, at time t, $\mathbb{E}(M_t) = \sigma \sqrt{2t/\pi}$. Thus, the

[6]In the Online Appendix, we show that our analysis can be extended to the case in which, for any alliance A, we have $\sigma_t^A = f^A(\{\sigma_{i,t}^A\}_{i \in A})$, with f^A a differentiable function. Comparative statics would naturally depend on alliances' technologies captured by $\{f^A\}_A$.

[7]Taylor et al. (1975) characterizes the maximal value of search with constant drift. The resulting value is far less amenable to further analysis as ours.

choice of search scope translates directly to the expected maximum value.

When any agent i stops at time τ, her resulting payoff is given by

$$M_\tau - \int_0^\tau c_i(\sigma_{i,t})dt,$$

where $\sigma_{i,t}$ is the timed search scope of individual i, which may depend on the alliances she is active in. In Section 6.1, we discuss an extension in which agents who stop later are penalized. For presentation simplicity, we keep rewards independent of the order of exits for most of our study.

Agents observe one another's search. In particular, whenever agents stop searching, other agents realize their search will continue within a smaller alliance. Any progress made after an agent stops searching does not impact her payoffs.

3.2 Strategies and Equilibrium

At any time t, the state of the environment is summarized by X_t, M_t, and A_t, where A_t is the active alliance of agents still searching.

A strategy for agent i dictates the chosen search scope over time and the stopping policy. Formally, it is a pair of functions (σ_i^A, τ_i^A), where $A \subseteq \{1, ..., N\}$ and $i \in A$. In principle, (σ_i^A, τ_i^A) may depend on time, as well as the entire path of observed project values and corresponding maxima. Let $\{\mathcal{F}_t\}$ denote the natural filtration induced by the governing Brownian motion. Agents' strategies are adapted to this filtration.

We restrict attention to Markov strategies. That is, we assume agents use strategies of the form (σ_i^A, τ_i^A) that depend only on the state variables X_t, M_t, and A_t. Formally, $\sigma_i^A : \mathbb{R}^2 \to [\underline{\sigma}, \bar{\sigma}]$, and τ_i^A is a random variable over $\mathbb{R}_+$ such that $\Pr(\tau_i^A = t|\mathcal{F}_t) = \Pr(\tau_i^A = t|X_t, M_t)$ for all i.

We further assume that a continuous stopping boundary determines when each agent halts her search. Formally, for all i and all alliances A such that $i \in A$, the stopping policy takes the following form:

$$\tau_i^A = \inf\{t \geq 0 : X_t = g_i^A(M_t)\},$$

where $g_i^A(\cdot)$ is a continuous function. This formulation implicitly implies that, upon indifference, agents exit the search. Our assumption that stopping boundaries are continuous is without loss of generality as long as any agent is willing to search on her own, which we show in the Online Appendix.[8]

Given $\left\{(\sigma_j^A, \tau_j^A)\right\}_{j \neq i}$, agent i's best-response strategy simply maximizes her expected payoff given this profile. Formally, it is determined by solving the following problem for each A such that $i \in A$:

$$\sup_{\tau^A, \left\{\sigma_{i,t}^A\right\}_{t=0}^{\tau_i}} \mathbb{E}_{\left\{(\sigma_j^A, \tau_j^A)\right\}_{j \in A \setminus \{i\}}} \left[M_{\tau^A} - \int_0^{\tau^A} c_i\left(\sigma_{i,t}^A\right) dt \right].$$

An *equilibrium* is a profile of Markov strategies satisfying the assumptions above that constitute best responses for all agents.

We denote the continuation payoff at time t of agent i searching within an alliance A by $V_i^A(M_t, X_t)$. Whenever $A = \{i\}$ for some i, agent i searches on her own. In this case, we will often drop the set indication and simply refer to σ_i^i, g_i^i, and V_i^i. The characterization of the optimal solo search problem, offered by Urgun and Yariv (2020), serves as a natural benchmark for the impacts of team search and will be useful for our analysis:

Proposition 0 (Solo Search)**.** *When $A = \{i\}$, so that i searches on her own, the optimal search scope σ_i^i is constant and, when interior, solves:*

$$\frac{2c_i(\sigma_i^i)}{c_i'(\sigma_i^i)} = \sigma_i^i.$$

The optimal stopping boundary at any point t with an observed maximum M_t is:

$$g_i^i(M_t) = M_t - \frac{(\sigma_i^i)^2}{2c_i(\sigma_i^i)}.$$

The resulting continuation payoff is:

$$V_i^i(M_t, X_t) = M_t + \frac{c_i(\sigma_i^i)}{(\sigma_i^i)^2}\left(X_t - M_t + \frac{(\sigma_i^i)^2}{2c_i(\sigma_i^i)}\right)^2.$$

[8]In our setting, departing agents would never benefit from continuing the search in a smaller alliance: from an individual's perspective, the externalities offered by a larger alliance are always beneficial. We return to this point when discussing equilibria in our setting.

4 Equilibrium Team Search

In this section, we characterize the outcomes of team search. We identify the sequence of agents stopping their search, suggesting the likely pioneers and likely innovators in the market. We also describe the search scopes, stopping boundaries, and resulting payoffs.

4.1 Equilibrium Characterization

Given our restriction on agents' policy, it follows that any alliance A gets smaller at the minimal stopping time of its members. That is, the time τ^A at which the first members of A stop search, and the active alliance shrinks is given by $\tau^A = \min_{i\in A} \tau_i^A$. Equivalently,

$$\tau^A = \inf\{t \geq 0 : X_t = \max_{i\in A} g_i^A(M_t)\}.$$

Since we assumed agents use continuous stopping boundaries, we can write

$$\tau^A = \inf\{t \geq 0 : X_t = g^A(M_t)\},$$

where $g^A(M_t) \equiv \max_{i\in A} g_i^A(M_t)$ is continuous.

We start by identifying the equilibrium search scope. As it turns out, individual search scopes depend only on the active alliance and are constant as long as no one in the alliance terminates their search.

Proposition 1 (Team Search Scope)**.** *For any agent i in an active alliance A, equilibrium search scopes are constant, $\sigma_i^A(M_t, X_t) = \sigma_i^A$, and uniquely identified whenever interior by the system:*

$$\frac{2c_i(\sigma_i^A)}{c_i'(\sigma_i^A)} = \sigma^A = \sum_{i\in A} \sigma_i^A \quad \forall i \in A.$$

Whenever a unique interior solution exists for the set of equalities specified in the proposition, uniqueness of equilibrium search scopes follows directly from the log-convexity of costs. Multiplicity of equilibrium search scopes arises in our setting only when there are only boundary equilibria. In some sense, such multiplicity implies "extreme" free-riding. It is associated with some agents choosing either the minimal or the maximal possible search scope. Intuitively, if one agent selects high search scope, another might find it prohibitively costly to contribute and search minimally. In turn, the high-search scope agent could be best responding.[9]

Why are search scopes fixed as long as a certain alliance of agents is active? The rough intuition is the following. Consider an agent i in an active alliance A. Suppose i believes that all other agents j in the alliance search with scope σ_j^A. When away from agent i's stopping boundary, agent i can contemplate a small interval of time in which she is unlikely to hit her boundary. For that small interval, agent i considers the induced speed of the process: $\left(\sum_{k\in A}\sigma_k^A\right)^2$ and the cost she incurs, $c_i(\sigma_i^A)$. Ultimately, the agent aims at minimizing the cost per speed, or the overall cost to traverse any distance on the path,

$$\frac{c_i(\sigma_i^A)}{\left(\sum_{k\in A}\sigma_k^A\right)^2} = \frac{c_i(\sigma_i^A)}{(\sigma^A)^2}.$$

The identity in the proposition reflects the corresponding first-order condition.

A direct corollary of Proposition 1 pertains to the impacts of alliance size on individual and collective search scope. As alliances shrink with members departing, each individual increases her own search scope, but the alliance searches less expansively overall. That is,

Corollary 1 (Search Scope and Alliance Size)**.** *As an alliance shrinks, individual members' search scope increase, while total search scope decreases. That is, for any $i, j \in A$, we have $\sigma_i^{A\setminus\{j\}} \geq \sigma_i^A$ while $\sigma^A > \sigma^{A\setminus\{j\}}$.*

Intuitively, consider the individual minimization of cost per speed described above. For any individual i in an active alliance A, whenever total search scope σ^A increases, her marginal return from search scope decreases. Consequently, an increase in overall search scope would act to decrease individual search scope. In particular, were $\sigma^A < \sigma^{A\setminus\{j\}}$, each individual in $A\setminus\{j\}$ would decrease their search scope relative to when they search in A, leading to a reduced overall search scope and a contradiction. It must then be the case that $\sigma^A \geq \sigma^{A\setminus\{j\}}$ and, as the active alliance shrinks, its members all search more and more expansively.

[9]This is the case when, for example, search costs are linear for all agents.

Since individual search scope decrease within an alliance, the total search scope in any active alliance is smaller than that which would be generated by the alliance's members searching independently.[10] The corollary highlights a form of free-riding. Search scope is substitutable across individuals. The more agents searching, the less each one searches.

The corollary indicates that agents departing an alliance would never benefit from continuing search on their own, nor would they benefit from switching to search in a smaller alliance than the one they have left. In particular, our assumption that agents who cease search in an alliance simply reap the benefits from past discoveries rather than pursue further discoveries with other newly-departed agents is without loss of generality.

We now turn to the characterization of the stopping boundary. Our main result indicates that the order in which agents terminate their search is fixed and does not depend on the realized path of project values. Furthermore, the points at which agents cease their search are determined via a simple stopping boundary.

Proposition 2 (Alliance Stopping Boundary)**.** *There exists an equilibrium such that, for any agent i in any active alliance A,*

$$g_i^A(M) = M - \frac{(\sigma^A)^2}{2c_i(\sigma_i^A)}.$$

In particular, agent $i \in \arg\min_j \frac{(\sigma^A)^2}{2c_j(\sigma_j^A)}$ is the first to stop in any alliance A. Furthermore, equilibrium outcomes are generically unique.[11]

To glean some intuition for the structure of the equilibrium stopping boundary, consider some alliance A and suppose all agents believe that other members of the alliance will continue searching indefinitely at the search scope given by Proposition 1. Each individual agent i's optimization problem then boils down to a solo searcher's optimization, with the search of others simply affecting the effective cost

[10] Welfare is always lower when individuals search independently. In fact, any agent receives a higher payoff within the team than she would on her own. Indeed, any agent can emulate her solo-search policy when searching with the team. Others in the team searching can only raise her expected payoffs relative to what she would receive on her own. That would necessarily be the case when she best responds.

[11] Multiplicity of equilibrium outcomes may occur when there is more than one minimizer of $\frac{(\sigma^A)^2}{2c_j(\sigma_j^A)}$ in A.

they experience for any search scope. Namely, the induced cost of implementing search scope σ is then $c_i(\sigma - \sum_{j\in A, j\neq i} \sigma_j^A)$. From Proposition 0, the optimal stopping boundary of agent i would feature a drawdown. Denote that drawdown by d_i^A. Suppose $d_i^A = \min_{j\in A} d_j^A$. Consider then another iteration of best responses, where all agents use the drawdown stopping boundary calculated as above. Agent i would still be best responding since, from her perspective, others in the alliance would continue searching for as long as she does. Furthermore, while other agents may want to alter their stopping boundary, intuitively, none would want to cease search before agent i since that would contradict their desire to continue searching for at least as long as agent i in the first place.

This line of argument suggests that the stopping boundary of the first agent i to terminate search in any alliance A is determined uniquely in equilibrium. Multiplicity of equilibria arises from the stopping boundaries of other agents $j \in A$. Indeed, any agent j who stops strictly after agent i is indifferent across all stopping boundaries $g_j^A(\cdot)$ that satisfy $g_j^A(M) > g_i^A(M)$ for all M. Naturally, all such choices of stopping boundaries by agents other than i do not impact when the alliance first loses some of its members, nor the search scope while it is fully active. Consequently, equilibrium outcomes are unique.[12]

When all agents have the same costs and solutions are interior, equilibrium takes a simple form. Members of any team choose identical search scopes, as determined by Proposition 1. They also leave in unison—there is only one exit wave. Proposition 2 illustrates that joint departures may occur even when costs differ across individuals, a point we highlight in the next subsection as well.

As Proposition 2 indicates, when costs are heterogeneous, one agent leaving may trigger the departure of multiple agents. This suggests that targeted interventions, subsidizing the search of only particular agents, may impact the entire path of exit waves.

[12] Our analysis indicates a link to other cooperative solution concepts in the spirit of the core. At any point in time, were active agents free to form any coalition to pursue search, or cease search, the externalities present in our environment would imply a unique outcome corresponding to the equilibrium outcome we identify.

4.2 Well-ordered Costs

We now consider a particular setting of team search, which will prove useful for providing insights on the order by which agents terminate their search and the illustration of various comparative statics.

Suppose agents' cost functions are proportional to one another and point-wise ordered:

$$c = c_1\beta_1 = c_2\beta_2 \cdots = c_N\beta_N,$$

where $\beta_1 = 1 < \beta_2 < ... < \beta_N$. That is, agent 1 has the highest search costs, while agent N has the lowest search costs.

Proposition 1 implies that all agents in an active alliance choose the same search scope, assuming an interior solution exists. Suppose σ denotes the search scope all agents exert in the full alliance. It follows that

$$\frac{2c(\sigma)}{c'(\sigma)} = N\sigma.$$

As N increases, individual search scopes decrease. Let $\hat{N}$ be the maximal integer such that

$$\frac{2c(\underline{\sigma})}{c'(\underline{\sigma})} < \hat{N}\underline{\sigma}.$$

For any $N > \hat{N}$, there is no interior equilibrium. Furthermore, when there are $\hat{N}$ agents in the team, individual search scopes initially are roughly at their minimum $\underline{\sigma}$, while overall search scope is $\hat{N}\underline{\sigma}$.

Agents' search scope changes only when their alliance shrinks. In this special case, we can pin down the weak order by which agents stop their search without calculating their corresponding drawdown sizes, which greatly simplifies the analysis. Indeed, in equilibrium, agents with higher costs terminate search earlier. There is a subtlety, however. Despite agents' costs being strictly ordered, clustered exits may occur. To see when those happen, consider any active alliance A. Suppose $i = \arg\min_{j\in A}\beta_j$. That is, agent i is the alliance's member with the highest search cost. From Proposition 2, that agent will stop search first. The drawdown d_i^A governs when she exits.

Now consider the alliance $\tilde{A} = A \setminus \{i\}$ resulting from i's departure. For all remaining agents, there is then a new drawdown that governs their decision to stop search in alliance $\tilde{A}$. These new drawdowns are $\{d_j^{\tilde{A}}\}_{j\in\tilde{A}}$. The discrete drop in overall search scope induced by i's departure may imply that $d_j^{\tilde{A}} \leq d_i^A$ for some $j \in \tilde{A}$. This would intuitively occur when some agents' costs are close to i's. Let $D \subseteq \tilde{A}$ denote all these agents. It follows that, as soon as agent i terminates her search, so will all the agents in D.

Following this process recursively, we can identify the clustered exits that occur in equilibrium. Specifically, Proposition 2 implies that agent N exits no sooner than agent $N-1$,who exits no sooner than agent $N-2$, and so on. Consider any active alliance $\{j,\ldots,N\}$. If

$$d_j^{\{j,\ldots,N\}} \geq d_{j+1}^{\{j+1,\ldots,N\}}, d_{j+2}^{\{j+2,\ldots,N\}}, d_{j+k}^{\{j+k,\ldots,N\}}, \ldots,$$

then agents $j, j+1, j+2, \ldots, j+k$ will all terminate their search at the same time. Figure 1 depicts an example for $N = 10$ individuals. In the figure, once agent 1 leaves, agents 2 and 3 leave as well. Similarly, once agent 4 leaves, agent 5 leaves. And so on. Ultimately, the drawdowns that govern agents' departures correspond to the "upper envelope" of the graph depicting $d_j^{\{j,\ldots,N\}}$ as a function of j.

Despite agents' costs being strictly ordered, clustered exits are possible. In fact, when costs are close to one another, all agents might exit at once. Indeed, from Proposition 2, $d_j^{\{j\ldots N\}} = \frac{(\sigma^{\{j,\ldots,N\}})^2\beta_j}{2c(\sigma_j^{\{j,\ldots,N\}})}$. From Corollary 1, $\sigma^{\{j,\ldots,N\}}$ decreases in j, while $\sigma_j^{\{j,\ldots,N\}}$ increases in j. Therefore, for $\{\beta_j\}$ sufficiently close to one another, $d_1^{\{1,\ldots,N\}} > d_2^{\{2,\ldots,N\}} > \ldots > d_N^N$ and all agents exit at once.

Were we to decrease β_1, keeping $c_1\beta_1$ and all other parameters constant, the search costs of agent 1 would increase and she would stop sooner, potentially too soon for other agents to exit. Consequently, the number of exit waves would weakly increase. In contrast, were we to decrease β_N, keeping $c_N\beta_N$ and all other parameters constant, agent N's search costs would increase, making her more incline to exit when agent $N-1$ does. Consequently, the number of exit waves would weakly decrease.

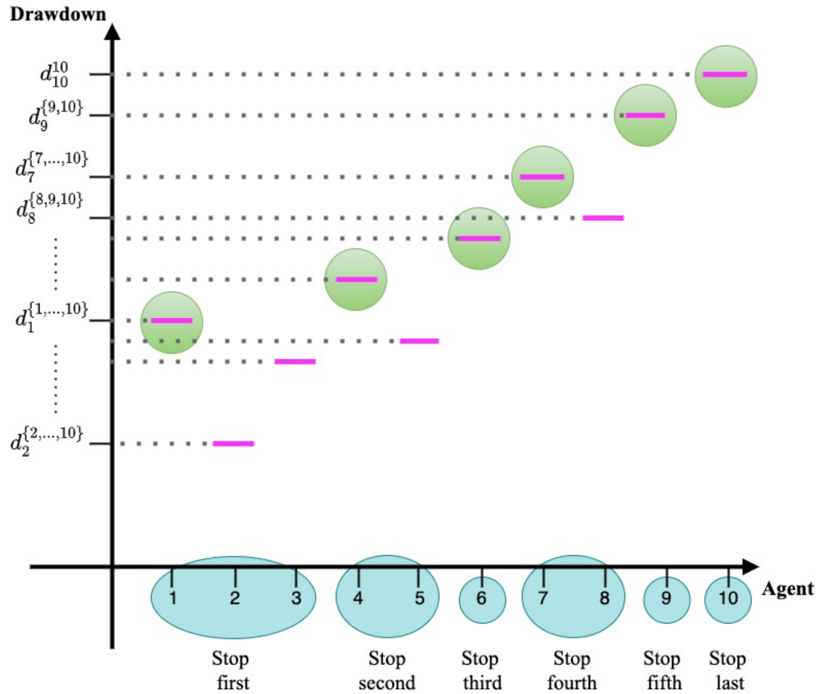

Figure 1: Equilibrium exit waves with well-ordered costs

5 The Social Planner's Problem

We now consider a social planner, who can dictate agents' search scopes and exit policies to maximize overall utilitarian efficiency of the team. This analysis highlights the type of inefficiencies that strategic forces in our joint search process imply. We show that the equilibrium search scopes exhibit different patterns than those prescribed by the social planner as alliances shrink: individuals increase their search scope in equilibrium, but decrease it under the socially optimal policy. Furthermore, the equilibrium order of exits may be inefficient.

5.1 The Social Objective

The social planner aims to maximize the agents' expected utilitarian welfare. The instruments at her disposal are the times at which various agents exit—the sequence of active alliances—and the search scopes within each active alliance.

Standard arguments allow us to restrict attention to Markovian policies for the social planner, see Puterman (2014). Formally, we consider a Markov decision problem in which the state at each date t is three-dimensional and comprised of (i) the set of active agents A_t, (ii) the current maximum M_t, and (iii) the current observed project value X_t. The social planner chooses a continuation alliance of agents—a subset of the current alliance A_t—and the search scope of each member in that alliance.

That is, the social planner has two Markovian controls. The first pertains to the selection of a continuation alliance, and can be denoted by $G(M,X,A) : \mathbb{R}^2 \times 2^N \mapsto 2^A$. The mapping G determines the subset of agents continuing the search from the current alliance as a function of the current state. In particular, if $G(M,X,A) = A$, the current alliance continues the search. If $\emptyset \neq G(M,X,A) \subsetneq A$, the alliance reduces in size. Whenever $G(M,X,A) = \emptyset$, no agent is left searching and the search terminates.

The second control of the social planner is the search scope chosen within an alliance, which can be written as $\sigma_i^A(M,X,A) : \mathbb{R}^2 \times 2^N \mapsto [\underline{\sigma}, \bar{\sigma}]$ for each $i \in A$.[13] Agents that already exited cannot be induced to choose positive search scope and do not participate in any future search: exit is irreversible. We therefore write $\sigma_i^A(M,X,A) = 0$ for each $i \notin A$. As before, for any active alliance A, we write:

$$\sigma^A(M,X,A) = \sum_{i \in A} \sigma_i^A(M,X,A)$$

and, as a shorthand, we drop the arguments when there is no risk of confusion.

Given these controls, we can now associate a stopping time for each active alliance A. This is the first time at which the alliance shrinks in size, that is:

$$\tau^A = \inf\{t \geq 0 : G(M_t, X_t, A) \neq A\}. \tag{1}$$

If an alliance A is never reached, we set $\tau^A = 0$.

Let $\tilde{A}_t$ denote the induced process of active alliances. For any active agent i, the time at which her search stops is given by

$$\tau_i = \inf\{t \geq 0 : i \notin G(M_t, X_t, \tilde{A}_t)\}.$$

[13] This notation is with a slight abuse, as A is both an argument and a superscript. Proposition 3 will clear this confusion as the only relevant argument will be shown as A and be denoted as a superscript.

This is the first time at which agent i is not included in an active alliance.

At any time t, the welfare of an individual $i \in \tilde{A}_t$, given the controls $\{G, \sigma_i\}$, is then

$$W_i(M_t, X_t, \tilde{A}_t | \sigma_i, G) = \mathbb{E}\left[M_{\tau_i} - \int_t^{\tau_i} c_i(\sigma_{i,s}^{\tilde{A}_s}) ds\right].$$

For any $i \notin \tilde{A}_t$, we set $W_i(M_t, X_t, \tilde{A}_t | \sigma_i, G) = 0$. We can state the problem of the social planner as:

$$\begin{aligned} W(M_t, X_t, \tilde{A}_t) &= \sup_{\{G, \sigma_i\}} \sum_i W_i(M_t, X_t, \tilde{A}_t | \sigma_i, G) \\ &= \sup_{\{G, \sigma_i\}} \sum_i \mathbb{E}\left[M_{\tau_i} - \int_t^{\tau_i} c_i(\sigma_{i,s}^{\tilde{A}_s}) ds\right]. \end{aligned}$$

We assume that whenever the social planner is indifferent between maintaining a certain set of agents searching or having them exit, she chooses the latter.

Given a pair of controls (G, σ), let $A_1 = N$ denote the first active alliance, containing all of the agents. Using (1), let $A_2 = A_{\tau^{A_1}}$ be the alliance that succeeds the initial alliance. That is, this is the alliance resulting from the first agents halting their search. In principle, A_2 could entail some randomness—depending on the path observed, different agents may be induced to stop their search. We then use (1) to define τ_{A_2}, the (random) time at which the second set of agents stops their search and define $A_3 = A_{\tau^{A_2}}$ as the (potentially random) resulting alliance. We can continue recursively to establish the (random) time τ^{A_k} at which the k'th set of agents stops search and by $A_{k+1} = A_{\tau^{A_k}}$ the (potentially random) resulting alliance. Let K denote the number of different active alliances the social planner utilizes till search terminates by all agents. This number is, in itself, potentially random. For any set of controls $\{G, \sigma_i\}$, we then have a sequence of active alliances $A_1, A_2, ..., A_K$ with associated stopping times $\tau^{A_1}, \tau^{A_2}, ..., \tau^{A_K}$.

Suppose our team-search problem starts at the state (M, X, A). We set $\tau^{A_0} = 0$ and $A_{K+1} = \emptyset$ so that the social planner's problem can now be written as:

$$W(M, X, A_1) = \sup_{\{G, \sigma_i\}} \mathbb{E}\left[\sum_{k=1}^{K} \left(|A_k \setminus A_{k+1}| M_{\tau^{A_k}} - \int_{\tau^{A_{k-1}}}^{\tau^{A_k}} \sum_{i \in A_k} c_i(\sigma_{i,t}^{A_k}) dt \right)\right].$$

Equivalently, we can write the problem recursively starting from any state (M, X, A_k) as:

$$W(M,X,A_k) = \sup_{\{G,\sigma_i\}} \mathbb{E}\left[|A_k \setminus A_{k+1}| M_{\tau^{A_k}} - \int_0^{\tau^{A_k}} \sum_{i \in A_k} c_i(\sigma_{i,t}^{A_k}) dt + W(M_{\tau^{A_k}}, X_{\tau^{A_k}}, A_{k+1}) \right].$$

Suppose the social planner finds it optimal to halt the search of agent i in an active alliance A when observing X and M. It would then also be optimal to halt the search of this agent when observing X' and M with any $X' < X$. The intuition is the following. The social planner's solution would be the same were the process shifted by a constant. Therefore, her choice when observing value X' and a maximum M is the same as when observing X and maximum value $M' \equiv M + X - X' > M$. When observing X and M', were the social planner to continue agent i's search for a small period of time, the optimal search scopes in the active alliance would coincide with those that she would pick for the same alliance were search continued when observing X and M. Now, the search scopes in each alliance are adapted and therefore depend only on the observed value, not explicitly on its achieved maximum. Indeed, as we soon describe, the separability inherent in the social planner's objective implies that the social planner sets search scopes to optimize the "speed" at which the alliance proceeds, which is independent of the current maximum. However, the likelihood of surpassing M' at this small interval of time is lower than the likelihood of surpassing M. Furthermore, the social planner could gain M' from the releasing agent i with the current observed maximum relative to the lower M she would get from releasing that agent when observing X and M. In particular, if it is optimal to halt agent i's search when observing X and M, it is also optimal to halt that agent's search when observing X and M'.

It follows that, whenever an alliance changes when observing X and M, it also changes when observing X' and M with $X' < X$. We can therefore write

$$\tau^A = \inf\{t \geq 0 : X_t = g^A(M_t)\},$$

where $g^A(M) = \sup\{X : G(M, X, A) \neq A\}$.[14] This kind of stopping time τ^A is commonly known as an Azéma-Yor stopping time (Azéma and Yor (1979)), with the

[14]We implicitly assume, without loss of generality, that whenever the social planner is indifferent between halting the search of a subset of agents or continuing their search, she chooses the former.

function g^A defining the corresponding stopping boundary.

For any active alliance A, we note that $g^A(M) < M$ for all M. In other words, it is never optimal to stop that alliance at any t such that $M_t = X_t$. If an alliance searches for a non-trivial amount of time at its inception, say at time t_0, it must be that $M_{t_0} \geq X_{t_0}$. Given our consideration above, the alliance would then continue searching jointly even were the planner to observe, at some time t, the value X_t and recorded maximum of M_t with $X_t = M_t = M_{t_0}$. But then the same should hold when $M_t = X_t = y$, with arbitrary y; this corresponds to a shifted problem and does not alter welfare considerations.[15]

5.2 Optimal Team Search

Our first result illustrates that the social planner chooses constant search scopes for each active alliance. Furthermore, the specification of these search scopes differs from that dictated by equilibrium.

Proposition 3 (Optimal Search Scope)**.** *Search scopes within an alliance are constant and dependent only on the alliance's composition. Furthermore, scopes are uniquely identified whenever interior by the system:*

$$\frac{2\sum_{i\in A} c_i(\sigma_i^A)}{\sum_{i\in A}\sigma_i^A} = c_i'(\sigma_i^A) \quad \forall i \in A.$$

The proposition implies that socially optimal search scopes are higher than those prescribed in equilibrium. The intuition for this result resembles that provided for equilibrium choices. For any active alliance A, the social planner considers the induced speed of the process, given by $\left(\sum_{k\in A}\sigma_k^A\right)^2$ and the cost she incurs, $\sum_{k\in A} c_k(\sigma_k^A)$. The social planner then aims at minimizing the cost per speed, or the overall cost to traverse any distance on the path,

$$\frac{\sum_{k\in A} c_k(\sigma_k^A)}{\left(\sum_{k\in A}\sigma_k^A\right)^2} = \frac{\sum_{k\in A} c_k(\sigma_k^A)}{(\sigma^A)^2}.$$

[15]This would not hold were the social planner's objective function concave in the maximum values of the project, taking into account risk attitudes of agents. Risk attitudes introduce new complexities to our setting, see Urgun and Yariv (2020) for a discussion of their impact on single-agent decisions. Their investigation would be an interesting direction for the future.

The identity in the proposition reflects the corresponding first-order condition. Due to log-convexity of the costs, when alliance A is active, each alliance as a whole searches weakly more under the social planner's solution. Intuitively, the social planner internalizes the positive externalities entailed by agents' contributions to the scope of search and thus specifies greater overall search investments. Uniqueness and comparative statics of the socially optimal search scopes follow from similar arguments to those used for equilibrium choices in Section 4.

Proposition 3 illustrates that overall search scope in any active alliance is lower in equilibrium than is socially optimal. The positive externalities search entails suggest that, in equilibrium, each *individual* of an active alliance invests in lower search scope than is socially optimal. Furthermore, in equilibrium, Corollary 1 indicated that, as alliances shrink, remaining agents increase their search scope. The impacts of agents departing are quite different in the social planner's solution. As members depart, the externalities of each remaining agent decline, as there are fewer others her search scope help. Consequently, the socially optimal search scope of each individual agent *declines*. We therefore have the following corollary:

Corollary 2 (Optimal Scope and Alliance Size)**.** *Suppose the equilibrium and social planner's search scopes are interior. Then, in any alliance, an agent's equilibrium search scope is lower than that agent's search scope in the social planner's solution. Furthermore, in the welfare maximizing solution, each agent's search scope decreases as her alliance shrinks in size.*

The sequencing of alliances and their search duration also differ between the social planner's solution and the corresponding equilibrium. The next proposition provides some features of those.

Proposition 4 (Optimal Alliance Sequencing)**.** *The socially optimal sequence of alliances is deterministic. For any deterministic sequence of alliances $A_1,...,A_k$ exerting optimal search scopes, the socially optimal stopping boundaries are drawdown stopping boundaries. That is, for each alliance A_k, $g^{A_k}(M) = M - d_{A_k}$ with $d_{A_k} \in R_+$. Furthermore, the drawdown sizes $\{d_{A_k}\}$ exhibit a recursive structure: for any k,*

$$d_{A_k} = \frac{|A_k \setminus A_{k+1}|}{2\left(\frac{\sum_{i\in A_k} c_i(\sigma_i^{A_k})}{(\sigma^{A_k})^2} - \frac{\sum_{i\in A_{k+1}} c_i(\sigma_i^{A_{k+1}})}{(\sigma^{A_{k+1}})^2}\right)}.$$

Why does the social planner use drawdown stopping boundaries for various alliances? Intuitively, for any active alliance A_k, the social planner considers the marginal group of agents $A_k \setminus A_{k+1}$ whose search will be terminated next. The relevant marginal added cost per speed for that group is then

$$\frac{\sum_{i \in A_k} c_i(\sigma_i^{A_k})}{(\sigma^{A_k})^2} - \frac{\sum_{i \in A_{k+1}} c_i(\sigma_i^{A_{k+1}})}{(\sigma^{A_{k+1}})^2}.$$

Each of these agents would receive the established maximum once they depart, thereby generating a multiplier of $|A_k \setminus A_{k+1}|$ of the maximum in the social planner's objective. The resulting stopping boundary then emulates that of a single decision maker, as in Proposition 0, with scaled up returns to each maximum established when the alliance shrinks, and adjusted costs as above.

To glean some intuition into the deterministic nature of the sequence of alliances, suppose that the social planner, starting with some active alliance A, proceeds to either alliance A' or alliance A'', depending on the realized path of the underlying process, with $A', A'' \subset A$. Following our discussion above, both transitions—from A to A' and from A to A''—are associated with a drawdown stopping boundary, with drawdown sizes of d' and d'', respectively. If $d' < d''$, starting from alliance A, the social planner would always shrink the alliance to A' as the relevant stopping boundary would always be reached first. Similarly, if $d'' < d'$, the social planner would always reduce the alliance to A''. In other words, different drawdown stopping boundaries never cross one another, and so the path of alliances is deterministic.

Propositions 3 and 4 suggest that the general structure of efficient search is similar to that conducted in equilibrium. Agents depart the search process in a pre-specified order and do so using drawdown stopping boundaries. Furthermore, within each active alliance, search scopes are constant over time. Nonetheless, the optimal sequence of active alliances and their corresponding drawdown sizes and the search scopes chosen do not generally coincide with those prescribed by equilibrium.

Certainly, agents who search exert positive externalities on others searching. Naturally, then, the social planner exploits these externalities by extending the time individuals spend searching. In fact, the expressions derived for the optimal and equilibrium alliance drawdown sizes imply directly the following.

Corollary 3 (Longer Optimal Search)**.** *For any alliance A_k, the drawdown chosen by the social planner for that alliance is weakly larger than the equilibrium drawdown of the same alliance.*

The drawdown nature of the stopping boundaries allows us to write the resulting welfare of the social planner in terms of the optimal drawdowns and search scopes. Namely, suppose $A_1, \ldots, A_K$ is the optimal sequence of alliances with associated drawdown sizes $d_{A_1}, ..., d_{A_K}$. When search starts at $X_0 = M_0 = 0$, we have:

$$W(0,0,N) = \sum_{m=1}^{K} \left((d_{A_m})^2 - (d_{A_{m-1}})^2\right) \frac{\sum_{i \in A_m} c_i(\sigma_i^{A_m})}{(\sigma^{A_m})^2},$$

where we set $d_{A_0} = 0$ (for details, see the Online Appendix).

The results of this section provide some features of the optimal solution. However, they do not offer a general characterization of the optimal sequence of alliances, which is the result of a challenging combinatorial optimization problem—in principle, the planner needs to consider all possible exit patterns, corresponding to ordered partitions of the team. Although it is possible to solve this problem algorithmically, a sharper characterization requires more structure on the environment's fundamentals. In the next subsection, we impose such a structure and solve the social planner's problem completely, illustrating the optimal sequence of alliances and contrasting it with that emerging in equilibrium.

5.3 Optimal Team Search with Well-ordered Costs

Suppose, as in Section 4.2, that agents' cost functions are proportional to one another and point-wise ordered as follows:

$$c_1\beta_1 = c_2\beta_2 = \cdots = c_N\beta_N,$$

where $\beta_1 = 1 < \beta_2 < ... < \beta_N$.

We start by showing that the social planner uses a similar sequencing of active alliances to that used in equilibrium.

Lemma 1 (Optimal and Equilibrium Alliance Sequence)**.** *In the social planner's solution, agent i never terminates search before agent j if $i > j$. In particular, whenever* *agent i terminates search before agent j in equilibrium, the social planner terminates agent i's search either with, or before, agent j's.*

Intuitively, the social planner optimally terminates the search of agents with the highest search costs first, so agent 1's search is terminated no later than agent 2's search, which is terminated no later than agent 3's, etc. This mimics, "weakly," the order governed by equilibrium. Nonetheless, the social planner's sequencing need not echo that prescribed by equilibrium since clustered exits could differ dramatically, as we soon show.

It will be useful to introduce the following notation for our characterization of the socially optimal sequence of alliances. Let:

$$B_k = \{k, k+1, ..., N\} \quad \text{for all } k = 1, ..., N.$$

Lemma 1 and our equilibrium characterization implies that the optimal sequence of active alliances has to correspond to a subset of $\{B_k\}_{k=1}^N$. This already suggests the computational simplicity well-ordered costs allow. For instance, instead of considering $2^N - 1$ alliances that could conceivably be the last ones active, we need to consider only N.

For $B' \subsetneq B$, we denote by $d_{B \to B'}$ the socially optimal drawdown size associated with alliance B, when it is followed by alliance B', as described in Proposition 4. In particular, $d_{B \to \emptyset}$ denotes the optimal drawdown of an alliance B when it is last active alliance.

We are now ready to characterize the optimal sequence of alliances in this setting.

Proposition 5 (Optimal Alliance Sequence with Well-Ordered Costs)**.** *The optimal sequence of alliances is identified as follows:*

- *There is a unique maximizer of $\{d_{B_k \to \emptyset}\}_{k=1}^N$. Let $L_1 = \underset{k=1,...,N}{\arg\max} d_{B_k \to \emptyset}$. The last active alliance is B_{L_1}, with $L_1 \leq N$. If $L_1 = 1$, all agents optimally terminate their search at the same time. Otherwise,*
- *There is a unique maximizer of $\{d_{B_k \to B_{L_1}}\}_{k=1}^{L_1 - 1}$. Let $L_2 = \underset{k=1,...,L_1-1}{\arg\max} d_{B_k \to B_{L_1}}$. The penultimate active alliance is B_{L_2}, with $L_2 < L_1$. If $L_2 = 1$, there are optimally only two active alliances: B_1 followed by B_{L_1}. Otherwise,*

- *Proceed iteratively until reach L_n, where $L_n = 1$. The socially optimal order of alliances is given by $B_1, B_{L_{n-1}}, \dots, B_{L_1}$.*

Intuitively, the optimal sequence of alliances is constructed recursively as follows. Consider first the case in which an alliance's search is terminated jointly. That is, once search terminates for one of the alliance's members, it is terminated for all others as well. Our analysis in the previous section suggests that, restricted in this way, the social planner would optimally determine the stopping time using a drawdown stopping boundary. Naturally, any possible alliance would be associated with a different optimal drawdown as such. Higher drawdown sizes correspond alliances the planner would prefer to have searching for longer periods. It is therefore natural to suspect that the alliance corresponding to the highest such drawdown size is the last active alliance. Since we already determined that optimal search exits occur in "weak" order, with agent i never exiting after agent $i+1$, it suffices to consider drawdown sizes as such corresponding to each alliance B_k.[16] This allows us to determine the last active alliance chosen by the social planner, B_{L_1}.

Once B_{L_1} is identified, we can proceed similarly to identify the penultimate active alliance. Namely, we consider all plausible super-sets of B_{L_1} and find the optimal drawdown size were the social planner constrained to transition directly to B_{L_1}. The alliance generating the maximal such drawdown size is the one the planner would want to keep searching for the longest period of time, foreseeing her optimal utilization of the next alliance B_{L_1}. That is the penultimate alliance. We continue recursively until reaching the maximal active alliance B_1.

5.4 Comparing Exit Waves in an Exponential World

In order to illustrate that contrast between the structure of equilibrium and socially optimal exit waves, we now consider a particular example. Suppose the team is comprised of three agents, $N = 3$, and assume cost functions are exponential and well ordered:

$$c(\sigma) = c_1(\sigma) = e^{b\sigma} = \beta_2 c_2(\sigma) = \beta_3 c_3(\sigma),$$

[16] This simplifies the computation problem substantially. Instead of considering $2^N - 1$ alliances, we need to consider only N.

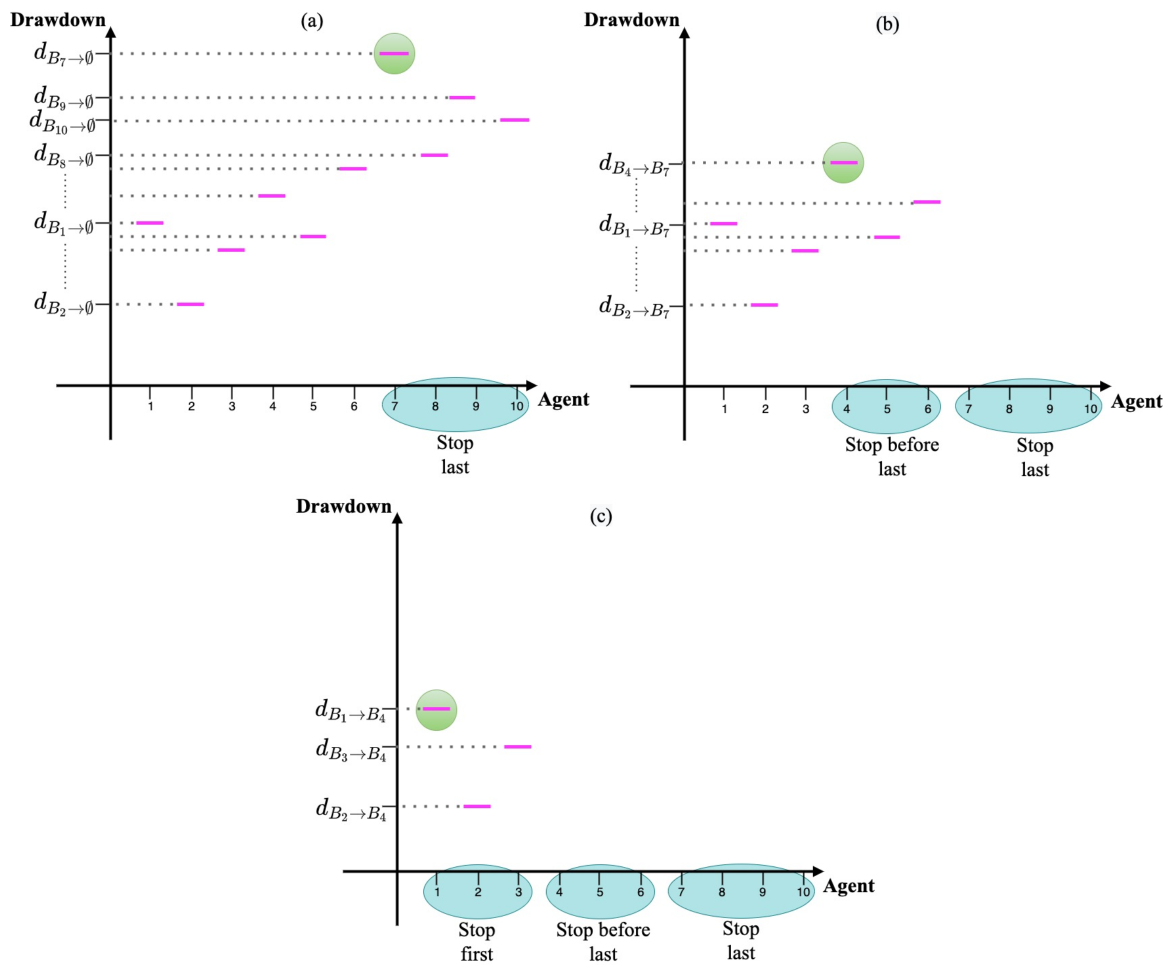

FIGURE 2: Socially optimal exit waves

where $1 < \beta_2 < \beta_3$ and $b > 0$. As we show in the Online Appendix, the characterization of the equilibrium and optimal exit waves is particularly simple in this case. In this case, there are four possible structures for exit waves: all agents can leave at once; agent 1 might leave first, followed the clustered exit of the lower-cost agents 2 and 3; agents 1 and 2 might leave together, followed by agent 3; or agents may exit at different points.

Figure 3 focuses on the case in which the social planner would cluster all agents' exits when $b = 1$ (each tick on the axes correspond to one unit of the corresponding multiplier, so that both β_2 and β_3 range from 0 to 24). The figure depicts the different regions of β_2 and β_3 combinations that generate the four possible structure of equilibrium exit waves.[17]

When the cost multipliers are sufficiently close to one another, agents exit in unison even in equilibrium. When β_2 is sufficiently close to 1, but β_3 is sufficiently higher, agent 3 has substantially lower search costs. Since agents 1 and 2 do not internalize their externalities on agent 3, they prefer to leave early on, generating two exit waves. Similarly, when β_2 and β_3 are sufficiently high but close to one another, agents 2 and 3 have similar search costs, which are substantially lower than those of agent 1. Again, agent 1, who does not internalize her externalities, prefers to exit earlier and two exit waves occur in equilibrium. Last, when agents' costs are sufficiently different, equilibrium dictates agents exiting at different points, resulting in three exit waves, even when externalities are sufficiently strong so that the social planner would prefer to have the agents search together till they all exit. Naturally, for sufficiently high β_2 and β_3, the wedge in costs is big and even the social planner would prefer to split agents' exits. The Online Appendix contains similar figures for other exit-wave structures chosen by the social planner.

6 Conclusions and Discussion

This paper analyzes team search patterns. We show that the equilibrium and socially optimal search scopes are constant within an alliance. However, as alliance members depart, individual search scopes increase in equilibrium and decrease

[17] Since $\beta_3 > \beta_2$, all regions are above the gray 45 degree diagonal line. We use $\{1,2,3\}$ to denote one clustered exit wave including all agents; $\{1,2\},\{3\}$ to denote an exit wave consisting of agents 1 and 2, followed by the exit of agent 3; and so on.

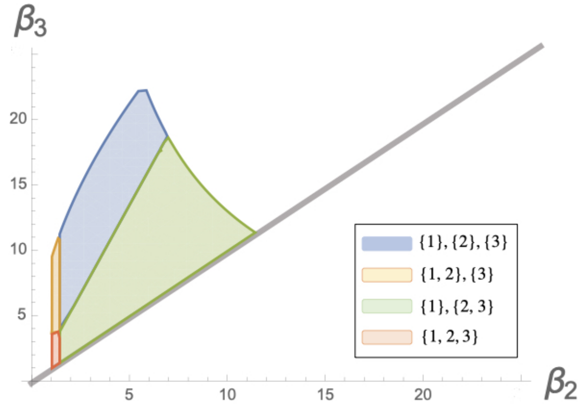

FIGURE 3: Equilibrium exit wave patterns when the optimal policy entails one exit wave including all three agents

under the optimal policy. We also characterize the deterministic path of exit waves generated in equilibrium. In particular, even when team member are fully heterogeneous, clustered exits may occur. The optimal path of exit waves shares some features with the equilibrium path in terms of the structrure of stopping boundaries that govern departures. However, search externalities naturally prolong optimal search in teams.

In this section, we consider several extensions of our model. We illustrate the impacts of explicit rewards for innovating early, the limitations introduced by a fixed search scope that cannot be altered, and the implications of our model to settings in which search observations are independent over time.

6.1 Equilibrium with Penalties for Later Innovations

In this section we consider an extension of our model where stopping earlier grants one an advantage. For example, a firm that produces the first product of its type might capture a market segment that is later more challenging to capture. Similarly, researchers arguably get additional credit for being the first to suggest a modeling framework or a measurement technique. The presence of such "first-mover advantages" introduces some known effects from the industrial organization literature. As it turns out, their consideration does not appear to qualitatively alter some of our results or the techniques needed to derive them.

For simplicity, we consider here a team of only two agents. We assume that the first agent to stop, say at time t, receives M_t. The second agent to stop, say at time $s > t$, receives αM_s, with $\alpha \leq 1$. If both agents stop at the same time t, they both receive M_t.[18] As we show in the Online Appendix, the order of exits will again be deterministic. Furthermore, as long as both agents are searching jointly, the search scope and the initial stopping boundary is identical to the case analyzed so far, where $\alpha = 1$. Thus, if there is a unique exit wave when $\alpha = 1$, that is still the case when $\alpha < 1$.

Suppose there are two distinct exit waves with $\alpha = 1$. Then, there is a *leader*—the agent who exits early—and a *follower*—the agent who exits later. The leader's stopping boundary $g_L(\cdot)$ remains her equilibrium stopping boundary regardless of α and is governed by the drawdown identified in Proposition 2. Both the leader and the follower's search scopes, when searching together or separately, also follow identical considerations to those pertaining to the $\alpha = 1$ case and given by Proposition 1. In contrast, the follower's stopping boundary does change since her rewards are scaled down by α.

In order to characterize the follower's stopping boundary, denote the costs of the leader by $c_L(\cdot)$ and those of the follower by $c_F(\cdot)$. Let σ_L denote the leader's search scope when searching within the full alliance, σ_T denote the total search scope in that alliance, the full team, and σ_F denote the follower's optimal solo search scope. Similar calculations to those underlying Proposition 2 yield the follower's stopping boundary $g_F(\cdot)$:

$$g_F(M) = \begin{cases} M - \frac{\alpha\sigma_F^2}{2c_F(\sigma_F)} \text{ if } M < \overline{M} \text{ and } \frac{\alpha\sigma_F^2}{2c_F(\sigma_F)} > \frac{\sigma_T^2}{2c_L(\sigma_L)}, \\ g_L(M) \text{ otherwise,} \end{cases}$$

where

$$\overline{M} = \frac{1}{1-\alpha}\frac{c_F(\sigma_F)}{\sigma_F^2}\left(\frac{\alpha\sigma_F^2}{2c_F(\sigma_F)} - \frac{\sigma_T^2}{2c_L(\sigma_L)}\right)^2.$$

[18]The analysis naturally extends to N agents via a decreasing sequence of discounts: $\alpha_0 = 1 \geq \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_N$. In addition, one could consider more continuous version of this setup, where the second agent who stops at time $s > t$ receives $M_t + \alpha(M_s - M_t)$. That model generates qualitatively similar results, but is more cumbersome to analyze.

To glean some intuition, consider the follower's problem after the leader's departure. The follower faces a similar problem to the individual agent's problem, with identical search costs and rewards scaled down by α. This case falls within the analysis in Urgun and Yariv (2020). As mentioned, the search scope is unaffected by the attenuation rewards, but the drawdown size is scaled linearly by α—as α declines, the rewards from search become less meaningful, and the follower ceases search more willingly. Naturally, for sufficiently low α, search continuation would not be worthwhile altogether, regardless of the maximal observation achieved when the leader exits. That corresponds to the drawdown used by the follower alone, $\frac{\alpha\sigma_F^2}{2c_F(\sigma_F)}$, being smaller, or even more demanding, than the full alliance's drawdown, $\frac{\sigma_T^2}{2c_L(\sigma_L)}$. In that case, the stopping boundary of the leader governs the exit of both. In addition, when the maximal observation M achieved when the leader exits is high enough, the loss from leaving at a later point, $(1-\alpha)M$ is substantial for any $\alpha < 1$.[19] For sufficiently high M, search continuation would again not be profitable. Furthermore, as α increases, the threshold level $\overline{M}$ increases. To summarize, for the follower to continue search after the leader, α needs to be sufficiently high and the current maximum sufficiently small, the restrictions seen in the formulation above.

Importantly, when later innovations are penalized, there are no preemption motives. The main impact is on later innovators, who face weakened incentives to search. In particular, larger exit waves occur for a larger set of parameters. In particular, the main messages of the paper extend directly to such settings.

Certainly, one could consider more elaborate models of competition between innovators: allowing for non-Markovian strategies, time-dependent first-mover advantages, and so on. We view these as interesting directions for future research.

6.2 Fixed Scope

Throughout the paper, we assume that search scope is controlled and can be adjusted as active alliances shrink. When scope is not controlled, which is the case when $\underline{\sigma} = \bar{\sigma} = \sigma$, such adjustments are not possible. In this case, agents left to their own devices as well as the social planner can choose only the time at which they

[19]Specifically, the gain from continuation for the follower is given by $(1-\alpha)M + (d_L - d_F)^2 \frac{c_F}{\sigma_F^2}$, where d_L and d_F are the drawdown sizes for the leader and the follower, respectively.

stop.

For illustration, consider the case of two agents, $N = 2$, and the case of well ordered costs, where $c \equiv c_1 = c_2\beta$.

Suppose first that σ is fixed at the optimal solo search scope, so that $\frac{2c(\sigma)}{c'(\sigma)} = \sigma$. In this case, agents would like to reduce their search scope when starting their search as a team, but are unable to do so. Consequently, the equilibrium drawdown size of the full alliance of both players decreases relative to that characterized in Proposition 1. In particular, the full alliance is more demanding for its search continuation and shrinks within a shorter expected time.

As another polar case, suppose σ is fixed at the optimal search scope for the full alliance. In this case, if the equilibrium prescribes simultaneous exit of both agents, the restriction to a fixed σ has no bearing and search outcomes coincide with those resulting in the case of unconstrained search scope. However, when agent 2 continues her search after agent 1 terminates her search, she cannot adjust her search scope upward as she desires according to Proposition 1. Since she is forced to use a sub-optimal search scope, her optimal drawdown size declines relative to one she would set were search scope unconstrained. That is, agent 2's solo search time *goes down* in expectation.

If a social planner can control which fixed search scope the agents use, which should be used? Suppose costs are close enough so that equilibrium would prescribe simultaneous exit by both agents; Namely, $\beta \leq 4$. As we show in the Online Appendix, welfare maximizing fixed scope is the one corresponding to either agent's optimal *solo* search scope.

6.3 Independent Samples

Our study is motivated by many applications in which there is inherent correlation in the search process—research and policy progress frequently builds on prior discoveries. Nonetheless, it may seem natural to consider the problem of team search even when observations are independent. While some of the methods we use carry over, it turns out sample independence introduces some important complications.

For simplicity, consider a discrete-time model in which, at each period, an independent sample is drawn, its costs associated with its variance as we have assumed throughout. The single-agent version of this model is analyzed in Urgun and Yariv

(2021). In such a setting, recall plays no role: since the environment is stationary, if new maximal rewards are not collected when they are first observed, there is no reason for agents to return and collect them in future periods. Exit then occurs when observations are high enough. Namely, within an alliance, each agent is associated with a threshold such that any observation exceeding that threshold would lead her to reap those rewards and terminate her search. That is the case both in equilibrium and when a social planner governs the search. As a consequence, the path of alliances entails inherent randomness: with some probability, an alliance generates a sufficiently high observation that leads all members to exit; with some probability, the alliance generates an observation that leads only a subset to depart. While the techniques introduced in the paper are still valid, and the comparative statics we identify on search scopes still hold, the characterization of the optimal path of alliances becomes more challenging due to this randomness. For illustration, we include a formal treatment of two-member teams in the Online Appendix.

A Appendix

A.1 Proofs for Equilibrium Team Search

As mentioned, the proof of Proposition 0 follows directly from Proposition 2 of Urgun and Yariv (2020).

First we note a useful lemma, commonly known as "reflection on the diagonal". This lemma allows us to omit the partial derivatives pertaining to M in the control problem in the various Hamilton-Jacobi-Bellman(HJB) equations that we will derive. Proofs of this result can be found in various sources, including Dubins, Shepp, and Shiryaev (1994), Urgun and Yariv (2020) and Peskir (1998) among others and hence omitted.

Lemma A.1. *The infinitesimal generator of the two dimensional process $Z = (M, X)$ satisfies the following:*

1. *If $M_t > X_t$, then $\mathcal{A}_Z^{\sigma_t} = \mathcal{A}_X^{\sigma_t} = \frac{1}{2}(\sigma_t)^2 \frac{\partial^2}{\partial X^2}$.*
2. *If $M_t = X_t$, then $\frac{\partial V}{\partial M} = 0$.*

Proof of Proposition 1. For any agent i in an alliance A, the value function takes the following form

$$V_i^A(M,X) = \mathbb{E}\left[\int_0^{\tau^A} c_i(\sigma_{i,t}|M,X)dt + \mathbb{E}\left[V_i^A(M_{\tau^A}, g^A(M_{\tau^A})\right]\right], \tag{2}$$

where

$$\mathbb{E}\left[V_i^A(M_{\tau^A}, g^A(M_{\tau^A})\right] = \begin{cases} M_{\tau^A} \text{ if } g_i^A(M_{\tau^A}) = g^A(M_{\tau^A}) \\ V_i^{A\setminus\{j\}} \text{ if } g_j^A(M_{\tau^A}) = g^A(M_{\tau^A}) > g_i^A(M_{\tau^A}), j \in A \end{cases}.$$

In words, with Markov strategies, agent i's expected value is derived from two components: the cost accrued until her alliance shrinks, and the continuation value once that happens. If the alliance shrinks with agent i's departure, her continuation value is simply the maximum value when she exits.

In order to proceed we first need the following lemma.

Lemma A.2. *When agents use Markov strategies, in any equilibrium, the total search scope is Markov with respect to X, but is independent of M. That is, in any equilibrium, for any alliance A, there exists $\sigma^A(\cdot)$ such that $\sigma_t^A(M,X) = \sum_{i\in A}\sigma_i^A(M,X)$ is path-wise equivalent to $\sigma^A(X)$.*[20]

Proof of Lemma A.2. Suppose each agent i uses a Markov strategy $\sigma_i^A(M,X)$ for all alliances A she might belong to. The resulting equilibrium search scope when alliance A is active is given by $\sigma^A(M_t,X_t) = \sum_{i\in A}\sigma_i^A(M_t,X_t)$. By definition, Markov controls are adapted. Furthermore, the full path of values implies the maximal recorded values at any point. Therefore, it is without loss of generality to represent the equilibrium control as $\sigma^A(X_{[0,t]})$.

Consider any time t before $\tau^{\{1,\dots,N\}}$ is realized in equilibrium. We can describe the path of values as the following Ito process:

$$X_t = \int_0^t \sigma^{\{1,\dots,N\}}(M_s,X_s)dB_s.$$

Since we have assumed that each individual scope is bounded, by Theorem 3 part

[20]That is, the diffusion defined by $dX_t = \sigma_t^A(M_t,X_t)dB_t$ follows the same probability law as that defined by $dY_t = \sigma^A(Y_t)dB_t$.

b of Krylov (1973) there exists an equivalent process of values with non-random coefficients satisfying the following stochastic differential equation (SDE) that has the same probability distribution at every t,[21]

$$dX_t = \sigma^{\{1,\dots,N\}}(X_t)dB_t.$$

Moreover, given that all agents are using Markov stopping times, the total alliance stopping times are Markov as well. Without loss of generality, let A denote the first alliance after τ^N (in principle, different paths may yield different alliances that follow the full one). Then, by the strong Markov property, for any $\tau^N < \tilde{t} < \tau^A$, we have

$$X_{\tilde{t}} = X_{\tau^{\{1,\dots,N\}}} + \int_{\tau^N}^{\tilde{t}} \sigma_s^A dB_s.$$

Then again, by Krylov (1973), there is an equivalent process, starting from X_τ^N that has non-random coefficients that has the same distribution as

$$dX_t = \sigma^A(X_t)dB_t.$$

Clearly, using strong Markov property in a similar fashion, we can continue for any alliance until the last agent stops. Thus, for any alliance A, there is an equivalent non-random alliance search scope, which depends on only X. ■

Given the Markov structure of the problem, the HJB equation for any agent i in an active alliance A before that alliance gets smaller, with current values of M and X, takes the following form:

$$\sup_{\sigma_i}\left[\frac{1}{2}\left(\sigma_i + \sum_{j\neq i}\sigma_j^A(M,X)\right)^2 \frac{\partial^2 V_i^A(M,X)}{\partial X^2} - c_i(\sigma_i)\right].$$

[21] This results shows the existence of such a process. In fact, one can extend the result of Gyöngy (1986) utilizing time independence of our value process to achieve an explicit unique selection.

The corresponding first-order condition then yields

$$\left(\sigma_i + \sum_{j\neq i} \sigma_j^A(M,X)\right) \frac{\partial^2 V_i^A(M,X)}{\partial X^2} = c_i'(\sigma_i).$$

The following lemma helps complete the proof.

Lemma A.3. *For each agent i, the HJB equation satisfies the following*

$$\frac{\partial^2 V_i^A(M,X)}{\partial X^2} = \frac{2\mathbb{E}\left[c_i(\sigma_i^A(M,X))\right]}{(\sigma^A(X))^2}.$$

Proof of Lemma A.3. As already shown, X_t can be written as the solution to an SDE of the following form:

$$dX_t = \sigma^A(X_t)dB_t,$$

where A is the active alliance at time t.

Let $g(M)$ be a continuous function and $\tau_g = \inf\{t \geq 0 : X_t \leq g(M_t)\}$. From Pedersen and Peškir (1998) we know that

$$\begin{aligned}\mathbb{E}\left[\int_0^{\tau_g} 1dt|M,X\right] =& 2\frac{M-X}{M-g(M)}\int_{g(M)}^{X} \frac{(x-g(M))}{\left(\sigma^A(x)\right)^2}dx + 2\frac{X-g(M)}{M-g(M)}\int_X^M \frac{(M-x)}{\left(\sigma^A(x)\right)^2}dx \\ &+ 2\frac{X-g(M)}{M-g(M)}\int_M^\infty \frac{1}{x-g(x)}\left(\int_{g(x)}^{x} \frac{(y-g(x))}{\left(\sigma^A(y)\right)^2}dy\right)e^{-\int_M^x \frac{1}{y-g(y)}dy}dx.\end{aligned}$$

Suppose $\sigma_i^A(M,X)$ varies non-trivially with M. As defined, $\sigma^A(\cdot)$ is independent of M. For any agent i in active alliance A, starting from any time with observed M and X, consider the following "sped up" process, measurable with respect to the original filtration

$$d\tilde{X}_t^i = \frac{\sigma^A(X_t)}{\sqrt{c_i(\sigma_i^A(M_t,X_t))}}dB_t.$$

Let $\tilde{M}_t^i$ denote the maximal observed value for this diffusion. Utilizing Krylov (1973) again, there is an equivalent "sped up" process $dX_t^i = \tilde{\sigma}_i^A(X_t)dB_t$ that has equivalent paths to $\tilde{X}_t^i$. Thus, from agent i's perspective, the expected time for alliance A to shrink in size can be calculated as follows:

$$\mathbb{E}_{\tilde{X}^i}\left[\int_0^{\tau^A} 1dt|\tilde{M}^i = M, \tilde{X}^i = X\right] = 2\frac{M-X}{M-g(M)}\int_{g(M)}^{X} \frac{(x-g(M))}{(\tilde{\sigma}_i^A(x))^2}dx + 2\frac{X-g(M)}{M-g(M)}\int_X^M \frac{(M-x)}{(\tilde{\sigma}_i^A(x))^2}dx$$
$$+2\frac{X-g(M)}{M-g(M)}\int_M^{\infty}\frac{1}{x-g(x)}\left(\int_{g(x)}^{x}\frac{(y-g(x))}{(\tilde{\sigma}_i^A(y))^2}dy\right)e^{-\int_M^x \frac{1}{y-g(y)}dy}dx.$$

Differentiating the right-hand side and substituting for $\tilde{\sigma}_i^A$ then implies:

$$\frac{\partial^2 \mathbb{E}_{\tilde{X}^i}\left[\int_0^{\tau^A} 1dt|\tilde{M}^i = M, \tilde{X}^i = X\right]}{\partial X^2} = \frac{2\mathbb{E}\left[c_i(\sigma_i^A(M,X))\right]}{(\sigma^A(X))^2}.$$

This expected time is equal to the costs accumulated on the original process for alliance A to reduce in size:

$$\mathbb{E}_{\tilde{X}^i}\left[\int_0^{\tau^A} 1dt|\tilde{M}^i = M, \tilde{X}^i = X\right] = \mathbb{E}\left[\int_0^{\tau^A} c_i(\sigma_i(M_t,X_t))dt|M,X\right].$$

Since τ^A is a Markov stopping time, by the strong Markov property, differentiating (2) with respect to X leads to the following:

$$\frac{\partial^2 V_i^A(M,X)}{\partial X^2} = \frac{\partial^2 \mathbb{E}\left[\int_0^{\tau^A} c_i(\sigma_i^A(M_t,X_t))dt|M,X\right]}{\partial X^2}.$$

The result then follows. ■

Using the identity in Lemma A.3 and the HJB of each agent i in alliance A yields the following constraint at any time t:

$$\frac{2\mathbb{E}\left[c_i(\sigma_i^A(M_t,X_t)|X_t = X)\right]}{c_i'(\sigma_i^A(M_t,X_t))} = \sigma_i^A(M_t,X_t) + \sum_{j\neq i}\sigma_j^A(M_t,X_t).$$

Since c_i is log-convex, the left-hand side is strictly decreasing in $\sigma_i^A(M_t,X_t)$. As we established above, for the solution $\sigma_i^A(M,X)$, we have $\sum_{j\in A}\sigma_j^A(M,X) = \sigma^A(X)$ so that the right-hand side is independent of of M_t. It follows that $\sigma_i^A(M_t,X_t)$ is independent of M_t.

The fact that the left-hand side is strictly decreasing implies that the dependence on X must be trivial for any alliance of at least two individuals, while the independence of a singleton alliance follows from Urgun and Yariv (2020). The system of constraints then simplifies to:

$$\frac{2c_i(\sigma_i^A)}{c_i'(\sigma_i^A)} = \frac{2c_j(\sigma_j^A)}{c_j'(\sigma_j^A)} = \ldots = \sum_{i \in A} \sigma_i^A.$$

■

Proof of Corollary 1. First, we show as the alliances shrink, total search scope cannot increase. Suppose not and assume there exists an alliance A and an agent i such that $A \cup \{i\}$ generates lower overall search scope compared to that generated by alliance A. That is, $\sigma^A > \sigma^{A \cup \{i\}}$. From uniqueness of interior solutions, for each agent $j \in A$, we must have

$$\frac{2c_j(\sigma_j^A)}{c_j'(\sigma_j^A)} = \sigma^A \quad \text{and} \quad \frac{2c_j(\sigma_j^{A \cup \{i\}})}{c_j'(\sigma_j^{A \cup \{i\}})} = \sigma^{A \cup \{i\}}.$$

Since $\sigma^A > \sigma^{A \cup \{i\}}$ and each of the cost functions is log-convex, it must be the case that $\sigma_j^A \le \sigma_j^{A \cup \{i\}}$. Thus,

$$\sum_{j \in A} \sigma_j^A \le \sum_{j \in A} \sigma_j^{A \cup \{i\}} < \sum_{j \in A} \sigma_j^{A \cup \{i\}} + \underline{\sigma} \le \sigma^{A \cup \{i\}},$$

in contradiction.

The comparative statics pertaining to individuals' search scope follows immediately from log-convexity of the cost functions. ■

Proof of Proposition 2. The statement of Proposition 2 is a combination of the following claims.

Claim A.1. *For any given alliance A with $i \in A$, if $g_i^A(M^*) = \max_{j \in A} g_j^A(M^*)$ for some M^*, then $g_i^A(M) = \max_{j \in A} g_j^A(M)$ for all M.*

Proof of Claim. The proof of the claim relies on the following lemma.

Lemma A.4. *Suppose agent $i \in A$ has the highest stopping boundary at a given observed M, X. Then $g_i^A(M)$ is a drawdown stopping boundary.*

Proof of Lemma A.4. Suppose $\max_{j\in A} g_j^A(M) = g_i^A(M)$. When observing M, consider any value X such that $g_i^A(M) \leq X \leq M$. One of two events may occur for the stopping threshold to change. First, a new maximal value $M' > M$ could be observed. Alternatively, a sufficiently low value $X \leq g_i^A(M)$ could be observed so that agent i leaves the alliance.

The Green function on the interval $[a, b]$ is defined as follows:

$$G_{a,b}(x,y) = \begin{cases} \frac{(S(b)-S(x))(S(y)-S(a))}{S(b)-S(a)} & \text{if } a < y < x < b \\ \frac{(S(b)-S(y))(S(x)-S(a))}{S(b)-S(a)} & \text{if } a < x < y < b \end{cases}$$
$$= \begin{cases} \frac{(b-x)(y-a)}{b-a} & \text{if } a < y < x < b \\ \frac{(b-y)(x-a)}{b-a} & \text{if } a < x < y < b \end{cases},$$

where $S(\cdot)$ denotes the scale function of a diffusion. Since our process has no drift, $S(x) = x$.

Following standard techniques (for details, see Urgun and Yariv (2020)), we can write the equilibrium value function of agent i in the following recursive fashion:

$$V_i^A(M,X) = M\frac{M-X}{M-g_i^A(M)} + V_i^A(M,M)\frac{X-g_i^A(M)}{M-g_i(M)} - \int_{g_i^A(M)}^M G_{g_i^A(M),M}(x,y)c_i(\sigma_i^A(y))\frac{2}{(\sigma^A(y))^2}dy.$$

Rearranging terms, we get:

$$V_i^A(M,M) - M = \frac{M-g_i^A(M)}{X-g_i^A(M)}\left[V_i^A(M,X) - M + \int_{g_i^A(M)}^M G_{g_i^A(M),M}(x,y)c_i(\sigma_i^A(y))\frac{2}{(\sigma^A(y))^2}dy\right].$$

Since agent i is optimally terminates her search at $g_i^A(M)$, smooth pasting must hold at $g_i^A(M)$. The derivative of the continuation value as $X \to g_i^A(M)$ can be written as $\lim_{X\to g_i^A(M)} \frac{V_i^A(M,X)-M}{X-g_i^A(M)}$. By smooth pasting, this must equal the derivative of the value from stopping, $\frac{\partial}{\partial x}M = 0$.

Consider then the above equality for $V_i^A(M,M)$. Taking the limit as $X \to g_i^A(M)$, we get

$$V_i^A(M,M) = M + \int_{g_i(M)}^M (M-y)c_i(\sigma_i^A(y))\frac{2}{(\sigma^A(y))^2}dy.$$

This, in turn, implies that

$$V_i^A(M,X) = M + \int_{g_i^A(M)}^{X} (X-y)c_i(\sigma_i^A(y))\frac{2}{(\sigma^A(y))^2}dy.$$

Now differentiating $V_i^A(M,X)$ with respect to M and evaluating the derivative at $X = M$ yields the following ordinary differential equation (ODE) for $g_i^A(M)$:

$$g_i^A(M)' = \frac{(\sigma^A)^2}{2c_i(\sigma_i^A)\big(M - g_i^A(M)\big)},$$

which leads to the following solution:

$$g_i^A(M) = M - \frac{(\sigma^A)^2}{2c_i(\sigma_i^A)}.$$

This is a drawdown stopping boundary with drawdown size $d_i^A := \frac{(\sigma^A)^2}{2c_i(\sigma_i^A)}$. ■

We can now proceed with the claim's proof. Towards a contradiction, suppose that $\max_{j\in A} g_j^A(M) = g_i^A(M)$ for some M. Suppose $M' = \inf_{\hat{M}>M}\{M | i \notin \arg\max_{j\in A} g_j^A(\hat{M})\}$ and suppose that for some $\varepsilon > 0$, for any $\hat{M} \in (M', M'+\varepsilon)$, for some $k \neq i$, we have $\max_{j\in A} g_j^A(M') = g_k^A(M') > g_i^A(M')$. From continuity of the stopping boundary and Lemma A.4 we reach,

$$g_i^A(M) = M - \frac{(\sigma^A)^2}{2c_i(\sigma_i^A)} \quad \text{and} \quad g_k^A(M') = M' - \frac{(\sigma^A)^2}{2c_k(\sigma_k^A)}.$$

From our choice of i and k we must have that $\frac{(\sigma^A)^2}{2c_i(\sigma_i^A)} \leq \frac{(\sigma^A)^2}{2c_k(\sigma_k^A)}$ and $\frac{(\sigma^A)^2}{2c_i(\sigma_i^A)} > \frac{(\sigma^A)^2}{2c_k(\sigma_k^A)}$, in contradiction. ■

Claim A.2. *Suppose that for some i in an active alliance of A, $\frac{(\sigma^A)^2}{2c_i(\sigma_i^A)} \leq \frac{(\sigma^A)^2}{2c_j(\sigma_j^A)}$ for all $j \in A$. Then i is the first to exit alliance A.*[22]

Proof of Claim. Suppose $\frac{(\sigma^A)^2}{2c_i(\sigma_i^A)} \leq \frac{(\sigma^A)^2}{2c_j(\sigma_j^A)}$ for all $j \in A$ but that agent i is not one of the first agents to exit from alliance A for some path of observed values. For that

[22] If there are multiple agents who satisfy the condition, all exhibiting the same drawdown size, they all exit jointly, weakly before others.

path, agent i ceases her search when active at a smaller alliance $A \setminus K$. Without loss of generality, suppose agent j exits alliance A first (if there are multiple such agents, we can pick any), when observing M and X. From Lemma A.4, agent j's stopping boundary is characterized by a drawdown. However, from the Claim's restriction,

$$M - \frac{(\sigma^A)^2}{2c_j(\sigma_j^A)} \leq M - \frac{(\sigma^A)^2}{2c_i(\sigma_i^A)}.$$

In order for agent i to be stopping first when in alliance $A \setminus K$, observing M' and X', but not in alliance A, it follows from the claim above that

$$M' - \frac{(\sigma^{A\setminus K})^2}{2c_i(\sigma_i^{A\setminus K})} < M' - \frac{(\sigma^A)^2}{2c_j(\sigma_j^A)}.$$

By Corollary 1, $\sigma_i^{A\setminus K} \geq \sigma_i^A$. Since search costs are strictly increasing, this implies that $c_i(\sigma_i^{A\setminus K}) \geq c_i(\sigma_i^A)$. This implies that

$$M - \frac{(\sigma^{A\setminus K})^2}{2c_i(\sigma_i^{A\setminus K})} > M - \frac{(\sigma^A)^2}{2c_i(\sigma_i^A)},$$

a contradiction. ∎

Combining the two claims and the characterization in the Lemma A.4 leads to the statement of the proposition.

∎

A.2 Proofs for the Social Planner's Solution

Proof of Proposition 3. Let $\left\{\sigma_i^A(M,X,A)\right\}$ and $G(M,X,A)$ correspond to a solution to the social planner's problem. Consider any alliance A_k at some observed values and let A_{k+1} denote the potentially empty random alliance dictated by this optimal solution. Optimality implies that the induced search scopes with A_k should solve:

$$\sup_{\{\sigma_{i,t}\}_{i\in A_k}} \mathbb{E}\left[|A_k \setminus A_{k+1}| M_{\tau^{A_k}} - \int_0^{\tau^{A_k}} \sum_{i\in A_k} c_i(\sigma_{i,t}^{A_k}) dt\right].$$

As in our equilibrium analysis, we can use Krylov (1973) to conclude that the timed overall search scope within alliance A_k is Markovian and depends non-trivially only on X. We therefore write this scope as $\sigma^{A_k}(X)$. Since individual costs are convex, this implies that individual search scopes within A_k are also Markovian and depend non-trivially only on X. We write these as $\left\{\sigma_i^{A_k}(X)\right\}_{i\in A_k}$. This further implies that the overall search costs within alliance A_k, given by $\sum_{i\in A_k} c_i(\sigma_i^{A_k}(X))$, can also depend non-trivially only on X. As in the proof of Lemma A.3, we can "speed up" the process by $\frac{\sigma^{A_k}(X)}{\sqrt{\sum_{i\in A_k} c_i(\sigma_i^{A_k})(X)}}$ to conclude that

$$\mathbb{E}\left[\int_0^{\tau^{A_k}} \sum_{i\in A_k} c_i(\sigma_{i,t}^{A_k})dt|M,X\right] = 2\frac{M-X}{M-g^{A_k}(M)}\int_{g^{A_k}(M)}^{X} \frac{(x-g^{A_k}(M))\sum_{i\in A_k} c_i(\sigma_i^{A_k})(x)}{(\sigma^{A_k}(x))^2}dx$$
$$+2\frac{X-g^{A_k}(M)}{M-g^{A_k}(M)}\int_X^M \frac{(M-x)\sum_{i\in A_k} c_i(\sigma_i^{A_k})(x)}{(\sigma^{A_k}(x))^2}dx$$
$$+2\frac{X-g^{A_k}(M)}{M-g^{A_k}(M)}\int_M^\infty \frac{1}{x-g^{A_k}(x)}\left(\int_{g^{A_k}(x)}^{x} \frac{(y-g^{A_k}(x))\sum_{i\in A_k} c_i(\sigma_i^{A_k})(y)}{(\sigma^{A_k}(y))^2}dy\right)e^{-\int_M^x \frac{1}{y-g^{A_k}(y)}dy}dx.$$

The continuation HJB for the social planner is given by:

$$\sup_{\{\sigma_i\}_{i\in A_k}}\left[\frac{1}{2}\left(\sum_{i\in A_k}\sigma_i^{A_k}(X)\right)^2\frac{\partial^2 W(M,X,A_k)}{\partial X^2}-\sum_{i\in A_k} c_i(\sigma_i^{A_k}(X))\right].$$

Again replacing the sup with the appropriate FOC we end up with the following system:

$$\left(\sum_{i\in A_k}\sigma_i^{A_k}(X)\right)\frac{\partial^2 W_k(M,X,A_k)}{\partial X^2} = c_i'(\sigma_i^{A_k}(X)) \quad \forall i\in A_k.$$

The expression for the continuation value described in the text implies:

$$\frac{\partial^2 W(M,X,A_k)}{\partial X^2} = \frac{\partial^2\mathbb{E}\left[\int_0^{\tau^{A_k}} \sum_{i\in A_k} c_i(\sigma_i^{A_k}(X))dt|M,X\right]}{\partial X^2}.$$

It follows that

$$\frac{2\sum_{i\in A_k} c_i(\sigma_i^{A_k}(X))}{\sum_{i\in A_k}\sigma_i^{A_k}(X)} = c_j'(\sigma_i^{A_k}(X)) \quad \forall j\in A_k.$$

From log-convexity of costs, whenever an interior solution exists, it is unique. In particular, the optimal search scopes are independent of the observed values and are constant over time for each active alliance.

■

The proof of Corollary 2 utilizes the proof of Proposition 4 and therefore follows it.

Proof of Proposition 4. The proof follows from two lemmas:

Lemma A.5. *If the set of agents dropping from an alliance is independent of the observed path, then each alliance has a stopping boundary identified by a drawdown size* d_{A_k}.

Proof of Lemma A.5. Let A_K be the final alliance in the social planner's problem, with cardinality $|A_K|$. The social planner's problem when left with alliance A_K, and observing maximum M and current value X, takes the following form:

$$W_K(M,X) = \sup_{\tau^K, \{\sigma_i\}_{i \in A_K}} \mathbb{E}\left[|A_K| M_{\tau^K} - \int_0^{\tau^K} \sum_{i \in A_K} c_i(\sigma_i^{A_K}) dt \mid M, X \right].$$

This is tantamount to a single-searcher problem, where search rewards are scaled by $|A_K|$. From Urgun and Yariv (2020), the stopping boundary is given by:

$$g^{A_K}(M) = M - d_{A_k},$$

where $d_{A_k} = \frac{|A_K| \sigma^{A_K}}{2 \sum_{i \in A_K} c_i(\sigma_i^{A_K})}$.

Consider the social planner's problem when the penultimate alliance A_{K-1} is active and the observed maximum and value are M and X, respectively:

$$W_{K-1}(M,X) = \sup_{\tau^{K-1}, \{\sigma_i\}_{i \in A_{K-1}}} \mathbb{E}\left[|A_{K-1} \setminus A_K| M_{\tau^{K-1}} + W_K(M_{\tau^{K-1}}, g^{A_{K-1}}(M_{\tau^{K-1}})) \mid M, X \right]$$
$$- \mathbb{E}\left[\int_0^{\tau^{K-1}} \sum_{i \in A_{K-1}} c_i(\sigma_i^{A_{K-1}}) dt \mid M, X \right].$$

By optimality of the stopping time τ^{K-1}, we have smooth pasting of $W_{K-1}(M,X)$

and $W_K(M,X)$. Therefore,

$$W_{K-1}(M, g^{A_{K-1}}(M)) = |A_{K-1} \setminus A_K|M + W_K(M, g^{A_{K-1}}(M)),$$
$$\frac{\partial W_{K-1}(M, g^{A_{K-1}}(M))}{\partial X}|_{X=g^{A_{k-1}}} = \frac{\partial(|A_{K-1} \setminus A_K|M + W_K(M, g^{A_{K-1}}(M)))}{\partial X}|_{X=g^{A_{K-1}}(M)}.$$

Similar to our equilibrium analysis, and using the notation for the Green function introduced there, we can write the welfare maximization problem as

$$W_{K-1}(M,X) = |A_{K-1} \setminus A_K|M + W_K(M, g^{A_{K-1}}(M))\frac{M-X}{M-g^{A_{K-1}}(M)}$$
$$+ W_{K-1}(M,M)\frac{X-g_{A^{K-1}}(M)}{M-g^{A_{K-1}}(M)} - \int_{g^{A_{K-1}}}^{M} G_{g^{A_{K-1}}(M),M}(x,y)\frac{2\sum_{i\in A_{K-1}} c_i(\sigma_{i,t}^{A_{K-1}})}{(\sigma^{A_{K-1}})^2}dy.$$

Letting X approach $g^{A_{K-1}}(M)$, smooth pasting and rearranging yields: [23]

$$W_{K-1}(M,X) = |A_{K-1} \setminus A_K|M + W_K(M, g^{A_{K-1}}(M))$$
$$+\Big(X-g^{A_{K-1}}(M)\Big)\int_{g^{A_K}(M)}^{g^{A_{K-1}}(M)} \frac{2\sum_{i\in A_K} c_i(\sigma_i^{A_K})}{(\sigma^{A_K})^2}dx + \int_{g^{A_{K-1}}(M)}^{X}(X-y)\frac{2\sum_{i\in A_{K-1}} c_i(\sigma_i^{A_{K-1}})}{(\sigma^{A_{K-1}})^2}dy,$$

Using the closed-form representation of W_K leads to:

$$W_{K-1}(M,X) = |A_{K-1}|M + \frac{1}{2}(g^{A_{K-1}}(M) - g^{A_K}(M))^2\frac{2\sum_{i\in A_K} c_i(\sigma_i^{A_K})}{(\sigma^{A_K})^2}$$
$$+\Big(X-g^{A_{K-1}}(M)\Big)(g^{A_{K-1}}(M) - g^{A_K}(M))\frac{2\sum_{i\in A_K} c_i(\sigma_i^{A_K})}{(\sigma^{A_K})^2}$$
$$+\frac{1}{2}(X-g^{A_{K-1}}(M))^2\frac{2\sum_{i\in A_{K-1}} c_i(\sigma_i^{A_{K-1}})}{(\sigma^{A_{K-1}})^2}.$$

To generate an ODE that identifies $g^{A_{K-1}}(M)$, we take the derivative with respect to M that, evaluated at $X = M$, should equal 0. After some algebraic manipulations,

[23]The equality follows from

$$\frac{\partial(|A_{K-1} \setminus A_K|M + W_K(M, g^{A_{K-1}}(M)))}{\partial X}|_{X=g^{A_{K-1}}(M)} = \Big(X-g^{A_{K-1}}(M)\Big)\int_{g^{A_K}(M)}^{g^{A_{K-1}}(M)} \frac{2\sum_{i\in A_K} c_i(\sigma_i^{A_K})}{(\sigma^{A_K})^2}dx.$$

this ODE take the form

$$\frac{dg^{A_{K-1}}(M)}{dM} = \frac{|A_{K-1} \setminus A_K|}{2\left(M - g^{A_{K-1}}(M)\right)\left(\frac{\sum_{i\in A_{K-1}} c_i(\sigma_i^{A_{K-1}})}{(\sigma^{A_{K-1}})^2} - \frac{\sum_{i\in A_K} c_i(\sigma_i^{A_K})}{(\sigma^{A_K})^2}\right)}.$$

It is straightforward to verify that the unique solution for this ODE satisfying the value-matching condition takes the form $g^{A_{K-1}}(M) = M - d_{A_{K-1}}$, where

$$d_{A_{K-1}} = \frac{|A_{K-1} \setminus A_K|}{2\left(\frac{\sum_{i\in A_{K-1}} c_i(\sigma_i^{A_{K-1}})}{(\sigma^{A_{K-1}})^2} - \frac{\sum_{i\in A_K} c_i(\sigma_i^{A_K})}{(\sigma^{A_K})^2}\right)}.$$

In particular, the optimal stopping boundary is a drawdown stopping boundary.

Proceeding inductively, for any alliance $m \le K$, the continuation value when M and X are observed can be written as:

$$\begin{aligned} W_m(M,X) =& |A_m \setminus A_{m+1}|M + W_{m+1}(M, g^{A_m}(M)) \\ &+ \left(X - g^{A_m}(M)\right)\sum_{k=m}^{K-1}\left(\int_{g^{k+1}(M)}^{g^k(M)} \frac{2\sum_{i\in A_{k+1}} c_i(\sigma_i^{A_{k+1}})}{(\sigma^{A_{k+1}})^2} dx\right) \\ &- \int_{g^{A_m}(M)}^{X} (X-y)\frac{2\sum_{i\in A_m} c_i(\sigma_i^{A_m})}{(\sigma^{A_m})^2} dy. \end{aligned}$$

We can then repeat the steps above to generate an analogous ODE for $g^{A_m}(M)$ and verify that it is uniquely identified as a drawdown stopping boundary. Namely, $g^{A_m}(M) = M - d_{A_m}$, where

$$d_{A_m} = \frac{|A_m \setminus A_{m+1}|}{2\left(\frac{\sum_{i\in A_m} c_i(\sigma_i^{A_m})}{(\sigma^{A_m})^2} - \frac{\sum_{i\in A_{m+1}} c_i(\sigma_i^{A_{m+1}})}{(\sigma^{A_{m+1}})^2}\right)}.$$

■

Lemma A.6. *The set of agents dropping from an alliance is deterministic. That is for any alliance A, for all pairs $(M,X),(M',X')$ such that $G(M,X,A) \neq A$, we have $G(M,X,A) = G(M',X',A)$.*

Proof of Lemma A.6. We prove this result by induction on the size of the initial team N, regardless of the starting values of the maximum and the project. The claim follows immediately for $N = 1$. In that case, the agent uses a drawdown stopping boundary and the only way for the singleton alliance to change is for the agent to terminate her search.

For the inductive step, assume that for any initial team of size $N-1$ or less the optimal alliance sequence is deterministic. By Lemma A.5, each of these alliances is associated with a drawdown stopping boundary. Let A_1 be an alliance of size N. The continuation value when M and X are observed is:

$$W_1(M,X) = \mathbb{E}\left[M_{\tau^1} + \max_{A_2 \subsetneq A_1}\{|A_1 \setminus A_2| M_{\tau^1} + W_{A_2}(M_{\tau^1}, g^{A_1}(M_{\tau^1}))\} - \int_0^{\tau^1} \sum_{i \in A_k} c_i(\sigma_i^{A_1}) dt\right].$$

Suppose that, for some path, the social planner optimally transitions from alliance A_1 to a strictly smaller alliance $A_2 \neq \emptyset$. In particular, alliance A_2 contains fewer than N agents. By the inductive hypothesis, the sequence that ensues is path independent. We can therefore write the continuation value as:

$$\begin{aligned} W_1(M,X) =& |A_1 \setminus A_2| M + W_2(M, g^{A_1}(M)) \\ &+ \left(X - g^{A_1}(M)\right) \sum_{m=2}^{K-1} \left(\int_{g^{A_{m+1}}(M)}^{g^{A_m}(M)} \frac{2\sum_{i \in A_{m+1}} c_i(\sigma_i^{A_{m+1}})}{(\sigma^{A_{m+1}})^2} dx \right) \\ &+ \int_{g^{A_1}(M)}^{X} (X - y) \frac{2\sum_{i \in A_1} c_i(\sigma_i^{A_1})}{(\sigma^{A_1})^2} dy. \end{aligned}$$

As before, this yields an ODE characterizing $g^{A_1}(M)$ and a unique solution of the form $g^{A_1}(M) = M - d_{A_1}$, where

$$d_{A_1} = \frac{|A_1 \setminus A_2|}{2\left(\frac{\sum_{i \in A_1} c_i(\sigma_i^{A_1})}{(\sigma^{A_1})^2} - \frac{\sum_{i \in A_2} c_i(\sigma_i^{A_2})}{(\sigma^{A_2})^2}\right)}.$$

Towards a contradiction, suppose that at some other path a different alliance is optimally chosen to follow the full alliance A_1. Call that alliance $\hat{A}_2 \neq A_2$. Similar argument would then imply that the stopping boundary for A_1 is given by

$g^{A_1}(M) = M - \hat{d}_{A_1}$, where

$$\hat{d}_{A_1} = \frac{|A_1 \setminus \hat{A}_2|}{2\left(\frac{\sum_{i\in A_1} c_i(\sigma_i^{A_1})}{(\sigma^{A_1})^2} - \frac{\sum_{i\in \hat{A}_2} c_i(\sigma_i^{\hat{A}_2})}{(\sigma^{\hat{A}_2})^2}\right)}.$$

Suppose $d_{A_1} = d_{\hat{A}_1}$. Without loss of generality, assume $\sigma^{A_2} \geq \sigma^{\hat{A}_2}$. We cannot have $\hat{A}_2 \subseteq A_2$. Indeed, if that were the case, the social planner would be indifferent between keeping the agents in $A_2 \setminus \hat{A}_2$ at points at which either $\hat{A}_2$ or A_2 are chosen to continue. However, our tie-breaking rule implies that such indifferences are broken in favor of stopping; that is, in such cases, the smaller set $\hat{A}_2$ would be chosen. Therefore, $\hat{A}_2 \nsubseteq A_2$ and there exists an agent $i \in \hat{A}_2 \setminus A_2$. Suppose that whenever the social planner transitions from A_1 to A_2, she instead transitions to $A_2 \cup \{i\}$, maintaining the search scopes of members of A_2 as before and having agent i search with the lowest scope $\underline{\sigma}$ for a sufficiently small interval of time. In that interval of time, everyone in A_2 benefits. When alliance $\hat{A}_2$ is picked, agent i uses at least as high a search scope, while benefitting from lower overall search scope. In particular, agent i benefits as well from this change.

Suppose $d_{A_1} \neq d_{\hat{A}_1}$. In this case, the two stopping boundaries identified above, $M - d_{A_1}$ and $M - d_{\hat{A}_1}$ never intersect, in contradiction. ■

Combining the two lemmas leads to the conclusion of the proposition. ■

Proof of Corollary 2. To prove the corollary, we introduce superscripts *eq* and *sp* to denote the equilibrium and social planner's solution, respectively (these are suppressed otherwise, when there is low risk of confusion). We use the following set of lemmas. For these, we assume interior equilibrium and social planner search scopes, as presumed in the corollary.

Lemma A.7. *Any active alliance A has a higher search scope under the social planner's solution compared to the equilibrium.*

Proof of Lemma A.7. Towards a contradiction, suppose there exists an alliance A such that $\sum_{i\in A} \sigma_i^{A,eq} > \sum_{i\in A} \sigma_i^{A,sp}$. The social planner's solution satisfies

$$\frac{2\sum_{i\in A} c_i(\sigma_i^{A,sp})}{c_j'(\sigma_j^{A,sp})} = \sum_{i\in A} \sigma_i^{A,sp} \quad \forall j \in A,$$

which implies that

$$\frac{2c_j(\sigma_j^{A,sp})}{c_j'(\sigma_j^{A,sp})} < \sum_{i\in A} \sigma_i^{A,sp} < \sum_{i\in A} \sigma_i^{A,eq} \quad \forall j \in A.$$

From log-convexity of costs, the left-hand side of this inequality increases as $\sigma_j^{A,sp}$ decreases. For the equilibrium constraint to hold, $\sigma_j^{A,eq} < \sigma_j^{A,sp}$ for all $j \in A$, in contradiction. ■

Lemma A.8. *In the social planner's solution, for any alliance A and any agent $i \notin A$, $\sigma^{A,sp} < \sigma^{A\cup\{i\},sp}$.*

Proof of Lemma A.8. Suppose not. Then, there exists an alliance A and an individual $i \notin A$ such that $\sigma^{A,sp} \geq \sigma^{A\cup\{i\},sp}$. Then, for all $l \in A$, we must have

$$\frac{2\sum_{j\in A} c_j(\sigma_j^{A,sp})}{c_l'(\sigma_l^{A,sp})} = \sigma^{A,sp} \quad \text{and} \quad \frac{2\sum_{j\in A\cup\{i\}} c_j(\sigma_j^{A\cup\{i\},sp})}{c_l'(\sigma_l^{A_k\cup\{i\},sp})} = \sigma^{A\cup\{i\},sp}.$$

Since search costs are strictly positive, the second equality implies that, for all $l \in A$,

$$\frac{2\sum_{j\in A} c_j(\sigma_j^{A\cup\{i\},sp})}{c_l'(\sigma_l^{A\cup\{i\},sp})} < \frac{2\sum_{j\in A\cup\{i\}} c_j(\sigma_j^{A\cup\{i\},sp})}{c_l'(\sigma_l^{A\cup\{i\},sp})} = \sigma^{A\cup\{i\},sp} \leq \sigma^{A,sp}.$$

Log-convexity of costs implies that the left-hand side of this inequality increases as $\sigma_l^{A\cup\{i\},sp}$ decreases. The social planner's constraint for alliance A then implies that $\sigma_l^{A,sp} < \sigma_l^{A\cup\{i\},sp}$ for all $l \in A$, in contradiction to the last inequality. ■

Lemma A.9. *In the social planner's solution if A_k and A_{k+1} are consecutive active alliances in the social planner's solution, then for any i in A_{k+1}, we have $\sigma_i^{A_k,sp} > \sigma_i^{A_{k+1},sp}$.*

Proof of Lemma A.9. From Proposition 4, if A_k and A_{k+1} are part of the optimal sequence, we have

$$\frac{\sum_{j\in A_k} c_j(\sigma_j^{A_k,sp})}{(\sigma^{A_k,sp})^2} - \frac{\sum_{j\in A_{k+1}} c_j(\sigma_j^{A_{k+1},sp})}{(\sigma^{A_{k+1},sp})^2} > 0.$$

Furthermore, from Proposition 3, for all $i \in A_{k+1}$,

$$\frac{2\sum_{j\in A_k} c_j(\sigma_j^{A_k,sp})}{c_i'(\sigma_i^{A_k,sp})} = \sigma^{A_k,sp} \quad \text{and} \quad \frac{2\sum_{j\in A_{k+1}} c_j(\sigma_j^{A_{k+1},sp})}{c_i'(\sigma_i^{A_{k+1},sp})} = \sigma^{A_{k+1},sp}.$$

Combining these yields, for all $i \in A_{k+1}$,

$$\frac{(c_i'(\sigma_i^{A_k,sp}))^2}{\sum_{j\in A_k} c_j(\sigma_j^{A_k,sp})} - \frac{(c_i'(\sigma_i^{A_{k+1},sp}))^2}{\sum_{j\in A_{k+1}} c_j(\sigma_j^{A_{k+1},sp})} = \frac{c_i'(\sigma_i^{A_k,sp})}{2\sigma^{A_k,sp}} - \frac{c_i'(\sigma_i^{A_{k+1},sp})}{2\sigma^{A_{k+1},sp}} > 0.$$

Since $\sigma^{A_k,sp} > \sigma^{A_{k+1},sp}$ from the previous lemma, for all $i \in A_{k+1}$, we must have $c_i'(\sigma_i^{A_k,sp}) > c_i'(\sigma_i^{A_{k+1},sp})$, which in turn implies $\sigma_i^{A_k,sp} > \sigma_i^{A_{k+1},sp}$. ■

To prove Corollary 2, we combine the three lemmas with Corollary 1. Consider any non-singleton alliance A on path for the equilibrium and the social planner's solution. For any $i \in A$, Corollary 1 implies that $\sigma_i^{A,eq} < \sigma_i^{\{i\},eq}$. From the lemmas above, $\sigma_i^{A,sp} > \sigma_i^{\{i\},sp}$. Since an individual searching on her own chooses the optimal search scope, $\sigma_i^{\{i\},eq} = \sigma_i^{\{i\},sp}$. We therefore have $\sigma_i^{A,sp} > \sigma_i^{A,eq}$. Furthermore, from Lemma A.9, in the welfare maximizing solution, each agent's search scope decreases as her alliance shrinks in size. ■

Proof of Corollary 3. For a contradiction suppose that for some alliance A_k, the equilibrium drawdown $d_{A_k}^{eq} > d_{A_k}^{sp}$. Now consider the set of agents $A_k \setminus A_{k+1}$ in the social planner's problem and consider the alternative strategy where each i in $A_k \setminus A_{k+1}$ searches with $\sigma_i^{A_k,eq}$ when the current gap $M - X$ is between $d_{A_k}^{eq} - d_{A_k}^{sp}$ instead of dropping at the level $d_{A_k}^{sp}$. The agents A_{k+1} search with $\sigma_i^{A_{k+1},sp}$ in an unchanged manner. To see that this leads to an improvement for each agent we need to consider the improvement to both groups. First observe that under this policy the agents in $A_k \setminus A_{k+1}$ are better off since they are searching with the same scope as they would in equilibrium, while the agents in A_{k+1} are searching with scope $\sigma_i^{A_{k+1},sp}$ but then by Corollary X $\sigma_i^{A_{k+1},sp} > \sigma_i^{A_k,eq}$, thus the agents in $A_k \setminus A_{k+1}$ are receiving even higher positive externalities compared to the equilibrium. Furthermore since the gap is larger than $d_{A_k}^{eq}$, in equilibrium the agents had a positive continuation value, which is now increased due to the positive externality for each agent. Second observe that under this policy agents in A_{k+1} are better off since the between $d^{sp}_{A_k} - d^{eq}_{A_k}$ their own search scope is unchanged, but they receive an additional positive externality from the agents in $A_k \setminus A_{k+1}$. Since both parties welfare can be improved in this manner it cannot be optimal to have $d^{eq}_{A_k} > d^{sp}_{A_k}$ for any A_k. ∎

A.3 Proofs for Optimal Sequencing with Well-ordered Costs

Proof of Lemma 1. As introduced in the proof of Corollary 2, we use here the superscripts *eq* and *sp* to denote the equilibrium and social planner's solution, respectively. When costs are well-ordered, in equilibrium, in any alliance, all agents utilize the same search scope. In particular, for any active alliance A and any $i, j \in A$, we have $\sigma_i^{A,eq} = \sigma_j^{A,eq}$. This implies that, in equilibrium, each agent k exits no later than agent $k-1$, for all $k = 2, \dots, N$. Indeed, in any active alliance A, the equilibrium stopping boundary is governed by drawdown size

$$d^{eq}_A = \min_{i \in A} \frac{(\sigma^{A,eq})^2}{2c_i(\sigma_i^{A,eq})} = \max_{i \in A} \frac{(\sigma^{A,eq})^2}{2c_i(\sigma_i^{A,eq})}.$$

Suppose, towards a contradiction, that there exists a pair i, j such that $i > j$, so that $\beta_i > \beta_j$, and the social planner has agent i terminate her search strictly before agent j. There are then two distinct alliances in the social planner's solution, A_k and A_m, with $k < m$, where $i, j \in A_k$ but $i \notin A_{k+1}$ and $j \in A_m$ but $j \notin A_{m+1}$.

As we showed, the social planner's solution associates a drawdown stopping boundary with each alliance. Denote the corresponding drawdown sizes d^{sp}_k and d^{sp}_m for A_k and A_m, respectively. Suppose that, instead, the social planner swaps the exits of agents i and j, exiting agent j from A_k whenever agent i was to cease her search and exit from A_k and exiting agent i from A_m whenever agent j was to cease her search and exit from A_m. Furthermore, the social planner can have agent i use the same search scope as agent j had originally in the alliances that follow A_k. The overall search scope in any alliance does not change after this modification. Consequently, expected search outcomes are unaltered. However, the overall cost decreases weakly in every alliance and strictly in all alliances $A_{k+1}, \dots, A_m$, contradicting the optimality of the proposed solution. ∎

Proof of Proposition 5. Recall that our results so far imply that the social planner can restrict attention to the choice between deterministic alliance sequences. Furthermore, given a deterministic sequence of alliances, Lemma A.5 identifies the optimal drawdown stopping boundaries associated with that alliance sequence. If the chosen sequence is suboptimal, some of its associated drawdown sizes might be negative or zero, implying the corresponding alliance is utilized for no length of time. This observation helps us to identify the optimal sequence. The proof of Proposition 5 follows from several lemmas. For any alliance B_k, regardless of whether it is on the social planner's optimal alliance sequence, we denote the optimal overall search scope within the alliance by $\tilde{\sigma}^k$ and the consequent overall search cost within that alliance by $\tilde{c}^k$.

Lemma A.10. *For any m, j, k such that $m < j < k$, if the welfare-maximizing sequence is such that B_k is preceded by B_m, then for any sequence where B_k is preceded by B_j, we have $d_{B_m \to B_k} > d_{B_j \to B_k}$.*

Proof of Lemma A.10. from the characterization of drawdowns in the well-ordered settings, $d_{B_m \to B_k} \neq d_{B_j \to B_k}$. Suppose that $d_{B_m \to B_k} < d_{B_j \to B_k}$. Since B_k is preceded by B_m in the optimal sequence, $d_{B_m \to B_k} > 0$. It then follows that $d_{B_j \to B_k} > 0$. This implies that it would be beneficial for the planner to have alliance B_m first transition to alliance B_j, and only then transition to alliance B_k. ■

Lemma A.11. *If $m < k$, $d_{B_m \to \emptyset} > d_{B_k \to \emptyset}$ implies $d_{B_m \to B_k} > d_{B_k \to \emptyset}$.*

Proof of Lemma A.11. $d_{B_m \to \emptyset} > d_{B_k \to \emptyset}$ implies

$$\frac{1}{N-m}\frac{\tilde{c}^m}{\tilde{\sigma}^m} < \frac{1}{N-k}\frac{\tilde{c}^k}{\tilde{\sigma}^k} \Longrightarrow \frac{1}{k-m}\left(\frac{\tilde{c}^m}{\tilde{\sigma}^m} - \frac{\tilde{c}^k}{\tilde{\sigma}^k}\right) < \frac{1}{N-k}\frac{\tilde{c}^k}{\tilde{\sigma}^k},$$

illustrating the claim. ■

Lemma A.12. *For any k such that $d_{B_k \to \emptyset} > d_{B_N \to \emptyset} > d_{B_{k+1} \to \emptyset}$, we have $d_{B_k, B_{k+1}} > d_{B_N \to \emptyset}$.*

Proof of Lemma A.12. Observe that $d_{B_k \to \emptyset} > d_{B_N \to \emptyset} > d_{B_{k+1} \to \emptyset}$ implies

$$(N-k+1)\frac{\tilde{c}^N}{\tilde{\sigma}^N} > \frac{\tilde{c}^{k+1}}{\sigma^{k+1}} \quad \text{and} \quad \frac{\tilde{c}^k}{\tilde{\sigma}^k} > (N-k)\frac{\tilde{c}^N}{\tilde{\sigma}^N}.$$

Simply summing the inequalities and reorganizing yields the implied statement. ■

Lemma A.13. *If $d_{B_k\to\emptyset} > d_{B_{k-1},B_k}$, then $d_{B_k\to\emptyset} > d_{B_{k-1}\to\emptyset} > d_{B_{k-1},B_k}$.*

Proof of Lemma A.13. From the first inequality, $d_{B_k\to\emptyset} > d_{B_{k-1},B_k}$, we have,

$$\frac{1}{N-k}\frac{\tilde{c}^k}{\tilde{\sigma}^k} < \frac{\tilde{c}^{k-1}}{\tilde{\sigma}^{k-1}} - \frac{\tilde{c}^k}{\tilde{\sigma}^k} \Longrightarrow \frac{N-(k-1)}{N-k}\frac{\tilde{c}^k}{\tilde{\sigma}^k} < \frac{\tilde{c}^{k-1}}{\tilde{\sigma}^{k-1}} \Longrightarrow d_{B_k\to\emptyset} > d_{B_{k-1}\to\emptyset}.$$

But this inequality implies that

$$\frac{1}{N-k}\frac{\tilde{c}^k}{\tilde{\sigma}^k} < \frac{1}{N-(k-1)}\frac{\tilde{c}^{k-1}}{\sigma^{\tilde{k}-1}} \Longrightarrow \frac{\tilde{c}^{k-1}}{\tilde{\sigma}^{k-1}} - \frac{\tilde{c}^k}{\tilde{\sigma}^k} > \frac{1}{(N-(k-1))}\frac{\tilde{c}^{k-1}}{\tilde{\sigma}^{k-1}} \Longrightarrow d_{B_{k-1}\to\emptyset} > d_{B_{k-1},B_k}.$$

■

Lemma A.14. *If k satisfies $\max_j d_{B_j\to\emptyset} = d_{B_k\to\emptyset}$, then any alliance B_l with $l < k$ cannot be the welfare maximizing last alliance.*

Proof of Lemma A.14. Suppose not, so that, form some $l < k$, alliance B_l is the last. Since B_k is strictly contained in B_l, from the characterization of drawdowns in the well-ordered settings, $d_{B_k\to\emptyset} \neq d_{B_l\to\emptyset}$. The social planner would, then, benefit from transitioning from B_l to B_k instead of exiting all members of B_l since $d_{B_k\to\emptyset} > d_{B_l\to\emptyset} > 0$, in contradiction. ■

Lemma A.15. *If k satisfies $\max_j d_{B_j\to\emptyset} = d_{B_k\to\emptyset}$, then any alliance B_l with $l > k$ cannot be the last.*

Proof of Lemma A.15. We use induction on the cardinality of the set B_k. The claim certainly holds when $|B_k| = 1$, so that $B_k = B_N = \{N\}$.

For the proof, it is useful to notice that our entire analysis does not hinge on the range of viable search scopes coinciding across agents. In fact, the analysis would go through in its entirety if each agent i had an individual range of scope $[\underline{\sigma}_i, \overline{\sigma}_i]$, as long as all solutions remain interior.

Assume the statement is true for sets up to cardinality n. We show the statement holds for $|B_k| = n+1$ (so that $k = N - n + 1$). By Lemma A.14, the last alliance cannot be B_j with $j < k$. Towards a contradiction, suppose that a smaller set B_m, with $m > k$, is the last alliance. From the inductive hypothesis, we must have $d_{B_m\to\emptyset} > d_{B_l\to\emptyset}$ for all $l > m$, as otherwise the social planner would benefit by inducing B_l to continue search instead of terminating it for all agents in B_m.

Suppose that $m < N$. Consider an equivalent problem, where alliance B_m is replaced with a single individual M that has cost function $\hat{c}(\cdot)$ defined so that $\hat{c}(\sigma)$ is the minimal overall cost in B_m required for implementing an overall search scope σ. That is, if

$$\sigma_m^{B_m}, \ldots, \sigma_N^{B_m} \in \arg \min_{\hat{\sigma}_j^{B_m} \in [\underline{\sigma}, \overline{\sigma}] \ \forall j, \sum_{j=m}^N \hat{\sigma}_j^{B_m} = \sigma} \sum_{j=m}^{N} c(\hat{\sigma}_j^{B_m}),$$

then $\hat{c}(\sigma) = \sum_{j=m}^N c(\hat{\sigma}_j^{B_m})$. Under this definition, $\sigma \in [(N-m+1)\underline{\sigma}, (N-m+1)\overline{\sigma}]$.

In the equivalent problem, we have m agents $1, 2, \ldots, m-1, M$.[24] From our construction so far, in the optimal solution to this problem, for any $j = 1, \ldots, m-1$, the corresponding drawdown size $d_{\{j,\ldots,M\} \to \emptyset}$ coincides with the optimally-set drawdown size $d_{B_{j \to \emptyset}}$ in our original problem. Furthermore, $d_{\{M\} \to \emptyset}$ coincides with $d_{B_{m \to \emptyset}}$ in our original problem. Therefore, $\max_{j \in \{1,\ldots,m-1,M\}} d_{\{j,\ldots,M\} \to \emptyset} = d_{\{k,\ldots,M\} \to \emptyset}$. By our induction hypothesis, $\{j, \ldots, M\}$ with $j > k$ cannot optimally be the last alliance, in contradiction.

Suppose now that $m = N$ and, towards a contradiction, assume B_N is the welfare maximizing last alliance. Now consider the sequence of welfare maximizing alliances B_p such that $B_p \subset B_k$. There are three cases to consider.

Case 1: For all $p \in \{k, \ldots, N-1\}$, the alliance B_p is part of the welfare-maximizing sequence. That is, agents terminate their search one by one starting from B_k onwards. Since B_N is the last alliance, we must have that

$$d_{B_N \to \emptyset} > d_{B_{N-1} \to B_N} > d_{B_{N-2} \to B_{N-1}} > \ldots > d_{B_k \to B_{k+1}}.$$

Applying Lemma A.13 repeatedly implies that

$$d_{B_N \to \emptyset} > d_{B_{N-1} \to \emptyset} > d_{B_{N-2} \to \emptyset} \ldots > d_{B_{k+1} \to \emptyset}.$$

The assumed maximality of $d_{B_k \to \emptyset}$ implies, in particular, that $d_{B_k \to \emptyset} > d_{B_N \to \emptyset}$ that, combined with the above, yields $d_{B_k \to \emptyset} > d_{B_N \to \emptyset} > d_{B_{k+1} \to \emptyset}$. By Lemma A.12, we then have that $d_{B_k \to B_{k+1}} > d_{B_N \to \emptyset}$. It follows that whenever agents in the active alliance B_k optimally stop searching, the social planner would benefit from halting all agents' search instead of proceeding with $B_{k+1}, B_{k+2}, \ldots, B_N$, in contradiction.

[24]Our assumption that all alliance optimally have members using an interior search scope guarantees that this fictitious agent would choose an interior search scope as well.

Case 2: There does not exist any $p \in \{k,...,N-1\}$ such that B_p is part of the optimal sequence. Thus, the penultimate alliance in the optimal sequence is B_l with $l < k$. From the maximality of $d_{B_k \to \emptyset}$, it follows that $d_{B_k \to \emptyset} > d_{B_N \to \emptyset}$. By Lemma A.11, $d_{B_k \to B_N} > d_{B_N \to \emptyset}$ and by Lemma A.10, $d_{B_l \to B_N} > d_{B_k \to B_N} > d_{B_N \to \emptyset}$. Thus, whenever agents in the active alliance B_l optimally stop searching, the social planner would benefit from halting all agents' search instead of proceeding with B_N, in contradiction.

Case 3: There exist $p, q \in \{k,...,N-1\}$ such that B_p is part of the optimal sequence but B_q is not. Here we have two subcases:

Subcase 1: B_{N-1} is the penultimate alliance. We must have $d_{B_{N-1} \to \emptyset} < d_{B_N \to \emptyset}$; otherwise, by Lemma A.11, we would have $d_{B_{N-1}, B_N} > d_{B_N \to \emptyset}$ and it would be suboptimal to utilize alliance B_N as the last alliance. From the maximality of $d_{B_k \to \emptyset}$ and Lemma A.10, for any $l < k$ such that B_l precedes B_{N-1} on the optimal path, $d_{B_l \to B_{N-1}} > d_{B_k \to B_{N-1}} > d_{B_{N-1} \to \emptyset}$. Finally, $d_{B_N \to \emptyset} > d_{B_{N-1} \to \emptyset}$ implies that

$$\frac{1}{2}\frac{\tilde{c}^{N-1}}{\tilde{\sigma}^{N-1}} > \frac{\tilde{c}^{N}}{\tilde{\sigma}^{N}} \implies \frac{1}{2}\frac{\tilde{c}^{N-1}}{\tilde{\sigma}^{N-1}} < \frac{\tilde{c}^{N-1}}{\tilde{\sigma}^{N-1}} - \frac{\tilde{c}^{N}}{\tilde{\sigma}^{N}} \implies d_{B_{N-1} \to \emptyset} > d_{B_{N-1} \to B_N}.$$

Thus,

$$d_{B_l \to B_{N-1}} > d_{B_k \to B_{N-1}} > d_{B_{N-1} \to \emptyset} > d_{B_{N-1} \to B_N}.$$

Therefore, whenever agents in the active alliance B_l optimally stop searching, the social planner would benefit from transitioning to B_N directly, thereby terminating the search of agent $N-1$ as well, instead of transitioning to B_{N-1} first, in contradiction.

Subcase 2: The penultimate alliance is B_p with $p \in \{k,...,N-2\}$. We can now emulate the argument above pertaining to the construction of an equivalent problem in which agents $\{p,...,N-1\}$ are viewed as one agent with appropriately induced search costs. We can then consider an equivalent problem with fewer agents to achieve a contradiction through our induction hypothesis. ■

The last two lemmas imply that the last alliance is given by B_k with $\max_j d_{B_j \to \emptyset} = d_{B_k \to \emptyset}$.

The proofs of the following Lemmas are a consequence of identical arguments to those in of Lemmas A.14 and A.15 and are therefore ommitted.

Lemma A.16. *Consider B_k where k is such that $d_{B_k \to B_{L_1}} > d_{B_j \to B_{L_1}}$ for all $j < L_1$ and B_{L_1} as the last alliance as identified above. Then any alliance with $l < k$ cannot be the welfare maximizing second to last alliance.*

Lemma A.17. *Consider B_k where k is such that $d_{B_k \to B_{L_1}} > d_{B_j \to B_{L_1}}$ for all $j < L_1$ and B_{L_1} as the last alliance as identified above. Then any alliance B_l with $L_1 > l > k$ cannot be the welfare maximizing second to last alliance.*

The proof of Proposition 5 then follows. Using the proposition's notation, B_{L_1} is the last alliance on the social planner's optimal path. Similarly, using the last two lemmas, the penultimate alliance is given by B_k where k is such that $d_{B_k \to B_{L_1}} > d_{B_j \to B_{L_1}}$ for all $j < L_1$. We can continue recursively to establish the Proposition's claim. ■

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