Safe Testing

Peter Grünwald[1], Rianne de Heide[2], and Wouter M. Koolen[3]

[1,2,3]Centrum Wiskunde & Informatica, Amsterdam, The Netherlands
[1,2]Leiden University, Leiden, The Netherlands
[1,2,3] {pdg, heide, wmkoolen}@cwi.nl

December 6, 2021

Abstract

We develop the theory of hypothesis testing based on the E-value, a notion of evidence that, unlike the P-value, allows for effortlessly combining results from several studies in the common scenario where the decision to perform a new study may depend on previous outcomes. Tests based on E-values are safe, i.e. they preserve Type-I error guarantees, under such *optional continuation*. We define growth-rate optimality (GRO) as an analogue of power in an optional continuation context, and we show how to construct GRO E-variables for general testing problems with composite null and alternative, emphasizing models with nuisance parameters. GRO E-values take the form of Bayes factors with special priors. We illustrate the theory using several classic examples including a one-sample safe t-test (in which the right Haar prior turns out to be GRO) and the 2×2 contingency table (in which the GRO prior is different from standard priors). Sharing Fisherian, Neymanian and Jeffreys-Bayesian interpretations, E-values and the corresponding tests may provide a methodology acceptable to adherents of all three schools.

1 Introduction and Overview

We wish to test the veracity of a *null hypothesis* $\mathcal{H}_0$, often in contrast with some *alternative hypothesis* $\mathcal{H}_1$, where both $\mathcal{H}_0$ and $\mathcal{H}_1$ represent sets of distributions on some given sample space. Our theory is based on E-*test statistics*. These are simply *nonnegative* random variables that satisfy the inequality:

$$\text{for all } P \in \mathcal{H}_0\text{:}\ \ \mathbf{E}_P[E] \leq 1. \tag{1}$$

We refer to E-test statistics as E-variables, and to the value they take on a given sample as the E-*value*, emphasizing that they are to be viewed as an alternative to, and in many cases an improvement of, the classical P-value. Note that *large* E-values correspond to evidence against the null: for given E-variable E and $0 \leq \alpha \leq 1$, we define the *threshold test corresponding to* E *with significance level* α, as the test that rejects $\mathcal{H}_0$ iff $E \geq 1/\alpha$. We will see, in a sense to be made more precise, that this test is *safe under optional continuation with respect to Type-I error*.

Motivation P-values and standard null hypothesis testing have come under intense scrutiny in recent years (Wasserstein et al., 2016, Benjamin et al., 2018). E-variables and safe tests offer several advantages. Most importantly, in contrast to P-values, E-variables behave excellently under *optional continuation*, the highly common practice in which the decision to perform additional tests partly depends on the outcome of previous tests. They thus seem particularly promising when used in meta-analysis (Ter Schure and Grünwald, 2019). A second reason is their enhanced *interpretability*: they have a very concrete (monetary) interpretation as 'evidence against the null' which remains valid even if one dismisses concepts such as 'significance' altogether, as recently advocated by Amrhein et al. (2019). A third is their flexibility: as we show in this paper, E-variables can be based on Bayesian prior knowledge, on earlier data, but also on minimax performance considerations, in all cases preserving frequentist Type I error guarantees.

Overall Contribution and Contents Although the concept is much older (Section 7), the interest in E-values and the related test martingales has exploded over the last three years (Wang and Ramdas, 2020, Vovk and Wang, 2021, Shafer, 2021, Henzi and Ziegel, 2021, Waudby-Smith and Ramdas, 2021, Orabona and Jun, 2021). In this paper, we further develop the theory of E-values, by providing general optimality criteria and show how to design E-variables that satisfy them. We do this on the basis of four ever more general versions of a single novel theorem, Theorem 1. In its first incarnation, in Section 2, it already tells us that one can design nontrivial, useful E-variables for a wide class of testing problems with composite null and alternative. This first instance relies on using a prior W_1 on the alternative $\mathcal{H}_1$. The ensuing E-variables, while guaranteeing frequentist Type-I error control, will have a GRO (*growth-rate optimality*) property under W_1. This GRO E-variable will be a Bayes factor with a special prior on the null. More general versions of the theorem allow us to construct E-variables when no prior on $\mathcal{H}_1$ is available. These satisfy either a direct worst-case optimality criterion (GROW) or a relative one (REGROW). In our example applications we restrict ourselves to classical testing scenarios such as 1-dimensional exponential families, the 2×2 contingency table, and the t-test. Importantly, the latter two have *nuisance* parameters and the GRO approach provides a generic methodology for dealing with them. For the t-test setting, GRO E-variables turn out to be Bayes factors based on the right Haar prior, as known from objective Bayes analyses (Berger et al., 1998). For the 2×2-setting, GRO E-values do not correspond to standard Bayes factors.

We then, in Section 5 and 6, investigate optional continuation, stopping and GRO in more detail, and we assess how competitive the E-variables we designed are compared to classical methods in terms of the amount of data needed before a certain desired power or growth rate can be reached. The final two sections put our work in context. We give a historical overview of E-value related work in Section 7 and then, in Section 8, taking a step back, we come to the inescapable conclusion that E-variables unify and correct ideas from the three main paradigms of testing: Fisherian, Neyman-Pearsonian and Jeffreysian.

But first, in the remainder of this introduction, we explain the three main interpretations of E-variables (Section 1.1), we briefly introduce our main theorem (Section 1.2) and, in Section 1.3, we explain the main advantage of E-variables over P-values in terms of optional continuation. We claim no technical novelty for this part, which mainly restates and reinterprets existing results. We defer to the appendices all longer proofs, as well as elaborations and specifics of several details that would distract from the main story.

1.1 The three main interpretations of E-variables

1. First Interpretation: Gambling The first and foremost interpretation of E-variables is in terms of *money*, or, more precisely, *Kelly* (1956) *gambling*. Imagine a ticket (contract, gamble, investment) that one can buy for 1\$, and that, after realization of the data, pays E \$; one may buy several and positive fractional amounts of tickets. (1) says that, if the null hypothesis is true, then one expects not to gain any money by buying such tickets: for any $r \in \mathbb{R}^+$, upon buying r tickets one expects to end up with $r\mathbf{E}[E] \leq r$ \$. Therefore, if the observed value of E is large, say $20 = 1/0.05$, one would have gained a lot of money after all, indicating that something might be wrong about the null.

2. Second Interpretation: Conservative P-Value, Type I Error Probability Recall that a (strict) P-value is a random variable P such that for all $0 \leq \alpha \leq 1$, all $P_0 \in \mathcal{H}_0$,

$$P_0(\mathrm{P} \leq \alpha) = \alpha. \tag{2}$$

A *conservative* P-value is a random variable for which (2) holds with '=' replaced by '$\leq$'. There is a close connection between (small) P- and (large) E-values:

Proposition 1. *For any given* E*-variable* E*, define* $\mathrm{P}_{[\mathrm{E}]} := 1/E$*. Then* $\mathrm{P}_{[\mathrm{E}]}$ *is a conservative* P*-value. As a consequence, for every* E*-variable* E*, any* $0 \leq \alpha \leq 1$*, the corresponding threshold-based test has Type-I error guarantee* α*, i.e. for all* $P \in \mathcal{H}_0$*,*

$$P(E \geq 1/\alpha) \leq \alpha. \tag{3}$$

Proof. **(of Proposition 1)** Markov's inequality gives $P(E \geq 1/\alpha) \leq \alpha \mathbf{E}_P[E] \leq \alpha$. □

While E-variables are thus conservative P-values, standard P-values satisfying (2) are by no means E-variables; if E is an E-variable and P is a standard P-value, and they are calculated on the same data, then we will usually observe $\mathrm{P} \ll 1/E$ so E gives *less* evidence against the null (Section 7).

Combining 1. and 2.: Optional Continuation Proposition 2 below shows that *multiplying* E-variables $E_{(1)}, E_{(2)}, \ldots$ for tests based on respective samples $\mathbf{Y}_{(1)}$, $\mathbf{Y}_{(2)}, \ldots$ (with each $\mathbf{Y}_{(j)}$ being the batch of outcomes for the j-th test), gives rise to new E-variables, even if the decision whether or not to perform the test resulting in $E_{(j)}$ was based on the value of earlier test outcomes $E_{(j-1)}, E_{(j-2)}, \ldots$ As a result (Corollary 1), *the Type I-Error Guarantee* (3) *remains valid even under this 'optional continuation' of testing.* Just as importantly, in contrast to P-values, E-variables satisfy an 'optional continuation principle': whether an observed E-value is valid or not does not depend on whether or not you would have performed an additional study and gathered additional evidence in situations that did not occur.

An indication that something like this might be true is immediate from our gambling interpretation: if we start by investing \$1 in $E_{(1)}$ and, after observing $E_{(1)}$, reinvest all our new capital $\$E_{(1)}$ into $E_{(2)}$, then after observing $E_{(2)}$ our new capital will obviously be $\$E_{(1)} \cdot E_{(2)}$, and so on. If, under the null, we do not expect to gain any money for any of the individual gambles $E_{(j)}$, then, intuitively, we should not expect to gain any money under whichever strategy we employ for deciding whether or not to reinvest (just as you would not expect to gain any money in a casino irrespective of your rule for re-investing and/or stopping and going home).

3. Third Interpretation: Bayes Factors For convenience, from now on we write the models $\mathcal{H}_0$ and $\mathcal{H}_1$ as

$$\mathcal{H}_0 = \{P_\theta : \theta \in \Theta_0\} \quad ; \quad \mathcal{H}_1 = \{P_\theta : \theta \in \Theta_1\},$$

where $\Theta_0, \Theta_1 \subset \Theta$, and $\{P_\theta : \theta \in \Theta\}$ represents a general family of distributions or random processes, defined relative to some given sample space and σ-algebra or filtration. $\mathbf{Y} = Y^N = (Y_1, \ldots, Y_N)$, a vector of N outcomes, represents our data. N may be a fixed sample size n but can also be a random stopping time. We assume that $\mathbf{Y}$ is measurable and, under every P_θ with $\theta \in \Theta$, has a probability density p_θ relative to some fixed underlying measure μ. In the Bayes factor approach to testing, one associates both $\mathcal{H}_j$ with a *prior* W_j, which is simply a probability distribution on Θ_j, and a *Bayes marginal probability distribution* P_{W_j}, with density (or mass) function given by

$$p_{W_j}(\mathbf{Y}) \coloneqq \int_{\Theta_j} p_\theta(\mathbf{Y}) \, \mathrm{d}W_j(\theta). \tag{4}$$

The *Bayes factor* is then given as:

$$\mathsf{BF} \coloneqq \frac{p_{W_1}(\mathbf{Y})}{p_{W_0}(\mathbf{Y})}. \tag{5}$$

Whenever $\mathcal{H}_0 = \{P_0\}$ is *simple*, i.e., a singleton, then the Bayes factor is also a (sharp) E-variable, since we must then have that W_0 is degenerate, putting all mass on 0, and $p_{W_0} = p_0$, and then for all $P \in \mathcal{H}_0$, i.e. for P_0, we have, assuming P_0 has strictly positive density,

$$\mathbf{E}_P[\mathsf{BF}] \coloneqq \int p_0(y) \cdot \frac{p_{W_1}(y)}{p_0(y)} \, \mathrm{d}\mu(y) = 1. \tag{6}$$

For such E-variables that are really simple-$\mathcal{H}_0$-based Bayes factors, Proposition 1 reduces to the well-known *universal bound* for likelihood ratios (Royall, 1997). When $\mathcal{H}_0$ is itself composite, most Bayes factors $\mathsf{BF} = p_{W_1}/p_{W_0}$ will *not* be E-variables any more, since for B to be an E-variable we require (6) to hold for *all* $P_\theta, \theta \in \Theta_0$, whereas in general it only holds for $P = P_{W_0}$. Nevertheless, Theorem 1 (in its first, simplest version in Section 2) implies that, under weak conditions, for every prior W_1 on Θ_1 there always exists a corresponding prior W_0 on Θ_0, for which $\mathsf{B} = p_{W_1}/p_{W_0}$ is an E-variable after all. More generally, in all our examples E-variables invariably take on a Bayesian form, though sometimes with highly unusual (e.g. degenerate) priors.

1.2 This Paper: Beyond Simple Nulls, Beyond Available Priors

In this paper, we focus on general, composite $\mathcal{H}_0$. The only assumption on $\mathcal{H}_0$ is the existence of densities as above — we make this assumption because it allows for a completely general characterization of GRO E-variables as in Theorem 1. Still, useful E-variables for nonparametric settings without densities do exist (Waudby-Smith and Ramdas, 2021, Orabona and Jun, 2021).

Theorem 1 in its first form in Section 2 tells us how to choose an E-variable that is optimal in the GRO sense if a prior W_1 on $\mathcal{H}_1$ is given. Roughly speaking, GRO means that the E-variable tends to grow fast under $\mathcal{H}_1$ as more data come in, thereby generating substantial

evidence against $\mathcal{H}_0$. In the generalizations in Section 3 and Section 4, we extend the GRO idea to E-variables when no such prior is available. Section 3 deals with a basic minimax optimality approach, which is appropriate if there is a single parameter of interest, a minimum relevant effect size, and no nuisance parameter. Section 4 describes *relative* minimax optimal E-variables, which are appropriate if there is no minimal effect size and/or nuisance parameters are present. This culminates in the fully general version of Theorem 1 in Section 4.3 which is also applicable to hypotheses with nuisance parameters that satisfy a group invariance, such as in the t-test. To show that the E-variable we propose for the t-test setting is indeed optimal we need an additional result, Theorem 2, which, to keep length at bay, we only state and do not prove here — the proof is in an earlier arXiv version of this paper and will be published elsewhere. But before embarking on these results, we explain the benefits of E-variable based tests in detail.

1.3 Optional Continuation

We defined E-variables for a single experiment. We now discuss sequential experimentation and how E-values can be combined to accumulate evidence against the null. To this end, let us imagine a sequence of random variables $\mathbf{Y}_{(1)}, \mathbf{Y}_{(2)}, \ldots$ representing the outcomes of experiments/studies. We will not (except for illustration purposes later on) make use of any internal structure of the $\mathbf{Y}_{(j)}$, which in particular may come to us as batches of varying lengths.

Definition 1. *Let $(E_{(m)})_m = E_{(1)}, E_{(2)}, \ldots$ be a non-negative random process adapted to filtration $(\mathcal{F}_{(m)})_m$ and let $\mathcal{H}_0$ be a collection of distributions for this process. We say that $(E_{(m)})_m$ is a sequence of* conditional E-variables *(for null model $\mathcal{H}_0$ and $(\mathcal{F}_{(m)})$) or a* conditional E-process *for short, if for all $m \in \mathbb{N}$,*

$$\text{for all } P \in \mathcal{H}_0\text{:} \quad \mathbf{E}_P[E_{(m)} \mid \mathcal{F}_{(m-1)}] \leq 1 \quad a.s.$$

In standard cases, $\mathcal{F}_{(m)}$ represent all that is known to us at time m (i.e. after having observed the m-th batch). In the simplest case this is simply $\sigma(\mathbf{Y}^{(m)})$, with $\mathbf{Y}_{(m)} = (\mathbf{Y}_{(1)}, \ldots, \mathbf{Y}_{(m)})$ the sequence of outcomes of previous studies, and we could then rewrite the expectation elementarily as $\mathbf{E}_P[E_{(m)} \mid \mathbf{Y}^{(m-1)}]$. More generally though, $\mathcal{F}_{(m)}$ may include nonstochastic side information such as 'there is money to do an additional study' or covariates; we briefly describe such extensions, as well as subtleties that may arise, in Appendix B.1.

Intuitively, conditional E-values measure the conditional evidence in round m against $\mathcal{H}_0$, and hence their running product measures the total evidence (such a running product would be a *test super-martingale* (Shafer et al., 2011), i.e. a nonnegative super-martingale with starting value ≤ 1, under every element of the null). We may turn this running product into one quantity by adding a stopping rule. The following result, a direct implication of Doob's optional stopping theorem (Williams, 1991) states that we obtain a fair measure of evidence.

Proposition 2. *Let $(E_{(m)})_m$ be a conditional E-process, i.e. a sequence of conditional E-variables adapted to $(\mathcal{F}_{(m)})_m$. Then the running product $(E^{(m)})_{(m)}$ with $E^{(m)} := \prod_{j=1}^{m} E_{(j)}$ is a test super-martingale w.r.t. each $P \in \mathcal{H}_0$. Moreover, for any stopping time τ (not necessarily finite), the stopped value $E^{(\tau)}$ is a (standard non-sequential) E-value for $\mathcal{H}_0$.*

Proposition 2 says that, no matter when we stop collecting batches of data, the resulting product is an E-variable and therefore a test based on it preserves Type-I error guarantees by Proposition 1.

Just-in-Time Conditional E-variables: Optional Continuation As we can see from Proposition 2, the stopped running product $E^{(\tau)}$ of a sequence of conditional E-variables only evaluates each member variable $E_{(m)}$ after m rounds, and only on the data $\mathbf{Y}_{(1)}, \ldots, \mathbf{Y}_{(m)}$ that actually happened. It is therefore perfectly fine (for the Type-I error safety guaranteed by combining Propositions 2 and 1) for us to construct $E_{(m)}$ on demand just before round m, as a function of all available information so far (including possibly both stochastically modeled and arbitrary variables), as long as we ensure the conditional safety property in Definition (1). This simple observation gives us tremendous flexibility for testing, much in contrast to traditional P-values where the sampling plan needs to be fixed up front. Note in particular that it allows *optional continuation*: the practice of deciding *after* an initial series of experiments whether to output the current accumulated evidence, or perform yet more experiments.

In a slight variation, we can also express the Type-I error safety under optional continuation as follows:

Corollary 1. (of Proposition 2): "Ville-Robbins"

$$\textit{For all } P \in \mathcal{H}_0\textit{, all } 0 < \alpha \leq 1\textit{:}\ \ P\left(\textit{there exists } m \textit{ such that } E^{(m)} \geq \frac{1}{\alpha}\right) \leq \alpha. \qquad (7)$$

Proof. Proposition 2 expresses that $E^{(1)}, E^{(2)}, \ldots$ is a super-martingale with starting value ≤ 1. The Ville-Robbins inequality (Ville, 1939) then implies (7). □

For use later on, we formally define a *threshold test based on non-negative process* $(E^{(m)})_{(m)}$ to be the random function that, when input m and level α, outputs REJECT if $E^{(m)} \geq 1/\alpha$ and outputs ACCEPT otherwise (in this general definition, $E^{(m)}$ can but does not need to be defined as a product of conditional E-variables). We say that a threshold test is *safe under optional continuation with respect to Type I error* if (7) holds. Thus, no matter when the data collecting and combination process is stopped, the Type-I error probability is preserved. We relate optional continuation to the more common notion of 'optional stopping' in Section 5. Whereas Ville-Robbins stresses that we may greedily 'keep combining studies until we can reject or resources run out', it is just as important that our E-values keep providing valid Type-I error guarantees if the continuation rule is externally imposed or unknowable.

Example 1. Consider the simple scenario with a single underlying data stream $Y_1, Y_2, \ldots$ with the Y_i i.i.d. according to both $\mathcal{H}_0$ and $\mathcal{H}_1$, and assume for simplicity that $\mathcal{H}_0 = \{P_0\}$ is simple so that Bayes factors provide E-variables. For arbitrary prior W on Θ_1, define $e_{n,W}(Y^n) = p_W(Y^n)/p_0(Y^n)$ to be the Bayes factor as in (5) with prior W for Θ_1 applied to data Y^n.

Suppose we perform an initial study on sample $\mathbf{Y}_{(1)} := Y^{N_{(1)}} := (Y_1, \ldots, Y_{N_{(1)}})$ and we equip Θ_1 with prior $W_{(1)}$. We can use as our E-variable $E_{(1)}$ the Bayes factor $E_{(1)} := e_{N_{(1)},W_{(1)}}(\mathbf{Y}_{(1)})$. Suppose this leads to a first E-value $E_{(1)} = 18$ — promising enough for us to invest our resources into a subsequent study. We decide to gather $N_{(2)}$ data points leading to data $\mathbf{Y}_{(2)} = (Y_{N_{(1)}+1}, \ldots, Y_{N_{(1)}+N_{(2)}})$. For this second data batch, we will use an E-variable $E_{(2)} := e_{N_{(2)},W_{(2)}}(\mathbf{Y}_{(\mathbf{2})})$ for a new prior $W_{(2)}$. Crucially, we are allowed to choose both $N_{(2)}$ and $W_{(2)}$ as a function of past data $\mathbf{Y}^{(1)}$: clearly, because the underlying data stream was assumed i.i.d., $\mathbf{E}_{P_0}[E_{(2)} \mid \mathbf{Y}_{(1)}] \leq 1$ irrespective of our choice (here we use (6)),

and this allows us to use Proposition 2. If we want to stick to the Bayesian paradigm, we can choose $W_{(2)} := W_{(1)}(\cdot \mid \mathbf{Y}_{(1)})$, as the Bayes posterior for θ_1 based on data $\mathbf{Y}_{(1)}$ and prior $W_{(1)}$. Bayes' theorem shows that multiplying $E^{(2)} := E_{(1)} \cdot E_{(2)}$ (which gives a new E-variable by Proposition 2), satisfies

$$E^{(2)} = E_{(1)} \cdot E_{(2)} = \frac{p_{W_{(1)}}(\mathbf{Y}_{(1)}) \cdot p_{W_{(1)}(\cdot|\mathbf{Y}_{(1)})}(\mathbf{Y}_{(2)})}{p_0(\mathbf{Y}_{(1)}) \cdot p_0(\mathbf{Y}_{(2)})} = \frac{p_{W_{(1)}}(Y_1, \ldots, Y_{N_{(2)}})}{p_0(Y_1, \ldots, Y_{N_{(2)}})}, \tag{8}$$

which is exactly what one would get by Bayesian updating. This illustrates that, for simple $\mathcal{H}_0$, combining E-variables by multiplication can be done consistently with Bayesian updating.

The Local Perspective It might also be the case that it is not us who get the additional funding to obtain extra data, but rather some research group at a different location. If the question is, say, whether a medication works, the null hypothesis would still be $\mathcal{H}_0 = \{P_0\}$ but, if it works, its effectiveness might be slightly different due to slight differences in population. In that case, the research group might decide to use a different test statistic $E'_{(2)}$ which is again a Bayes factor, but now with an alternative prior W on θ_1 (for example, the original prior $W_{(1)}$ might be re-used rather than replaced by $W_{(1)}(\cdot \mid \mathbf{Y}_{(1)})$ — one might call this the *local* perspective. Even though this would not be standard Bayesian, $E_{(1)} \cdot E'_{(2)}$ would still be a valid E-variable, and Type-I error guarantees would still be preserved — and the same would hold even if the new research group would use an entirely different prior on Θ_1. And, after the second batch of data $\mathbf{Y}_{(2)}$, one might consider obtaining even more samples, each time using a different $W_{(j)}$, that is always allowed to depend on the past in arbitrary ways.

2 The GRO E-Variable

Mathematical Preliminaries In this and the coming sections we present our main result, Theorem 1. We first list all required mathematical notations and definitions. We invariably assume that some family $\{P_\theta : \theta \in \Theta\}$ of probability distributions for $\mathcal{Y}$ has been fixed and all P_θ with $\theta \in \Theta$ have densities relative to some underlying measure μ. When we write 'p is a (sub-) probability density', we mean it is a (sub-) probability density relative to μ, i.e. $p \geq 0$ and $\int p(\mathbf{Y}) d\mu = 1$ for a density and $\int p(\mathbf{Y}) d\mu \leq 1$ for a sub-density. In the latter case we call the measure P with density p a sub-probability distribution. We use $D(Q\|P)$ to denote the Kullback-Leibler (KL) Divergence between distributions Q and P (Cover and Thomas, 1991). We allow P (but not Q) to be a sub-probability distribution, with $D(P\|Q) = \mathbf{E}_{\mathbf{Y}\sim Q}[\ln q(\mathbf{Y})/p(\mathbf{Y})]$. We say that random variables U^* and U° are *essentially equal* if, for all $\theta \in \Theta$, $P_\theta(U^* = U^\circ) = 1$. We say that U^* *essentially uniquely* satisfies property PROP if all other random variables satisfying property PROP are essentially equal to U^*. When we write 'P has full support', we mean that its density p satisfies $p(\mathbf{Y}) > 0$ μ-almost everywhere. We assume some suitable σ-algebra including all singleton sets on Θ has been defined, and for $\Theta' \subset \Theta$ we let $\mathcal{W}(\Theta')$ be the set of all probability distributions (i.e., 'proper priors') on Θ' with this σ-algebra. Notably, $\mathcal{W}(\Theta')$ includes, for each $\theta \in \Theta'$, the degenerate distribution W which puts all mass on θ. We say that W *essentially uniquely* satisfies property PROP among $\mathcal{W}(\Theta')$ if for all other distributions $W' \in \mathcal{W}(\Theta')$ that satisfy PROP and all $\theta \in \Theta$, we have $P_\theta(p_W = p'_W) = 1$. $\mathcal{E}(\Theta_0)$ is defined as the set of all E-variables that can be defined on $\mathbf{Y}$ for Θ_0, i.e. all random variables satisfying (1). We frequently use

the fact (proof in Appendix B.2) that if $\Theta_0 = \{0\}$ is a singleton so that $\mathcal{H}_0$ is simple, then the class of E-variables corresponds exactly to the set of likelihood ratios relative to p_0:

$$\mathcal{E}(\{0\}) = \left\{ \frac{q(\mathbf{Y})}{p_0(\mathbf{Y})} \ : q \text{ is a sub-probability density for } \mathbf{Y} \right\}. \tag{9}$$

Our main theorem (proof in Appendix A.1) implies that nontrivial E-variables exist without any further conditions:

Theorem 1. *Suppose Q is a probability distribution with full support and with density q, and assume $\inf_{W_0 \in \mathcal{W}(\Theta_0)} D(Q\|P_{W_0}) < \infty$. Then there exists a (potentially sub-) distribution P_0^* with density p_0^* such that*

$$E^* := \frac{q(\mathbf{Y})}{p_0^*(\mathbf{Y})} \tag{10}$$

is an E-variable. Moreover, E^* satisfies, essentially uniquely,

$$\sup_{E \in \mathcal{E}(\Theta_0)} \mathbf{E}_{\mathbf{Y}\sim Q}[\log E] = \mathbf{E}_{\mathbf{Y}\sim Q}[\log E^*] = \inf_{W_0 \in \mathcal{W}(\Theta_0)} D(Q\|P_{W_0}) = D(Q\|P_0^*). \tag{11}$$

If the minimum is achieved by some W_0^*, i.e. $D(Q\|P_0^*) = D(Q\|P_{W_0^*})$, then $P_0^* = P_{W_0^*}$.

The full support condition is natural and discussed further in Appendix A.3. Following Barron and Li (1999) (see also (Csiszár and Matus, 2003)), we call P_0^* the *Reverse Information Projection (RIPr)* of Q on $\mathcal{P}(\Theta_0) = \{P_{W_0} : W_0 \in \mathcal{W}(\Theta_0)\}$. In all examples encountered in this paper, we have $P_0^* = P_{W_0^*}$, i.e. the minimum is achieved and its density integrates to 1 (it is possible to construct special $\mathcal{H}_0$ for which p_0^* integrates to strictly less than 1 (Harremoës, 2021), but we do not know whether this happens for any practically relevant $\mathcal{H}_0$). The following corollary (see Appendix A.1 for details) is useful in applications:

Corollary 2. *E^* is the only E-variable of Bayes factor/likelihood ratio form with q in the numerator. That is, for all $W_0 \in \mathcal{W}(\Theta_0)$: if P_{W_0} is not essentially equal to P_0^* then $q(\mathbf{Y})/p_{W_0}(\mathbf{Y})$ is not an E-variable. In particular this implies: (a) if $P_0^* = P_{W_0^*}$, then W_0^* achieves $\min_{W_0 \in \mathcal{W}(\Theta_0)} D(Q\|P_{W_0})$ essentially uniquely; and (b) if we have found an E-variable of form $q(\mathbf{Y})/p_{W_0}(\mathbf{Y})$ then W_0 must be essentially equal to W_0^*.*

2.1 The GRO criterion when $\mathcal{H}_1$ is simple

We now focus on the case with a given alternative $\mathcal{H}_1 = \{P_\theta : \theta \in \Theta_1\}$, and for now assume $\Theta_1 = \{\theta_1\}$ is a singleton. Applying Theorem 1 above with $Q = P_{\theta_1}$, we call the resulting E^* (or any essentially equal version of it) the θ_1-GRO E-variable, GRO standing for *growth-rate optimal*. We define the *growth rate achievable with* θ_1 as

$$\mathrm{GRO}(\theta_1) := \sup_{E \in \mathcal{E}(\Theta_0)} \mathbf{E}_{Y\sim P_{\theta_1}}[\log E] = D(P_{\theta_1}\|P_0^*), \tag{12}$$

with the equality following from Theorem 1 (we omit Θ_0 in the notations since, in contrast to Θ_1 or θ_1, Θ_0 will always be clear from context).

In general, there exist many nontrivial E-variables for a given $\mathcal{H}_0$. The θ_1-GRO E-variable is a special one that is optimal in a natural sense for the given $\mathcal{H}_1$: whereas in the Neyman-Pearson paradigm, one measures the quality of a test at a given significance level α by its

power, i.e. the probability of correct decision under θ_1, we will measure it by the *expected capital growth rate* under θ_1. This is different from power, yet there are close connections to which we return in Section 8.

To explain, we return to the monetary interpretation of E-values. The definition of E-variable ensures that we expect them to stay under 1 (one does not gain money) under any $P \in \mathcal{H}_0$. Analogously, one would like them to be constructed such that they can be expected to grow large as fast as possible (one gets rich, gets evidence against $\mathcal{H}_0$) under $\mathcal{H}_1$. Assuming for now that $\mathcal{H}_1 = \{P_{\theta_1}\}$ is simple, this suggests to define the optimal E-variable E^* as the one that maximizes $\mathbf{E}_{P_{\theta_1}}[f(E^*)]$ for some function that is increasing in E^*. At first sight it may seem natural to pick f the identity, but this can lead to adoption of an E-variable E^* such that $P_{\theta_1}(E^* = 0) > 0$. This choice, however, does not go together well with preserving evidence (capital) under optional continuation: if $E^*_{(1)}$ is 0 with positive probability, then it may happen that the evidence $E^{(m)} = \prod_{j=1}^m E_{(j)}$ obtained so far remains 0, no matter how large $E_{(j)}$ for $j \geq 1$ — akin to losing all one's money in the first round at a roulette table. A similar objection applies to any polynomial f, but it does not apply to the logarithm, which is the most natural choice for f: by the law of large numbers, a sequence of E-variables $E_{(1)}, E_{(2)}, \ldots$ based on i.i.d. $\mathbf{Y}_{(1)}, \mathbf{Y}_{(2)}, \ldots$ with, for all j, $\mathbf{E}_{\mathbf{Y}_{(j)} \sim P}[\log E_{(j)}] \geq L$, will a.s. satisfy $E^{(m)} := \prod_{j=1}^m E_{(j)} = \exp(mL + o(m))$, i.e. E will grow exponentially, and $L(\log_2 e)$ lower bounds the *doubling rate* (Cover and Thomas, 1991). Such strong exponential growth rates can only be given if we adopt the logarithm, which is a powerful reason for choosing it. Its use goes back to Kelly (1956); see also the extensive exposition by Shafer (2021).

Example 2. [2×2 Contingency Tables] Let $\mathcal{Y}^n = \{0,1\}^n$ and let $\mathcal{X} = \{a, b\}$ represent two categories. We start with a multinomial model $\mathcal{G}$ on $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$, extended to n outcomes by independence. We want to test whether the Y_i are dependent on the X_i. To this end, we condition every distribution in $\mathcal{G}$ on a fixed, given, $X^n = \mathbf{x}$ with $\mathbf{x} = (x_1, \ldots, x_n)$, and we let $\mathcal{H}_1$ be the set of (or a subset of the) conditional distributions on $\mathcal{Z}$ that thus result. We thus assume the design of $\mathcal{X}^n$ to be set in advance, but N_1, the number of ones, to be random; alternative choices are possible and would lead to a different analysis. Conditioned on $X^n = \mathbf{x}$, the likelihood of an individual sequence $\mathbf{y} \mid \mathbf{x}$ becomes:

$$p_{\mu_{1|a},\mu_{1|b}}(\mathbf{y} \mid \mathbf{x}) = p_{\mu_{1|a},\mu_{1|b}}(\mathbf{y} \mid \mathbf{x}) = \mu_{1|a}^{n_{a1}}(1-\mu_{1|a})^{n_{a0}} \cdot \mu_{1|b}^{n_{b1}}(1-\mu_{1|b})^{n_{b0}}, \tag{13}$$

where n_{ji} is the number of times outcome i was observed to fall in category j and $\mu_{1|j}$ is the probability of observing a 1 given category j. These densities define the full model $\{P_{\mu_{1|a},\mu_{1|b}} : (\mu_{1|a}, \mu_{1|b}) \in \Theta\}$ with $\Theta = [0,1]^2$. $\mathcal{H}_0$, the null model, simply has $(X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ independent, with $Y_i, \ldots, Y_n$ i.i.d. Ber(μ) distributed, $\mu \in \Theta_0 := [0,1]$, i.e. $p_\mu(\mathbf{y} \mid \mathbf{x}) = p_\mu(\mathbf{y}) = \mu^{n_1}(1-\mu)^{n-n_1}$ with $n_1 = n_{a1} + n_{b1}$. We defer description of the test of the full alternative $\{P_\theta : \theta \in \Theta_1\}$ with $\Theta_1 = (0,1)^2$ against $\mathcal{H}_0$ to Section 4.1. For now, we assume a simple $\mathcal{H}_1 = \{P_{\theta_1}\}$ for a specific $\theta_1 = (\mu_{1|a}, \mu_{1|b})$ with $\mu_{1|a} \neq \mu_{1|b}$. Turner et al. (2021) shows that the RIPr for P_{θ_1}, achieving the infimum in (11) is given by $P_0^* = P_{W_0^*}$ where W_0^* is the degenerate prior that puts all its mass on the single point $\mu^\circ = (n_a\mu_{1|a} + n_b\mu_{1|b})/(n_a + n_b)$. Thus, the θ_1-GRO E-variable has an intuitive form here, being given by

$$E^* = \frac{p_{\mu_{1|a},\mu_{1|b}}(\mathbf{Y} \mid \mathbf{x})}{p_{\mu^\circ}(\mathbf{Y} \mid \mathbf{x})}. \tag{14}$$

The fact that the RIPr is achieved by a point prior is quite specific to contingency tables. We also note that, while the expectation of E^* is bounded by 1 under all $P_\mu \in \mathcal{H}_0$, its actual distribution function varies with P_μ. This is in contrast to the t-test examples, in which the GRO E-variable turns out to have the same distribution under all distributions in the null.

2.2 GRO when prior on $\mathcal{H}_1$ is Available

We now take a Bayesian stance regarding $\mathcal{H}_1$ and, conditioned on $\mathcal{H}_1$, are prepared to represent our uncertainty by prior distribution W_1 on Θ_1. The marginal distribution of $\mathbf{Y}$ is then $P_{W_1}(\mathbf{Y})$. Applying Theorem 1 with P_{W_1} in the role of Q then leads to the "W_1-GRO E-variable" — it would be optimal in the GRO sense under prior W_1. This E-variable is a Bayes factor, but gives only a quasi-Bayesian notion of evidence since any prior W_1 on $\mathcal{H}_1$ that we wish to adopt forces us to adopt a particular corresponding prior $W_0^* \in \mathcal{H}_0$. One may perhaps consider this a small price to pay for creating a Bayes factor that, by its Type-I error safety under optional continuation, should be acceptable to frequentists as well. Moreover it is often recognized that priors on Θ_0 and Θ_1 should somehow be 'matched' to each other (Berger et al., 1998); we may view the RIPr construction as providing a reasonable(in the sense that it provides frequentist guarantees) matching.

Example 3. [Gaussian Location with Gaussian prior (z-test)] Now consider $\mathcal{H}_1$ according to which the Y_i are i.i.d. $\sim N(\mu, 1)$ for some $\mu \in \Theta_1 = \mathbb{R}$, so that $p_\mu(\mathbf{Y}) = p_\mu(Y_1, \ldots, Y_n) \propto \exp(-\sum_{i=1}^n (Y_i - \mu)^2/2)$. We let $\mathcal{H}_0 = \{P_0\}$. We perform a Bayes factor test using $E := p_W(\mathbf{Y})/p_0(\mathbf{Y})$ where we take the prior W to have Gaussian density $w(\mu) \propto \exp(-\mu^2/2)$. By (6) we know that E is not just a Bayes factor but also an E-variable. By straightforward calculation:

$$\log E = -\frac{1}{2}\log(n+1) + \frac{1}{2}(n+1) \cdot \breve{\mu}_n^2,$$

where $\breve{\mu}_n = (\sum_{i=1} Y_i)/(n+1)$ is the Bayes MAP estimator, which only differs from the ML estimator by $O(1/n^2)$: $\breve{\mu}_n - \hat{\mu}_n = \hat{\mu}_n/(n(n+1))$. If we were to reject Θ_0 when $E \geq 20$ (giving, by Proposition 1 a Type-I error guarantee of 0.05), we would thus reject if

$$|\breve{\mu}_n| \geq \sqrt{\frac{5.99 + \log(n+1)}{n+1}}, \text{ i.e. } |\hat{\mu}_n| \succeq \sqrt{(\log n)/n}, \tag{15}$$

where we used $2 \log 20 \approx 5.99$. Contrast this with the standard two-sided Neyman-Pearson (NP) test, which would reject (with $\alpha = 0.05$) if $|\hat{\mu}_n| > 1.96/\sqrt{n}$, or the one-sided test which would reject if $\hat{\mu}_n \geq 1.645/\sqrt{n}$ or the E-value based tests of the next section: the standard Bayesian test is significantly more conservative, requiring more data to achieve a similar conclusion. In Section 6 we investigate this further.

3 The GROW E-Variable

We now show how to construct good E-variables if $\mathcal{H}_1$ is composite and no prior on Θ_1 is available. We focus on variations of worst-case (minimax) growth optimality, but this is certainly not the only criterion that might be useful or valuable; see the discussion in Section 8. In the case of simple $\mathcal{H}_1 = \{P_{\theta_1}\}$, we aimed for E-variables that could be expected to grow as

fast as possible under P_{θ_1}. Analogously, we would now like them to be constructed such that they can be expected to grow large as fast as possible (one gets rich, gets evidence against $\mathcal{H}_0$) under *all* $P_1 \in \mathcal{H}_1$. We call E-variables satisfying this property GROW: *growth-rate optimal in worst-case*. We now discuss the simplest, 'raw' form of this criterion — in some settings a modification of this criterion, REGROW, which we discuss in the next section, is more suitable. GROW tells us to pick, among all E-variables relative to $\mathcal{H}_0$, the E^* that achieves the *worst-case optimal expected capital growth rate*

$$\text{GROW}(\Theta_1) := \sup_{E: E \in \mathcal{E}(\Theta_0)} \inf_{\theta \in \Theta_1} \mathbf{E}_{P_\theta}[\log E]. \tag{16}$$

Theorem 1, First Generalization *Suppose all $P_\theta, P_{\theta'}$ with $\theta, \theta' \in \Theta_1$ satisfy $D(P_\theta \| P_{\theta'}) < \infty$, and have full support. If*

$$\inf_{W_1 \in \mathcal{W}_1, W_0 \in \mathcal{W}(\Theta_0)} D(P_{W_1} \| P_{W_0}) = \inf_{W_0 \in \mathcal{W}(\Theta_0)} D(P_{W_1^*} \| P_{W_0}) < \infty,$$

(i.e. the minimum on the left is achieved by W_1^*) then then there exists an E*-variable*

$$E^* := \frac{p_{W_1^*}(\mathbf{Y})}{p_0^*(\mathbf{Y})}, \tag{17}$$

where p_0^* is the density of P_0^*, a (potentially sub-distribution satisfying $\inf_{W_0 \in \mathcal{W}(\Theta_0)} D(P_{W_1^*} \| P_{W_0}) = D(P_{W_1^*} \| P_0^*)$, and E^* achieves (16), i.e. it satisfies, essentially uniquely:

$$\inf_{\theta \in \Theta_1} \mathbf{E}_{\mathbf{Y} \sim P_\theta}[\log E^*] = \sup_{E \in \mathcal{E}(\Theta_0)} \inf_{\theta \in \Theta_1} \mathbf{E}_{\mathbf{Y} \sim P_\theta}[\log E] = D(P_{W_1^*} \| P_0^*).$$

If further $D(P_{W_1^*} \| P_0^*) = D(P_{W_1^*} \| P_{W_0^*})$ for some $W_0^* \in \mathcal{W}(\Theta_0)$, then $P_0^* = P_{W_0^*}$.

The earlier version of Theorem 1 is the special case we get if we set $\Theta_1 = \{\theta_1\}$ a singleton and $Q := P_{\theta_1}$. This generalized version expresses that the GROW E-variable is once again a Bayes factor — a special one in fact, between the components of the *joint information projection* $(P_{W_1^*}, P_0^*)$ (Figure 1).

3.1 One-parameter models with minimum relevant effect size

Let Θ be a connected subset of $\mathbb{R}$ indexing a 1-parameter parametric model $\{P_\theta : \theta \in \Theta\}$ with θ indicating the size of some effect. If, as is standard practice in e.g. medical statistics, we have a *minimum clinically relevant effect size* δ^+ and a status quo $\delta^- < \delta^+$ in mind, we want to test

$$\Theta_0 = \{\theta \in \Theta : \theta \leq \delta^-\} \text{ vs. } \Theta_1 = \{\theta \in \Theta : \theta \geq \delta^+\}. \tag{18}$$

In standard cases, often $\delta^- = 0$ and $\Theta_0 := \{0\}$.

Proposition 3. *Suppose there exists a 1-dimensional statistic $T = t(\mathbf{Y})$ such that the family of densities $\{p_\theta : \theta \in \Theta\}$ has a monotone likelihood ratio in T. Then for all $\delta^- < \delta^+$ with $\delta^-, \delta^+ \in \Theta$, the GROW* E*-variable relative to Θ_1 and Θ_0 as in (18), is given by $E^* = p_{\delta^+}(\mathbf{Y})/p_{\delta^-}(\mathbf{Y})$: it sets W_1^* and W_0^* to be degenerate priors, putting all their mass on δ^+ and δ^-, respectively.*

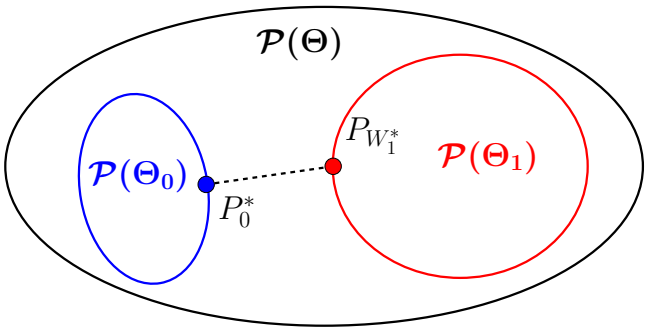

Figure 1: The Joint Information Projection (JIPr). Θ_0 and Θ_1 represent two non-overlapping models. $\mathcal{P}(\Theta') = \{P_W : W \in \mathcal{W}(\Theta')\}$, and $\mathcal{W}(\Theta')$ is the set of all priors over Θ'. Theorem 1 implies that the GROW E-variable between Θ_0 and Θ_1 is given by $P_{W_1^*}/P_0^*$, the Bayes factor between the two Bayes marginals that minimize KL divergence $D(P_{W_1}\|P_{W_0})$.

We now illustrate Proposition 3 for 1-dimensional exponential families, but stress that it can be applied to some other families (e.g. location families or the t-test setting in Section 4.3) as well.

Example 4. [GROW for 1-dimensional exponential families] Let $\{P_\theta \mid \theta \in \Theta\}$ represent a 1-parameter exponential family for sample space $\mathcal{Y}$, given in its mean-value parameterization, where Θ is a connected subset of (and possibly equal to) the full mean-value parameter space. Let $\delta^- < \delta^+$ with δ^-, δ^+ both in Θ. Both $\mathcal{H}_0 = \{P_\theta : \theta \in \Theta_0\}$ and $\mathcal{H}_1 = \{P_\theta : \theta \in \Theta_1\}$ with Θ_0, Θ_1 as in (18) are extended to outcomes in $\mathbf{Y} = (Y_1, \ldots, Y_n)$ by independence. Let $T = t(\mathbf{Y})$ be the sufficient statistic of the exponential family under consideration, i.e. $\mathbf{E}_{\mathbf{Y}\sim P_\theta}[t(\mathbf{Y})] = \theta$. It is well-known that the monotone likelihood property holds in the statistic T. It thus follows from Proposition 3 above that the GROW E-variable relative to Θ_1 and Θ_0 can be calculated as a likelihood ratio $E^* = p_{\delta^+}(\mathbf{Y})/p_{\delta^-}(\mathbf{Y})$ between two point hypotheses, even though Θ_1 and/or Θ_0 may be composite. Comparison of the ensuing test to the Neyman-Pearson and Bayes factor tests are given in Section 6.

4 The REGROW E-Variable: General Composite $\mathcal{H}_1$-Case

Theorem 1, Further Generalization *Let $f(\theta)$ be a function that is bounded on Θ_1; we abbreviate $f(W) := \mathbf{E}_{\theta\sim W}[f(\theta)]$. Suppose all $P_\theta, P_{\theta'}$ with $\theta, \theta' \in \Theta_1$ satisfy $D(P_\theta\|P_{\theta'}) < \infty$, and have full support. If $\inf_{W_1\in\mathcal{W}(\Theta_1), W_0\in\mathcal{W}(\Theta_0)}(D(P_{W_1}\|P_{W_0}) - f(W_1)) = \inf_{W_0\in\mathcal{W}(\Theta_0)} D(P_{W_1^*}\|P_{W_0}) - f(W_1^*) < \infty$ then then there exists an E-variable E^f given by*

$$E^f := \frac{p_{W_1^*}(\mathbf{Y})}{p_0^*(\mathbf{Y})} \tag{19}$$

where p_0^* is the density of P_0^*, a (potentially sub-) distribution such that $\inf_{W_0\in\mathcal{W}(\Theta_0)} D(P_{W_1^*}\|P_{W_0}) = D(P_{W_1^*}\|P_0^*)$, and E^f satisfies, essentially uniquely:

$$\inf_{\theta\in\Theta_1}\ (\ \mathbf{E}_{\mathbf{Y}\sim P_\theta}[\log E^f] - f(\theta)\ ) = \sup_{E\in\mathcal{E}(\Theta_0)} \inf_{\theta\in\Theta_1}\ (\ \mathbf{E}_{\mathbf{Y}\sim P_\theta}[\log E] - f(\theta)\ ) =$$
$$D(P_{W_1^*}\|P_0^*) - f(W_1^*). \tag{20}$$

If further $D(P_{W_1^*}\|P_0^*) = D(P_{W_1^*}\|P_{W_0^*})$ *for some* $W_0^* \in \mathcal{W}(\Theta_0)$*, then* $P_0^* = P_{W_0^*}$.
We call E^f the REGROW (standing for *relative* growth in the worst-case) E-variable relative to *offset* f. The previous version of Theorem 1 is the special case with f constant. The offset f will be useful when Θ_0 and Θ_1 are nested and no effect size can be stated in advance (Section 4.1) and/or when nuisance parameters are present (Section 4.2 and 4.3). All these cases can be handled essentially the same way (and we may in fact think of the case of nested models as a situation in which *all* parameters in Θ_1 are viewed as nuisance): we first consider a modified problem in which Θ_1 is reduced to a singleton. That is, we imagine that some oracle tells us "if $\mathcal{H}_1$ is the case, then the data are sampled from this specific θ_1^*. We then consider the corresponding $\text{GRO}(\theta_1^*) = \text{GROW}(\{\theta_1^*\})$ and view this as the desirable but unobtainable expected growth rate — the one we could have obtained if we had known θ_1^*. We may now aim for the E-variable such that, no matter what θ_1^* turns out to be, our expected growth is close to the optimum we could have obtained had we known θ_1^*. Thus, we want to be worst-case growth optimal *relative* to $f(\theta_1) := \text{GRO}(\theta_1) = \mathbf{E}_{P_{\theta_1}}[\log E^*_{\theta_1}] = \inf_{W_0 \in \mathcal{W}(\Theta_0)} D(P_{\theta_1}\|P_{W_0})$ (where we write $E^*_{\theta_1}$ for the GRO E-variable for testing $\{\theta_1\}$ vs. Θ_0 and the second equality follows by (12)). Plugging in this f and taking negatives on both sides, (20) now becomes:

$$\sup_{\theta_1 \in \Theta_1} \mathbf{E}_{\mathbf{Y}\sim P_{\theta_1}}\left[\log E^*_{\theta_1} - \log E^f\right] = \inf_{E \in \mathcal{E}(\Theta_0)} \sup_{\theta_1 \in \Theta_1} \mathbf{E}_{\mathbf{Y}\sim P_{\theta_1}}[\log E^*_{\theta_1} - \log E] = \mathbf{E}_{\theta_1 \sim W_1^*}\left[\inf_{W_0 \in \mathcal{W}(\Theta_0)} D(P_{\theta_1}\|P_{W_0})\right] - D(P_{W_1^*}\|P_0^*), \quad (21)$$

an expression that is always nonnegative, since, by definition of $E^*_{\theta_1}$, for any E-variable E, $\mathbf{E}_{\mathbf{Y}\sim P_{\theta_1}}[\log E^*_{\theta_1}] \geq \mathbf{E}_{\mathbf{Y}\sim P_{\theta_1}}[\log E]$. This shows that E^f can be thought of as a *minimax pseudo-regret* E-variable, regret being the loss of expected capital growth under $\mathcal{H}_1$ due to not knowing the underlying θ_1 in advance.

4.1 Composite $\mathcal{H}_1$, No Effect Size Known

Suppose we are interested in detecting whether there is any deviation at all from the null. There is no pre-stated effect size, and $\Theta_0 \subset \Theta_1 = \Theta$ are nested, or more generally, for all $\theta_1 \in \Theta_1$, $\inf_{\theta_0 \in \Theta_0} D(P_{\theta_1}\|P_{\theta_0}) = 0$. In this case, $\text{GROW}(\Theta_1) = 0$ and the GROW E-variable that achieves it is equal to $E^* = 1$, which will never give any evidence against $\mathcal{H}_0$, so clearly, the raw GROW approach is not useful. Instead, in this setting, the REGROW approach is a sensible generalization of successful existing approaches. We first establish this for simple nulls:

Simple Nulls If $\Theta_0 = \{0\}$ is simple, we have $\inf_{W_0} D(P_{\theta_1}\|P_{W_0}) = D(P_{\theta_1}\|P_0)$ and $D(P_{W_1^*}\|P_{W_0^*}) = D(P_{W_1^*}\|P_0)$, and terms in (21) involving $-\log p_0(\mathbf{Y})$ cancel. Further using the 1-to-1 mapping (9) between probability densities and E-variables for the case of point 0's to rewrite (21) and using $E^f = p_{W_1^*}(\mathbf{Y})/p_0(\mathbf{Y})$, (21) simplifies to:

$$\sup_{\theta \in \Theta_1} \mathbf{E}_{\mathbf{Y}\sim P_\theta}\left[-\log \frac{p_{W_1^*}(\mathbf{Y})}{p_\theta(\mathbf{Y})}\right] = \inf_q \sup_{\theta \in \Theta_1} \mathbf{E}_{\mathbf{Y}\sim P_\theta}\left[-\log \frac{q(\mathbf{Y})}{p_\theta(\mathbf{Y})}\right] = \sup_{W_1 \in \mathcal{W}(\Theta_1)} \mathbf{E}_{\theta \sim W_1}\left[D(P_\theta\|P_{W_1})\right], \quad (22)$$

where the infimum is over all sub-probability densities q over $\mathbf{Y}$. (22) is just the celebrated redundancy-capacity theorem (Cover and Thomas, 1991) of information theory, and it has a data-compression interpretation. In a nutshell, for any E-variable of the form $p_{W_1}(\mathbf{Y})/p_0(\mathbf{Y})$, the log of the evidence $\log p_{W_1}(\mathbf{Y})/p_0(\mathbf{Y})$ can be thought of as a difference between the code length needed to code the data using two lossless codes, one with lengths $-\log p_{W_1}$, associated with $\mathcal{H}_1$, and one with lengths $-\log p_0$, associated with $\mathcal{H}_0$. (22) expresses that when choosing $W_1 = W_1^*$, one associates $\mathcal{H}_1$ with the code that minimizes worst-case redundancy (the additional expected number of bits needed compared to an encoder that knows θ_1^*). This is in accordance with the *MDL* (Minimum Description Length) Principle, in which the code length difference between the same two codes is used to measure evidence (Rissanen, 1989, Barron et al., 1998, Grünwald and Roos, 2020).

Example 5. [Exponential Families with a point null: Jeffreys' Prior on Θ_1] To make this more concrete, let $\{P_\theta : \theta \in \Theta\}$ represent a d-dimensional exponential family given in either the mean or the canonical parameterization. We restrict the parameter set to Θ_1 that is a compact subset of the interior of Θ and let Θ_0 be a singleton subset in the interior of Θ_1. By standard properties of exponential families, the finite KL condition of Theorem 1 applies and the problem reduces to finding the prior W_1^* on Θ_1 that satisfies (22). (Clarke and Barron, 1994) showed that, for large n, this prior converges in an L_1-sense to Jeffreys' prior ('least favorable under entropy loss') , which is the main reason for adopting it in MDL model selection. They also showed that (22) and hence (21) is of size $(d/2)\log n + O(1)$. Thus, for point nulls and suitably truncated parameter spaces, this approach is consistent with the MDL Principle and with objective Bayes approaches based on Jeffreys prior.

Example 6. [2×2 Tables, Continued] If Θ_0 is not a singleton then the simplification of (21) to (22) is not possible, and numerical simulation can be used to determine (21) and the priors appearing therein. Consider for example the 2×2 model, but now with unrestricted $\Theta_1 = (0,1)^2$. This does satisfy the regularity conditions needed for Theorem 1 to be applicable (see Appendix A.3), but it has Θ_1 2-dimensional and Θ_0 1-dimensional. We saw in the previous example that for 1-vs. 0-dimensional exponential family models, (21) would take on value $(1/2)\log n + O(1)$, which suggests that it is the same here, for 2- vs. 1-dimensional, and this is indeed suggested by numerical simulations (Turner et al., 2021).

4.2 Composite $\mathcal{H}_1$, Nuisance Parameters Present

We now consider the common situation of models that can be parameterized by $\Theta = \{(\delta, \gamma) : \delta \in \Delta, \gamma \in \Gamma\}$ where δ is a single parameter of interest (for simplicity taken to be a scalar) and γ represents a nuisance parameter (scalar or vector). As in Section 3.1, we suppose that we want to test whether $\delta \geq \delta^+$ or $\delta \leq \delta^-$ for some $\delta^- < \delta^+$. We thus let

$$\Theta_0 = \{(\delta, \gamma) : \delta \leq \delta^-, \gamma \in \Gamma\}, \quad \text{vs.} \quad \Theta_1 = \{(\delta, \gamma) : \delta \geq \delta^+, \gamma \in \Gamma\}. \tag{23}$$

We will first consider the simplified problem in which we test $\Theta_{0,\delta^-} := \{(\delta^-, \gamma) : \gamma \in \Gamma\}$ vs. $\Theta_{1,\delta^+} := \{(\delta^+, \gamma) : \gamma \in \Gamma\}$, and later extend to (23). This simplified problem can be handled via a REGROW E-variable just like in the previous subsection, with now $\theta = (\delta^+, \gamma)$: we take $f((\delta^+, \gamma)) = \text{GRO}((\delta^+, \gamma))$ and apply Theorem 1 as in (21). This gives an E-variable $E^*_{\delta^+} := p_{W_1^*}(\mathbf{Y})/p_0^*(\mathbf{Y})$ with W_1^* a prior on $\{(\delta^+, \gamma) : \gamma \in \Gamma\}$. Using this REGROW rather than GROW approach reflects a particular interpretation of what it means for a parameter γ

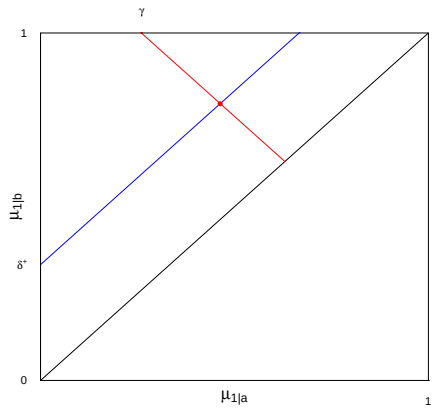

Figure 2: The 2×2 model. The diagonal represents the null, the decreasing line the set of parameters with nuisance parameter $\gamma = 1/4$ and the blue increasing line is Θ_{1,δ^+} for $\delta^+ = 1/3$.

to be nuisance: we have no idea of what the true γ might be and are therefore prepared to incur the same expected loss of growth for not knowing γ, irrespective of what γ is. Solving this problem for all $\delta^+ \geq \delta^-$ gives us a collection of E-variables $\mathcal{E}_{\geq \delta^-} := \{E^*_\delta : \delta \geq \delta^-\}$. Now suppose there exists another E-variable E^* such that

$$\sup_{E \in \mathcal{E}_{\geq \delta^-}} \inf_{\theta \in \Theta_1} \mathbf{E}_{P_\theta}[\log E] = \inf_{\theta \in \Theta_1} \mathbf{E}_{P_\theta}[\log E^*] \tag{24}$$

That is, we pick the worst-case optimal E-variable among $\mathcal{E}_{\geq \delta^-}$, thereby applying GROW on a meta-level as it were, after restricting ourselves to E-variables that are themselves REGROW for fixed δ and unknown γ. This E^* is then our choice for solving the original problem (23).

Example 7. [2×2 Tables, Continued] We can reparameterize $\{P_{\mu_{1|a},\mu_{1|b}} : (\mu_{1|a}, \mu_{1|b}) \in [0,1]^2\}$ as $\Theta = \{(\delta, \gamma) : \delta \in \Delta,\ \gamma \in [0,1]\ \}$ using $\gamma = (n_a \mu_{1|a} + n_b \mu_{1|b})/(n_a + n_b)$ as a nuisance parameter: the marginal probability of observing a 1. For δ we can take, for example, $\delta = \mu_{1|b} - \mu_{1|a}$ to be our notion of effect size, the *substantive difference*, with $\Delta = [-1, 1]$. Another popular choice, like substantive difference considered by Adams (2020), Turner et al. (2021) is $\delta = \log(\mu_{1|b}/(1-\mu_{1|b})) \cdot (1-\mu_{1|a})/\mu_{1|a}$, i.e. the log-odds ratio, but for simplicity we stick to the substantive difference here. We take a Θ_1 and Θ_0 relative to some effect size δ^+ and δ^- as in (23) above, where for simplicity we will take $\delta^- = 0$ and $\Delta = [0, 1]$ and also $n_a = n_b$ so that $\gamma = (\mu_{1|a} + \mu_{1|b})/2$. The situation is depicted in Figure 2, where we took, as an example, Θ_1 and Θ_0 defined relative to $\delta^+ = 1/3$ and $\delta^- = 0$.

Now, assume first that the true value of γ were given in advance. We would then be dealing with a one-parameter exponential family model represented by a straight decreasing line in Figure 2; the Figure illustrates this for $\gamma = 1/4$. We would then be in the situation of Section 3.1, Example 4, and find, for any δ^+, that $\text{GRO}((\delta^+, \gamma)) = \inf_{W_0 \in \mathcal{W}(\Theta_0)} D(P_{\delta^+,\gamma} \| P_{W_0}) = D(P_{\delta^+,\gamma} \| P_{0,\gamma})$ where the latter equality was already stated as (14) in Example 2.

Now we look at general, unknown γ. As suggested above, we first set $\delta := \delta^+$ and test $\Theta_{1,\delta}$ vs. $\Theta_{1,0}$ (the increasing lines in Figure 2), taking the REGROW E-variable relative to $f((\delta, \gamma)) = \text{GRO}((\delta, \gamma)) = D(P_{\delta,\gamma} \| P_{0,\gamma})$. Then the minimum W_1^* on $\Theta_{1,\delta}$ and W_0^* (with W_1^*

putting mass 1 on δ and spreading its mass over γ, and $P_0^* = P_{W_0^*}$) as in (21) are achieved and have finite support, and the finite KL condition of the theorem applies (see Appendix A.3). This solves the problem for testing $\Theta_{1,\delta}$ vs. $\Theta_{1,0}$ for $\delta = \delta^+$; by varying δ we get a collection of E-variables $\mathcal{E}_{\geq 0}$ containing an E_δ^* for each fixed $\delta \geq 0$. We then pick the E^* among $\mathcal{E}_{\geq 0}$ satisfying (24), which turns out to be equal to $E_{\delta^+}^*$, i.e. it is has a point mass on δ^+ again.

Discussion In our examples, we have so far used (and will keep using in the next section) the REGROW approach to first eliminate nuisance parameters, if they are present, followed by a GROW approach for the parameters of interest. We find in the application papers (Turner et al., 2021, Ter Schure et al., 2021) that this gives intuitive E-variables that perform well in practice, not just directly in the GRO sense but also in terms of secondary measures such as a power analysis (Section 6) or as the basis of anytime-valid confidence intervals (Turner et al., 2021). Still, it may not always be the best way to go. For example, a REGROW approach for the parameter of interest when a minimum effect size is given may sometimes be sensible as well. Let us consider this a bit further for (for simplicity) the case with a minimum effect size δ^+ but without nuisance parameters, as in Example 4. REGROW would amount here to using $E_{W_1}^* = p_{W_1}(\mathbf{Y})/p_{\delta^-}(\mathbf{Y})$ with prior W_1 spread out on the set $\Theta_1 = \{\delta : \delta \geq \delta^+\}$. Then we would get

$$\mathbf{E}_{P_\delta}[\log E_{W_1}^*] < \text{GROW}(\Theta_1) \text{ for } \delta \text{ close to } \delta^+ \; ; \mathbf{E}_{P_\delta}[\log E_{W_1}^*] \gg \text{GROW}(\Theta_1) \text{ for } \delta \gg \delta^+$$

so we would win if we are 'lucky' and it turns out that $\delta \gg \delta^+$. However, in practice we often deal with small sample sizes, and δ's that may very well be very close to δ^+. Then (as our experiments done for the papers above indicate) the difference in '$>$ ' above is non-negligible, and the GROW approach seems safer, since for the GROW E-variable we automatically have $\mathbf{E}_{P_\delta}[\log E_{W_1}^*] \geq \text{GROW}(\Theta_1)$ for all $\delta \in \Theta_1$.

4.3 Theorem 1 in Full: Application to Bayesian and Sequential t-test

If we try to apply Theorem 1 as above to the t-test, a prototypical nuisance setting, we run into the issue that the minimum KL is not achieved. This problem can be solved by extending the theorem further, allowing for densities on a *coarsening* of $\mathbf{Y}$. This is any random variable $\mathbf{V}$ that can be written as a function of $\mathbf{Y}$, i.e. $\mathbf{V} = g(\mathbf{Y})$ for some function g; we retrieve the previous version of Theorem 1 if we take g the identity and $\mathbf{V} = \mathbf{Y}$. We now present Theorem 1 in full generality, allowing for such coarsening and additionally for considering the best E-variable on a modified $\mathcal{H}_1$, consisting of any convex set of Bayes marginal distributions with priors on Θ_1. This is needed for accommodating the t-test. It also allows us to incorporate *robust Bayesian* settings (Berger, 1985, Grünwald and Dawid, 2004), but we will not further pursue those here.

In the theorem we use the following notation: for (sub-) distribution P for $\mathbf{Y}$, $P^{[\mathbf{V}]}$ denotes the marginal (sub-) distribution of P for $\mathbf{V}$, and p' denotes its density.

Theorem 1, Full Generality *Let $f(\theta)$ be a function that is bounded on Θ_1. Suppose all $P_\theta, P_{\theta'}$ with $\theta, \theta' \in \Theta_1$ satisfy $D(P_\theta \| P_{\theta'}) < \infty$, and have full support. Let $\mathcal{W}_1$ be a convex*

subset of $\mathcal{W}(\Theta_1)$*. If for some coarsening* $\mathbf{V}$ *of* $\mathbf{Y}$ *the following holds:*

$$\inf_{W_1 \in \mathcal{W}_1} \inf_{W_0 \in \mathcal{W}(\Theta_0)} \left(D(P_{W_1} \| P_{W_0}) - f(W_1) \right) =$$
$$\min_{W_1 \in \mathcal{W}_1} \inf_{W_0 \in \mathcal{W}(\Theta_0)} \Big(D(P^{[\mathbf{V}]}_{W_1} \| P^{[\mathbf{V}]}_{W_0}) - f(W_1) \Big) = \inf_{W_0 \in \mathcal{W}_0(\Theta_0)} D(P^{[\mathbf{V}]}_{W_1^*} \| P^{[\mathbf{V}]}_{W_0}) - f(W_1^*) < \infty, \quad (25)$$

then there exists an E*-variable*

$$E^f := \frac{p'_{W_1^*}(\mathbf{V})}{p_0^{*'}(\mathbf{V})}, \quad (26)$$

where $p_0^{*'}$ *is the density of* $P_0^{*[\mathbf{V}]}$*, a (potentially sub-) distribution for* $\mathbf{V}$ *that satisfies* $\inf_{W_0 \in \mathcal{W}(\Theta_0)} D(P_{W_1^*} \| P_{W_0}) = D(P^{[\mathbf{V}]}_{W_1^*} \| P_0^{*[\mathbf{V}]})$*.* E^f *satisfies, essentially uniquely:*

$$\inf_{W \in \mathcal{W}_1} \left(\mathbf{E}_{\mathbf{Y} \sim P_W}[\log E^f] - f(W) \right) = \sup_{E \in \mathcal{E}(\Theta_0)} \inf_{W \in \mathcal{W}_1} \left(\mathbf{E}_{\mathbf{Y} \sim P_W}[\log E] - f(W) \right) =$$
$$D(P^{[\mathbf{V}]}_{W_1^*} \| P_0^{*[\mathbf{V}]}) - f(W_1^*). \quad (27)$$

If further $D(P^{[\mathbf{V}]}_{W_1^*} \| P_0^{*[\mathbf{V}]}) = D(P^{[\mathbf{V}]}_{W_1^*} \| P^{[\mathbf{V}]}_{W_0^*})$ *for some* $W_0^* \in \mathcal{W}(\Theta_0)$*, then* $P_0^{*[\mathbf{V}]} = P^{[\mathbf{V}]}_{W_0^*}$*.*

The previous version of Theorem 1 is the special case obtained by setting $\mathcal{W}_1 = \mathcal{W}(\Theta_1)$, $\mathbf{Y} = \mathbf{V}$ and using linearity of expectation. We call E^f as in (27) the REGROW E-variable relative to offset f and set of priors $\mathcal{W}_1$. If f is constant (no offset), we call it $\mathcal{W}_1$*-GROW*, noting that it gives the worst-case optimal growth rate under all priors in $\mathcal{W}_1$.

The *t*-test Setting We return to the setting with a nuisance parameter with notation as in Section 4.2. Jeffreys (1961) proposed a Bayesian version of the t-test; see also (Rouder et al., 2009). We start with the models $\mathcal{H}_0$ and $\mathcal{H}_1$ for data $\mathbf{Y} = (Y_1, \ldots, Y_n)$ given as $\mathcal{H}_0 = \{P_{0,\sigma}(\mathbf{Y}) \mid \sigma \in \Gamma\}$; $\mathcal{H}_1 = \{P_{\delta,\sigma}(\mathbf{Y}) \mid (\delta, \sigma) \in \Theta_1\}$, where $\Delta = \mathbb{R}, \Gamma = \mathbb{R}^+$, $\Theta_1 := \Delta \times \Gamma$ and $\Theta_0 = \{(0, \sigma) : \sigma \in \Gamma\}$, and $P_{\delta,\sigma}$ has density (with $\overline{y} = \frac{1}{n}\sum_{i=1}^n y_i$)

$$p_{\delta,\sigma}(y) = \frac{1}{(2\pi\sigma^2)^{n/2}} \cdot \exp\left(-\frac{n}{2} \left[\left(\frac{\overline{y}}{\sigma} - \delta \right)^2 + \left(\frac{\frac{1}{n}\sum_{i=1}^n (y_i - \overline{y})^2}{\sigma^2} \right) \right] \right),$$

Jeffreys proposed to equip $\mathcal{H}_1$ with a Cauchy prior $W^{\mathrm{C}}[\delta]$ on the *effect size* δ, and both $\mathcal{H}_1$ and $\mathcal{H}_0$ with the scale-invariant prior measure with density $w^H(\sigma) \propto 1/\sigma$ on the variance. The same formula with the same prior on σ but other priors on δ was suggested by Lai (1976) with a non-Bayesian, martingale interpretation. Below we will see that, even though $w^H(\sigma)$ is improper (whereas the priors appearing in Theorem 1 are invariably proper), the resulting Bayes factor E^* is an E-variable. We then present Theorem 2 which shows that, for priors $W[\delta]$ with more than 2 moments, E^* in fact even is $\mathcal{W}_1$-GROW with $\mathcal{W}_1$ the set of all priors on $\delta \times \sigma$ with marginal δ, i.e. it has a worst-case optimal growth rate property relative to all distributions in $\mathcal{H}_1$ compatible with $W[\delta]$. Thus, a form of GROW-optimality holds for most priors $W[\delta]$ one might want to use, including standard choices (such as a standard normal) or the point prior we will suggest further below — but we do not know if it holds for the moment-less Cauchy proposed by Jeffreys.

Almost Bayesian Case: prior on δ available For any proper prior distribution $W[\delta]$ on δ and any proper prior distribution $W[\sigma]$ on σ, we define

$$p_{W[\delta],W[\sigma]}(y) = \int_{\delta\in\Delta}\int_{\sigma\in\Gamma} p_{\delta,\sigma}(y)\,\mathrm{d}W[\delta]\,\mathrm{d}W[\sigma],$$

as the Bayes marginal density under the product prior $W[\delta]\times W[\sigma]$.

For convenience later on we set the sample space to be $\mathcal{Y}^n = (\mathbb{R}\setminus\{0\})\times\mathbb{R}^{n-1}$, assuming beforehand that the first outcome will not be 0. Now we define $\mathbf{V} := (V_1,\ldots,V_n)$ with $V_i = Y_i/|Y_1|$. We have that $\mathbf{Y}$ determines $\mathbf{V}$, and $(\mathbf{V},Y_1)$ determines $\mathbf{Y} = (Y_1,Y_2,\ldots,Y_n)$. The distributions in $\mathcal{H}_0\cup\mathcal{H}_1$ can thus alternatively be thought of as distributions on the pair $(\mathbf{V},Y_1)$. $\mathbf{V}$ is "$\mathbf{Y}$ with the scale divided out": as is well-known (Lai, 1976, Berger et al., 1998) and easily shown (e.g. as a consequence of (29) below), under all $P\in\mathcal{H}_0$, i.e. all $P_{0,\sigma}$ with $\sigma>0$, $\mathbf{V}$ has the same distribution $P_0[\mathbf{V}]$ with density p'_0. In the same way, one shows that under all $P_{W[\delta],\sigma}$ with $\sigma>0$, $\mathbf{V}$ has the same pdf $p'_{W[\delta]}$ (which therefore does not depend on the prior on σ). We now get that, for all $\sigma>0$,

$$E^* := \frac{p'_{W[\delta]}(\mathbf{V})}{p'_0(\mathbf{V})} \tag{28}$$

satisfies $\mathbf{E}_{\mathbf{V}\sim P}[E^*]=1$ for all $P\in\mathcal{H}_0$, hence it is an E variable. Remarkably, this 'scale-free' E-variable coincides with the Bayes factor one gets if one uses, for σ, the prior $w^H(\sigma)=1/\sigma$ suggested by Jeffreys, and treats σ and δ as independent. That is (Lai, 1976, page 273) (since Lai does not give any details, we give a full derivation in Appendix B.4), we have

$$\frac{\int_\sigma p_{W[\delta],\sigma}(\mathbf{Y})w^H(\sigma)\,\mathrm{d}\sigma}{\int_\sigma p_{0,\sigma}(\mathbf{Y})w^H(\sigma)\,\mathrm{d}\sigma} = \frac{p'_{W[\delta]}(\mathbf{V})}{p'_0(\mathbf{V})} = E^*. \tag{29}$$

Despite its improperness, w^H induces a valid E-variable when used in the Bayes factor. The equivalence of this Bayes factor to E^* simply means that it manages to ignore the 'nuisance' part of the model and models the likelihood of the scale-free $\mathbf{V}$ instead. The reason this is possible is that w^H coincides with the right-Haar prior for this problem (Eaton, 1989, Berger et al., 1998), about which we will say more below. Amazingly, it turns out that the E-variable (29) has a GROW property (among all E-variables for data $\mathbf{Y}$, not just the coarsened $\mathbf{V}$!) under the weak condition that the prior $W[\delta]$ has a $(2+\epsilon)$th moment. This follows from our second main result, Theorem 2 below, whose (complicated) proof, to keep paper length at bay, is deferred to another paper (for now it can be found in the second arXiv version of the present paper). Let, for priors $W[\delta], W[\sigma]$, $P^{[\mathbf{V}]}_{W[\delta],W[\sigma]}$ be the marginal distribution on $\mathbf{V}$, i.e. the distribution with density $p'_{W[\delta],W[\sigma]}$.

Theorem 2. *Let $W[\delta]$ be a distribution on δ such that $\mathbf{E}_{\delta\sim W[\delta]}[|\delta|^{2+\epsilon}]<\infty$ for some $\epsilon>0$ (in particular this includes all degenerate priors with mass 1 on a single δ). Let $\mathcal{W}[\Gamma]$ be the set of all distributions $W[\sigma]$ on the variance σ. Let $\mathcal{W}_1$ be the set of all probability distributions on $\delta\times\sigma$ such that, for each $W'\in\mathcal{W}_1$, its marginal on δ coincides with $W[\delta]$. We then have:*

$$\inf_{W\in\mathcal{W}_1}\inf_{W[\sigma]\in\mathcal{W}[\Gamma]} D(P_W\|P_{0,W[\sigma]}) = \inf_{W[\sigma]\in\mathcal{W}(\Gamma)} D(P_{W[\delta],W[\sigma]}\|P_{0,W[\sigma]}) = D(P^{[\mathbf{V}]}_{W[\delta]}\|P^{[\mathbf{V}]}_0). \tag{30}$$

The theorem allows us to use Theorem 1 as above with constant $f(\delta, \sigma) = 0$ (note that $\mathcal{W}_1$ is convex) to conclude that E^* as in (28) is equal to E^f as in (27): the Bayes factor based on the right Haar prior, is not just an E-variable, but is even GROW relative to the set of all priors on $\delta \times \sigma$ that are compatible with $W[\delta]$.

REGROW-GROW safe t-test with minimum effect sizes Suppose we want to test Θ_1 vs. Θ_0 as in (23) with fixed effect sizes δ^+ and δ^- and with σ^2 in the role of γ. We proceed exactly as we did underneath (23): we first consider the test $\{(\delta^+, \sigma^2) : \sigma^2 > 0\}$ vs. $\{(\delta^-, \sigma^2) : \sigma^2 > 0\}$ for the fixed given δ^+ using the REGROW criterion with $f((\delta, \sigma)) = \text{GRO}(\delta, \sigma)$. We have (Koolen and Grünwald, 2021, Section 4.3) that $f(\delta^+, \sigma) = (n/2)\log(1 + \delta^{+2})$ is constant on σ. Therefore we can use Theorem 1 in its most general form above in combination with Theorem 2 (applied with point prior $W[\delta]$ on δ^+) to conclude that the REGROW E-variable is given by $E^*_{\delta^+} := p'_{\delta^+}(\mathbf{V})/p'_{\delta^-}(\mathbf{V})$. Since Proposition 3 is applicable to sets of distributions defined on $\mathbf{V}$ rather than $\mathbf{Y}$ (details in Appendix B.4), we find that, with $\mathcal{E}_{\geq \delta^+} := \{E^*_\delta : \delta \geq \delta^-\}$ that

$$\sup_{E \in \mathcal{E}_{\geq \delta^+}} \inf_{\sigma > 0, \delta \geq \delta^+} \mathbf{E}_{\mathbf{Y} \sim P_{\delta,\sigma}}[\log E] = \inf_{\sigma > 0, \delta \geq \delta^+} \mathbf{E}_{\mathbf{Y} \sim P_{\delta,\sigma}}[\log E^*_{\delta^+}]$$

so $E^*_{\delta^+}$ may be thought of as first applying REGROW, to get rid of the nuisance parameter, and then applying GROW — just like in the 2×2 Example 7.

Extension to General Group Invariant Bayes Factors In a series of papers (Berger et al., 1998, Dass and Berger, 2003, Bayarri et al., 2012), Berger and collaborators developed a theory of Bayes factors for $\mathcal{H}_0 = \{P_{0,\gamma} : \gamma \in \Gamma\}$ and $\mathcal{H}_1 = \{P_{\delta,\gamma} : \delta \in \Delta, \gamma \in \Gamma\}$ with a nuisance parameter (vector) γ that appears in both models and that satisfies a group invariance; the Bayesian t-test is the special case with $\gamma = \sigma, \Gamma = \mathbb{R}^+$ and with the scalar multiplication group and δ an 'effect size'. Other examples include regression based on mixtures of g-priors (Liang et al., 2008), testing a Weibull vs. the log-normal and many more Dass and Berger (2003). The reasoning of the first part of this section straightforwardly generalizes to all such cases: under some conditions on the prior on δ, the Bayes factor based on using the right Haar measure on γ in both models gives rise to an E-variable. We furthermore *conjecture* that in such testing problems, perhaps under some additional conditions, the resulting Bayes factor is even GROW relative to a suitably defined set $\mathcal{W}_1$; i.e. that a suitable analogue of Theorem 2 holds. Finding out whether this really is the case is a major goal for future work.

5 (RE)GRO(W), Optional Continuation and Stopping

We now address two related questions:

1. The GRO-criteria were chosen to optimize expected capital (logarithmic) growth 'locally', within a study. How well do GRO-criteria go together with optional continuation (OC) over several studies?

2. We focused on Type-I error safety under OC. Can we also get safety under *optional stopping* (OS), and what is the difference?

To make the questions concrete, consider the simplified set-up with a data stream $Y_1, Y_2, \ldots$ of which we observe subsequent batches, as in Example 1. We start with the filtration $(\mathcal{G}_n)_n$ on the underlying $Y_1, Y_2, \ldots$ with $\mathcal{G}_n = \sigma(Y^n)$. A conditional E-process is now determined by a finite or infinite sequence of almost-surely finite stopping times $0 = T_{(0)} < T_{(1)} < T_{(2)} < \ldots$ with $N_{(j)} := T_{(j)} - T_{(j-1)}$. These stopping times define a new filtration $\mathcal{F}_{(1)} := \sigma(Y^{T_{(1)}}), \mathcal{F}_{(2)} := \sigma(Y^{T_{(2)}}), \ldots$, the sequence of stopped σ-algebras. We call this the filtration at the *study level*, since $\mathcal{F}_{(j)}$ denotes all information available after j studies or trials have been completed — it is the filtration referred to in Proposition 2.

On the other hand, all our GRO criteria are really *specifications* of E-variables: given sample size n, they output an E-variable $S^{[n]}$ that is invariably of the form $S^{[n]} = q_n(U^n)/p_n(U^n)$ where $U^n = (U_1, \ldots, U_n)$, U_i is a coarsening of Y^i and q_n and p_n are densities for U^n. Now let us define $S_i := S^{[i]}/S^{[i-1]}$. In many of our examples (see below), the process $(S_i)_{i \in \mathbb{N}}$ turns out to be itself a conditional E-process adapted to $(\sigma(U^n))_n$, i.e. it satisfies for all $P \in \mathcal{H}_0$,

$$\mathbf{E}_P[S_i \mid \sigma(U^{i-1})] \leq 1. \tag{31}$$

We call a conditional E-process like this a *batch size-1 conditional E-process*, and we call any E-variable specification that corresponds to a batch-size 1 process in this way *sequentially decomposable*. Note that if (31) holds for all i, then with $S^{[i]} = S_i S^{[i-1]}$, all the $S^{[i]}$ are easily shown to be E-variables for Y^i. Sequential decomposability is the converse property, which sometimes but not always holds as well.

When we use a GRO-criterion (or any other E-variable specification) in practice, at the study level, we need to, given as input initial data sequence $\mathbf{Y}^{(j-1)} = (Y_1, \ldots, Y_{T_{(j-1)}})$ and stopping time $T_{(j)}$, output a prescription for defining $E_{(j)}$. As the following proposition shows, for sequentially decomposable E-specifications, it is natural to set $E_{(j)} := \frac{S^{[T_{(j)}]}}{S^{[T_{(j-1)}]}} = \prod_{i=T_{(j-1)}+1}^{T_{(j)}} S_i$. We call this the *sequential* E-assignment. More generally though, we may use the *local* E-assignment, in which we set $E_{(j)}$ equal to

$$\frac{q_{N_{(j)}}(U_{T_{(j-1)}+1}, \ldots, U_{T_{(j)}})}{p_{N_{(j)}}(U_{T_{(j-1)}+1}, \ldots, U_{T_{(j)}})} = \prod_{i=T_{(j-1)}+1}^{T_{(j)}} S_{i,(j)} \text{ with } S_{i,(j)} := \frac{q_{N_{(j)}}(U_i \mid U_{T_{(j-1)}+1}, \ldots, U_{i-1})}{p_{N_{(j)}}(U_i \mid U_{T_{(j-1)}+1}, \ldots, U_{i-1})} \tag{32}$$

i.e. we apply the specification that is given for $N_{(j)}$ general data points running from $n = 1$ to $N_{(j)}$ to specifically the $N_{(j)}$ data points in the j-th study, ignoring the data from all previous studies.

Parts 1–3 of the following proposition gives sufficient conditions for assignments of E-variables in a stream to be valid; Part 4 is explained further on.

Proposition 4.

1. *If the specification of E-variables $S^{[1]}, S^{[2]}, \ldots$ is sequentially decomposable, then the $E_{(j)}$ defined by sequential assignment define a valid conditional E-process, i.e. for each j, each $P \in \mathcal{H}_0$,*
$$\mathbf{E}_P[E_{(j)} \mid \sigma(U^{T_{(j-1)}})] \leq 1. \tag{33}$$
2. *If the data $Y_1, Y_2, \ldots$ are i.i.d., then the $E_{(j)}$ defined by local assignment are valid, i.e.,*
$$\mathbf{E}_P[E_{(j)} \mid \sigma(Y^{T_{(j-1)}})] \leq 1. \tag{34}$$

holds for these $E_{(j)}$ as well.

3. *If the specification is sequentially decomposable* and *the data are i.i.d. then any mixed local/sequential use of the* E*-variable specification is valid, i.e. for each j, (33) holds if for this j the sequential assignment was used and (34) holds if the local assignment was used. Moreover ...*

4. *... in this case the batch-size-1 process $E_1, E_2, \ldots$ defined by setting, for each j, for each $i \in \{T_{(j-1)+1}, \ldots, T_{(j)}\}$, E_i to respectively S_i as in (31) if $E_{(j)}$ was defined by sequential assignment, and $E_i := S_{i,(j)}$ as in (32) if $E_{(j)}$ was defined by local assignment, is compatible with $E_{(1)}, E_{(2)}, \ldots$ (i.e. for all m we have $\prod_{j=1}^{m} E_{(j)} = \prod_{i=1}^{T_{(m)}} E_i$) and it is a conditional* E*-process relative to filtration $(U_n)_n$.*

There are (a) cases in which sequential and local assignment coincide and are both valid; (b) cases in which sequential assignment is different from local and they are both valid; and (c) cases in which they are different and only one of them is valid. Examples of (a) are the GROW E-variable in Example 4 with $U_i = Y_i$, with p_i and q_i given respectively by p_{δ^+} and p_{δ^-}, and also the case that both $\mathcal{H}_0$ and $\mathcal{H}_1$ are simple and data are i.i.d. Koolen and Grünwald (2021) give additional examples with i.i.d. data, simple $\mathcal{H}_1$ and multivariate $\mathcal{H}_0$, including but not limited to the case with $\mathcal{H}_0$ being *convex*.

Examples of (b) include the case where a prior W_1 on Θ_1 is available as in Section 2.2 yet $\Theta_0 = \{0\}$ is simple and the Y_i are i.i.d. as in Example 1. Then they are not i.i.d. any more under P_{W_1} and the W_1-GRO assignment applied sequentially will set $E_{(2)} = p_{W_1|Y_{(1)}}(\mathbf{Y}_{(2)})/p_0(\mathbf{Y}_{(2)})$ with $W_1|Y_{(1)}$ the posterior of W_1. The W_1-GRO assignment applied locally will set $E_{(2)} = p_{W_1}(\mathbf{Y}_{(2)})/p_0(\mathbf{Y}_{(2)})$ and re-use the prior, which may be unavoidable if earlier data is not yet available and sometimes in fact the preferred choice (see Example 1). Another example of (b) are the t-test prescriptions of Section 4.3. The U_i for the second batch $E_{(2)}$ are then equal to $Y_i/|Y_{T_{(1)}+1}|$ under local assignment and $Y_i/|Y_1|$ under sequential assignment. An example of (c) is the 2×2 setting in which (after appropriate extension of the definition to conditional distributions $p(Y_i \mid X_i)$) the local assignment is valid, but the global one is not.

Question 1: Local vs. Global GRO If we consider the whole eventually-to-be-observed data sequence as a single batch and use the desired GRO criterion for that ('global' GRO), do we get the same evidence as if we would use the criterion for any more-fine grained decomposition of the same data into several batches (studies) and multiply the resulting E-values? The definition of sequential decomposability implies that this coherence property must hold if the GRO criterion is sequentially decomposable and we also do the assignment sequentially — that is, in case (a) above both a local and a sequential assignment will give coherence, in case (b), only a sequential assignment will give coherence. For the 2×2 case, even with simple $\Theta_1 = \{\theta_1\}$ (so all GRO-criteria coincide) we do not have sequential decomposability and we do not have the coherence property. To see the latter, think of a first batch $\mathbf{Y}_{(1)}$ with $n_a = n_b = 1$ and a second batch $\mathbf{Y}_{(2)}$ with $n_a = 1, n_b = 2$ and use (14) first locally to get E-variables $E_{(1)}$ for the first batch and $E_{(2)}$ for the second batch and set $E = E_{(1)} \cdot E_{(2)}$. Then use (14) 'globally' on the joint batch with $n_a = 2$ and $n_b = 3$ to get E-variable E' for the first five outcomes, and note that $E \neq E'$. This incoherence is not a serious problem though — see below.

Question 2: OC vs. OS Recall from Corollary 1 in Section 1.3 that for any nonnegative random process $E^{(1)}, E^{(2)}, \ldots$ adapted to $(\mathcal{F}_{(m)})_{(m)}$ we say that the corresponding threshold test is *safe under OC* (with respect to Type-I error) if the Ville-Robbins inequality (7) holds. Extending this definition in the natural way, we say, for a *batch-size-1* process of nonnegative random variables $E^{[1]}, E^{[2]}, \ldots$, i.e. adapted to the finest level of data $(\mathcal{G}_n)_n$, that the corresponding threshold test is safe under OS (with respect to Type-I error) if again the Ville-Robbins inequality holds (with $E^{(n)} := E^{[n]}$).

Thus, in the simplified setting in which a steady stream of data arrives, the mathematical difference[1] between safety under OC and under OS is simply whether stopping at will while preserving error guarantees is possible only at the end of each study (OC) or whether we can also stop at any point within a study (OS): OC can then simply be viewed as OS at the 'level' of filtration $\mathcal{F}_{(n)}$ rather than $\mathcal{G}_{(n)}$. Note that in practice, the data within a study might come in unordered and then OS in our definition, i.e. at the level of $\mathcal{G}_{(n)}$, is undefined.

Clearly, by Corollary 1 the running products $S^{[1]}, S^{[2]}, \ldots$ of a batch-size-1 conditional E-process $S_1, S_2, \ldots$ then give tests that are safe under OS with respect to Type-I error. Part 4 of the proposition above gives an explicit sufficient condition for when the batch-size-1 process $E_1, E_2, \ldots$ that at all times $T_{(1)}, T_{(2)}, \ldots$ agrees (if evidence is combined by multiplication) with the study-level process $E_{(1)}, E_{(2)}, \ldots$ gives a batch-size-1 conditional E-process, thus providing safety under OS. We see that OS is possible under both local and global assignments under both Cases (a) and (b) above, but again not for the 2×2 setting.

The Main Open Question concerning GRO In practice we may very well be in a situation with a single data stream as above, OS is desirable (see the next section for why it would be), yet we want to use a GRO criterion for a setting such as the 2×2 case for which neither local nor sequential assignment allows for it. Even if OS is not required or desired, we would ideally like to achieve the global GRO criterion, which cannot be achieved if local and global GRO do not coincide such as again in the 2×2 case. We may then in some cases try the following approach, which for simplicity we only describe for the REGROW criterion: let E_n^f be the REGROW E-variable (19) achieving (20) with $f(\theta) = \mathrm{GRO}(\theta)$ for samples of size n. We try to find a sequence of conditional E-variables $E_1, E_2, \ldots$ such that E_i is a conditional E-variable for Y_i given Y^{i-1} and the product E-variable $E^{[n]} := \prod_{i=1}^n$ achieves (20) to within some fixed ϵ for all n, i.e.

$$\inf_{\theta \in \Theta_1} \left(\mathbf{E}_{\mathbf{Y} \sim P_\theta}[\log E^{[n]}] - \mathrm{GRO}(\theta) \right) \geq \inf_{\theta \in \Theta_1} \left(\mathbf{E}_{\mathbf{Y} \sim P_\theta}[\log E_n^f] - \mathrm{GRO}(\theta) \right) - \epsilon. \tag{35}$$

By construction, the conditional E-process $E_1, E_2, \ldots$ allows for optional stopping, and if we can find $E_1, E_2, \ldots$ such that ϵ is small for all n, , we can say that the process is 'almost' global REGROW in the desired sense.

Example 8. Turner et al. (2021) successfully use this idea for the 2×2 model. We illustrate this confining ourselves for simplicity to a stream of paired data, i.e. $X_1 = a, X_2 = b, X_3 = a, X_4 = b, \ldots$. First, we note that directly applying the idea above will not work. To see this,

[1] More generally, we refer to OC also when the decision of whether to start a new study at all, and if so, where to get the data (e.g. what hospital) and what E-variable to use are made just-in-time, only after previous studies were complete — a setting that seems quite different from those in which we usually speak about 'optional stopping'.

consider simple $\Theta_1 = \{(\mu_{1|a}.\mu_{1|b})\}$. According to the composite null, the Y_i are i.i.d. Bernoulli with parameter $\mu \in [0,1]$, but the only E-variable for such a $\mathcal{H}_0$ and Y_i is the trivial $E \equiv 1$. Thus we would get all E_i equal to 1, and zero growth. However, if we analyze the data in batches of size 2, so set $Y_1' = (Y_1, Y_2), Y_2' = (Y_3, Y_4)$, put a suitable prior W on $\Theta_1 = (0,1)^2$ and take as $E_i' = \mathrm{e}(Y_i')$ the $(\mu_{1|a}, \mu_{1|b})$-GRO-E-variable for $Y_i' = (Y_{2i-1}, Y_{2i})$ then $E'^{[n]} = \prod_{i=1}^n E_i'$ is the $(\mu_{1|a}, \mu_{1|b})$-GRO-E-variable for all n. Moreover, if we now consider the full alternative $\Theta_1 = (0,1)^2$ and put a suitable prior W on it, and take E_j' the $W \mid Y_{j-1}'$-GRO E-variable, $W \mid Y_{j-1}'$ being the posterior based on $Y_{(j-1)}'$, then we numerically find that $E'^{[n]} = \prod_{i=1}^n E_i'$ is, for all but the smallest n, very close to the REGROW E-variable E_n^f for that n, i.e. it achieves (35) for small ϵ.

The example raises an important question: under what conditions (on model, minimal batch sizes and the like) can we use a sequence of local GRO-E variables that behave optimally (as in the examples when local GRO is global GRO) or almost optimally (as in the 2×2 example above with batch size 2) in a global GRO sense? Answering this question more fully is a major avenue for future research, since it ultimately determines how widely applicable the GRO criteria really are.

6 Competitiveness: GRO and Power

How competitive are E-values compared to existing methods, in particular: what sample size should we minimally plan for in a study so that we may expect a useful result? The answer depends on whether one looks at E-values purely as measures of evidence, without an accept/reject decision attached, or whether one considers such decisions after all. We consider each in turn.

E-Values as Evidence E-values may be viewed simply as a measure of evidence, extending the evidential interpretation of likelihood ratios (Royall, 2000). They are then certainly competitive in every sense: for simple $\mathcal{H}_0$ they coincide with likelihood ratios and Bayes factors, and will give thus as much evidence as these notions do; for composite $\mathcal{H}_0$, GRO(W) E-variables are designed to give as much expected log-evidence against $\mathcal{H}_0$ as possible without violating the optional continuation requirement — in practice in some cases giving a bit more, and in some cases a bit less evidence against the null than standard Bayes factors (see Turner et al. (2021) for a practical example).

Now suppose we have a minimal effect size δ in mind, and we plan a study in which obtaining data is expensive. What sample size should we plan for? One reasonable option is to pick a certain *target growth* L (essentially the logarithm of Shafer (2021)'s notion of "implied target") and then determine the sample size at which we expect to gain L. To give a simple illustration, consider the 1-dimensional exponential family case of Example 4 with $\Theta_0 = \{0\}$. We know that, for a sample of size n , under all $\theta_1 \in \Theta_1$ with $\Theta_1 = \{\theta_1 : \theta_1 \geq \delta\}$, we have $\mathrm{GROW}(\Theta_1) = nD(P_\delta \| P_0)$ where $D(P_\delta \| P_0)$ is the KL divergence for 1 outcome. We then calculate n_{GROW} as the smallest n such that $nD(P_\delta \| P_0) \geq L$, i.e. $n_{\mathrm{GROW}} = \lceil L / D(P_\delta \| P_0) \rceil$. In the Gaussian location model, $D(P_\delta \| P_0) = \delta^2/2$ so this gives $n = \lceil 2L/\delta^2 \rceil$. We return to the question of choosing L below.

E-Values for Decisions We can also use E-values in the traditional setting, in which a study ends with an accept/reject decision — with the proviso that any decision is provisional, since there always is an option to continue and combine the results with a new study. For better or worse, this is the paradigm that researchers often have to work in, and within this paradigm they will inevitably be interested in the power for the experiment ahead. They will then plan for a certain sample size n to achieve such power, with a minimal relevant effect size δ in mind. As long as the E-variables themselves are chosen according to one of the GRO criteria, such a use of power as a 'secondary' criterion used merely to determine sample size is consistent with the E-variable approach.

In order for E-variables to be embraced by practitioners, we would hope that the sample sizes required to achieve a certain desired power with GRO-E-variables would then be competitive with existing standard approaches based on Neyman-Pearson tests. We now study whether this is the case. For simplicity we consider this question only for the Gaussian location model of Example 3, where $\mathcal{H}_0$ is the standard normal $N(0,1)$ and $\mathcal{H}_1$ the set $\{p_\mu : \mu \in \Theta\}$ of normals with variance 1. All results readily generalize to 1-dimensional exponential families.

Power: planning for a Fixed n For comparison, recall that a standard one-sided NP test at level α would reject if $\hat{\mu} \geq z_\alpha/\sqrt{n}$ with z_α the $(1-\alpha)$-quantile of the standard normal with $z_{0.05} = 1.645, z_{0.01} = 2.33$. By standard calculation (see Appendix B.6), under an alternative with mean $\geq \delta$, the sample size needed with this test to get power at least $1-\beta$ satisfies $n_{\text{NP}} = C_{\text{NP}}/\delta^2$ with $C_{\text{NP}} = (z_\alpha + z_\beta)^2$; for $\alpha = 0.05, \beta = 0.2$ we get $C_{\text{NP}} \approx 6.180$. For the same $\Theta_1 = \{\mu : \mu \geq \delta\}$, we can also calculate the sample size needed to get power $1-\beta$ using the GROW E-variable of Example 4. If we use a fixed sample size n, we reject if $\log p_\delta(Y^n)/p_0(Y^n) \geq -\log \alpha$. By a simple calculation, for $\alpha > 1/2$, the smallest n at which we have power at least $1-\beta$, is given by setting

$$n_{\text{GROW-FIXED}} = 2 \cdot \frac{-\log \alpha}{\delta^2} \cdot \left(1 + \frac{z_\beta}{z_\alpha}\right)^2 = c_\alpha n_{\text{NP}} \text{ with } c_\alpha = \frac{2 \cdot (-\log \alpha)}{z_\alpha^2}. \tag{36}$$

We have $c_{0.05} \approx 2.2$; $c_{0.01} \approx 1.7$ and c_α very slowly converges to 1 in the limit $\alpha \downarrow 0$: up to a constant factor of about two we need the same amount of data as in a classical approach, and the width of the induced confidence interval is of the same order. We can therefore choose a GROW E^* that is qualitatively more similar to a standard NP test than a standard Bayes factor approach: using instead a standard Bayesian prior W on Θ_1 with the W_1-GRO E-variable has the advantage of not needing to specify any δ in advance, but as can be seen from (15), the number of samples to get power $1-\beta$ is larger by a logarithmic factor (Appendix B.6).

We also note that the evidential target growth and the maximal power approach are not contradictory: for any particular choice of α and β, there is a choice of L such that the planned for sample sizes become the same function of δ (but of course the L resulting from $\alpha = 0.05, \beta = 1 - 0.8$ will be just as arbitrary as these choices were in the first place).

Power with Optional Stopping — a Tragedy of the Commons? No matter the considerable advantages of being safe under optional continuation, the factor of about 2 of extra data needed to get a desired power might scare away practitioners from adopting the E-variable approach. The situation changes completely once one adopts optional stopping. As we

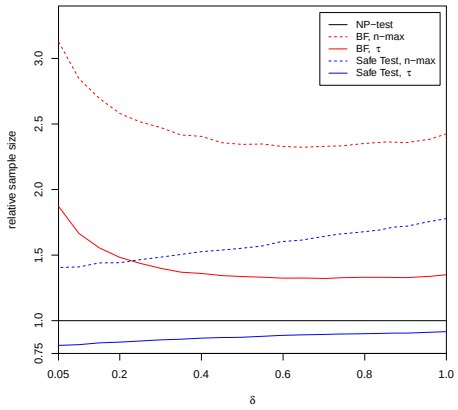

Figure 3: Factor of additional data needed to obtain a power of $1-\beta = 80\%$ compared to a Neyman-Pearson test for the z-test setting, as a function of effect size (mean) for various E-value based tests for the Bayesian E-variable as in Example 3 with normal prior and the GROW E-variable with minimum relevant effect size δ as in Example 4 under optional stopping, both in expectation ($\tau_{0.8}$) and in the worst-case ($n_{\max}(0.8)$) (for very small and very large δ, the normal prior that we used for the Bayesian E-variable may not be the prior of choice, but the results are representative for other priors one might use at δ around 0.5).

saw in the previous section, many testing problems allow us to use E-variables that remain safe under optional stopping — and we can use the most aggressive stopping rule that stops as soon as either $E_n \geq -\log \alpha$ (and we reject) or a pre-set maximum $n_{\max}$ is achieved (and we reject if $E_{n_{\max}} \geq -\log \alpha$ and otherwise accept). A simple but quite accurate approximation of the resulting stopping time τ_1 for i.i.d. data in the GROW setting of Example 4, when setting $n_{\max}$ to ∞ is given by using Wald's equality in a manner first set out by Breiman (1961); Ter Schure et al. (2021) give details. It gives for data $Y_1, Y_2, \ldots \sim P_\delta$, that $\mathbf{E}_{P_\delta}[\tau_1] \approx (-\log \alpha)/D(P_\delta \| P_0)$ with $D(P_\delta \| P_0)$ the KL divergence for a single outcome. For the Gaussian location family $D(P_\delta \| P_0) = \delta^2/2$, and we get $\mathbf{E}_{P_\delta}[\tau_1] \approx 2(-\log \alpha)/\delta^2$. Comparing to (36), this gives that with $n_{\max} = \infty$ (so that the power of our test is 1), the expected stopping time will be in fact already smaller than the fixed stopping time we get with the Neyman-Pearson approach at power $1-\beta$ set to 0.8. In practice we will choose $n_{\max}(\beta)$ to be the smallest number so that the overall procedure has power $1-\beta$, resulting in a stopping time $\tau_\beta = \min\{\tau_1, n_{\max}(\beta)\}$. The expected stopping time is then really even smaller. Figure 3 demonstrates this for the Gaussian location family. As the figure illustrates for the case $\beta = 0.8$, we have $n_{\max}(\beta) = C_{\beta,\delta}/\delta^2$ and $\mathbf{E}[\tau_\beta] = C'_{\beta,\delta}/\delta^2$ for $C_{\beta,\delta}$ and $C'_{\beta,\delta}$ that remain within constant bounds as δ varies. We can in fact heuristically derive analytic expressions (integrals) for the limits $C_{\beta,0}$ and $C'_{\beta,0}$ for $\delta \downarrow 0$ by rescaling the log-likelihood process $(\log p_\delta(X^n)/p_0(X^n))_n$ to become compatible with a Brownian motion with drift, see Appendix B.6. These give, for $\beta = 0.2$, $C_{\beta,0} = 8.5936$ and $C'_{\beta,0} = 4.971$ in accordance with Figure 3. Experiments with a logrank test (Ter Schure et al., 2021), t-test (in the vignette of R package (Ly et al., 2020)) and 2×2 tables (Turner et al., 2021) all confirm the picture that arises from Figure 3: with E-variables based on optional stopping one needs on average *less* data to achieve a certain desired power, but one needs

to prepare for *more* data in the worst-case. Taking stock, we can conclude that if current standard null hypothesis tests were replaced by E-value-based tests, and the standard practice to determine study sizes were replaced by the one above, and the percentage of studies in which the alternative is true is not too small, the world would need on average about the same or even a bit less data than it does now, to reach substantially more robust conclusions and better meta-analyses. Yet — at least as long as scientists insist on power and significance requirements — each individual study would have to *plan* for substantially more data, giving researchers an incentive not to adapt these new methods. We see this *Tragedy of the Commons* as one of the biggest obstacles for uptake of E-variables in practical settings.

7 Earlier and Related Work

E-Variables, Test Martingales, General Novelty As seen in Section 1.3, E-variables are the building blocks of *test (super-) martingales*, which go back to Ville (1939), the paper that introduced the modern notion of a martingale. E-variables themselves have probably been originally introduced by Levin (of P vs NP fame) (1976) (see also (Gács, 2005)) under the name *test of randomness*, but Levin's abstract context is quite different from ours. Independently discovered by Zhang et al. (2011), they were later analyzed by Shafer et al. (2011), Shafer and Vovk (2019), Vovk and Wang (2021), Shafer (2021); all these authors used different names for the concept (we originally called them 'S-value'). Although a good case can be made for *bets* and *betting scores* (Shafer, 2021), the literature seems to converge to E-variables and -values. Here the E may either stand for evidence or for expectation.

Test martingales themselves have been thoroughly investigated by Shafer et al. (2011), Shafer and Vovk (2019). They themselves underlie AV (anytime-valid) P-values (Johari et al., 2021), AV tests (which we call 'tests that are safe for optional stopping') and AV confidence sequences. The latter were recently developed in great generality by A. Ramdas and collaborators; see e.g. (Balsubramani and Ramdas, 2016, Howard et al., 2021). Both AV tests and confidence sequences have first been developed by H. Robbins and his students (Darling and Robbins, 1967, Lai, 1976, Robbins, 1970). Like we do for E-variables, Ramdas et al. (and also e.g. Pace and Salvan (2019)) stress the promise of the AV notions for a safer kind of statistics that is significantly more robust than standard testing and confidence interval methodology.

Just like regular tests can be turned into confidence intervals by varying the null and 'inverting' the resulting tests, AV confidence sequences can be created by starting with a collection of test martingales, one for each null, and then varying the null[2]. The work on AV tests and confidence sequences is therefore very similar in spirit to ours, with our work stressing analysis at the level of batches of data rather than individual data points, and with the AV work bringing out the difference to Bayesian approaches more explicitly (AV $1-\alpha$-confidence intervals are typically wider than Bayesian $1-\alpha$-credible sets). In fact we do not claim any real novelty for the 'safe' or 'AV' setting per se: the real novelty of this paper is in the four versions of Theorem 1. As far as we know, these results are new, with the exception of a special case of the simplest version of Theorem 1 (Section 2): the case of discrete outcome spaces, simple $\mathcal{H}_1$ and convex $\mathcal{H}_0$ was already formulated and proved by Zhang et al. (2011).

[2]It is sometimes claimed that problematic aspects of null hypothesis testing are mostly due to the very idea of a 'null hypothesis' or a significance level (Cumming, 2012, McShane et al., 2019). Without wanting to take sides in this issue, we note that standard confidence intervals are just as unsafe under optional continuation as standard Neyman-Pearson hypothesis tests.

Beyond GRO The above works often rely on E-variables E^* that, while having good growth properties ($\mathbf{E}_{P_1}[\log E^*]$ is large) are not GRO in any of our senses ($\mathbf{E}_{P_1}[\log E^*]$ is not as large as possible). For example, Ter Schure et al. (2021) provides a supermartingale as a product of E-variables that is a partial rather than a standard likelihood, amounting to a coarsening of the data that ignores aspects of the data generating process that are very hard to model. We emphasize that our development of GRO notwithstanding, we certainly acknowledge that there can be many E-variables such as these that are useful and have good but not optimal growth properties.

Relation to Sequential Testing *Sequential testing* (Lai, 2009), pioneered by Wald (1947) and Barnard and developed much further by H. Robbins and his students, is mathematically very similar to, but conceptually quite different from, testing based on test martingales and (therefore) E-variables. Sequential tests are made for streams of data $Y_i, Y_2, \ldots$ as in Example 1 and Section 5 and are based on random processes $(S_i)_{i\in\mathbb{N}}$ such that, for each i, S_i is a conditional E-variable given $Y_1, \ldots, Y_{i-1}$ under $\mathcal{H}_0$, and $1/S_i$ is a conditional E-variable given $Y_1, \ldots, Y_{i-1}$ under $\mathcal{H}_1$. Of course, this two-sided E-variable property only holds in quite special cases — roughly under the same conditions as Proposition 3 (monotone likelihood ratio), i.e. in our Example 4 and for the t-test with point prior on δ^+, δ^-. In such a setting, the sequential test based on $S_1, S_2, \ldots$ with prespecified parameters α, β proceeds by calculating $S_1, S_2, \ldots$ and stopping at τ^*, the smallest τ at which either $S_\tau \geq (1-\beta)/\alpha$ ('accept') or $S_\tau \leq (1-\alpha)/\beta$ ('reject'). Wald showed that this test has Type I error probability bounded by α and Type II error bounded by β. The reason one can stop at a smaller threshold ($(1-\beta)/\alpha$ rather than $1/\alpha$) is that one *has* to stop at τ^*, Thus, the method does not allow for optional stopping in our sense: conceptually, sequential tests were designed for variable sample sizes but very special, pre-specified stopping times. Still, much of the work in sequential testing can be re-cycled to obtain test martingales and E-values — but usually not vice versa since most E-variables are not 'two-sided'.

Related Work on Relating P-values and E-variables Shafer et al. (2011), Shafer and Vovk (2019), give a general formula for *calibrators* f (see also Vovk (1993), Sellke et al. (2001) for earlier work in this direction). These are decreasing functions $f : [0,1] \to [0,\infty]$ so that for any P-value P, $E := f(\mathrm{P})$ is an E-variable. The choice of any such calibrator is essentially arbitrary, but, following Shafer (2021), let us consider one that is especially simple: $f(p) = 1/\sqrt{p} - 1$. For example, for any calibrator f suggested for practice, rejection under the E-variable based test with significance level $\alpha = 0.05$, so that $E \geq 20$, would then correspond to reject only if $\mathrm{P} \leq f^{-1}(20) = 1/441 = 0.0023$, requiring a substantial amount of additional data for rejection under a given alternative. Note that the E-variables we developed for *given* models in previous sections are more sensitive than such generic calibrators though. For example, consider the normal location family of the previous section. With the calibrator above, we would reject if $\hat{\mu} \geq z_{0.0023}/\sqrt{n} \approx 2.8/\sqrt{n}$. The amount of data to plan for to obtain power 80% would then be $\approx (2.8/1.65)^2 n_{\mathrm{NP}} \approx 3.0 n_{\mathrm{NP}}$, whereas for the E-value based on the normal likelihood ratio we would need $\approx 2.2 n_{\mathrm{NP}}$, and even significantly less under optional stopping.

8 Could Fisher, Jeffreys and Neyman Have Agreed on a *Currency* for Testing?

The three main approaches towards null hypothesis testing are Jeffreys' Bayes factor methods, Fisher's p-value-based testing and the Neyman-Pearson method. In the paper *Could Fisher, Jeffreys and Neyman Have Agreed on Testing?*, Berger (2003) noted that, while these three methodologies seem superficially highly contradictory, there exist methods that have a place within all three. The developments in this paper lead to the inevitable conclusion that e-variable based testing — although it differs in some technical respects from Berger's proposals — is very much in the same spirit:

Concerning the *Neyman-Pearson approach*: e-variables lead to tests with Type-I error guarantees at any fixed significance level α, which is also the first requirement of a Neyman-Pearson test — requiring safety under optional continuation or optional stopping simply enforces the requirement to hold over a non-pre-specified sequence of studies, which is a natural requirement in scientific applications. Since there is then no single study any more, the concept of 'power' looses its centrality (and may be upgraded to requiring power one), and growth-rate optimality is a natural quantitative refinement. The fact that a high growth-rate corresponds to a high value of $\mathbf{E}_{P_1}[\log E]$ under $\mathcal{H}_1$ whereas a high power corresponds to a high probability that $P_1(\log E \geq -\log \alpha)$ also shows that GRO and power remain intimately connected in e-variable theory as well.

Concerning the *Fisherian approach*: in this approach, p-values are interpreted as indicating amounts of evidence against the null, and their definition does not need to refer to any specific alternative $\mathcal{H}_1$. Exactly the same holds for e-values: the basic interpretation 'a large e-value provides evidence against $\mathcal{H}_0$' holds no matter how the e-variable is defined, as long as it satisfies (1). If they are defined relative to $\mathcal{H}_1$ that is close to the actual process generating the data they will grow fast and provide a lot of evidence, but the basic interpretation holds regardless. In contrast to evidence based on standard p-values however, (a) e-based evidence has a very concrete additional interpretation in terms of money (the higher E, the more money one has gained in a game that would not allow one to make money under the null); (b) it remains valid under optional continuation, and (c) unlike the p-value, it is compatible with (provides the same evidence as) likelihood ratios do in simple-vs.-simple testing — the one case where the use of likelihood ratio as evidence is standard.

Concerning the *Bayesian approach*: despite their monetary interpretation, *all* e-variables that we encountered can also be written as Bayes factors, and Theorem 1 strongly suggests that this is a very general phenomenon. Subjective prior knowledge can be accounted for using the W_1-GRO e-variable (Section 2.2), whereas minimax optimal GROW and REGROW e-variables sometimes correspond to 'objective' Bayes approaches based on Jeffreys' and/or right-Haar priors. Still, there seem to be two fundamental differences: first, in a standard Bayesian analysis, one would require error guarantees and safety under OC *under the prior* instead of under all $P \in \mathcal{H}_0$, and second, one would insist on using full, standard likelihoods — whereas e-variables may also be based on partial (Ter Schure et al., 2021) or Dawid's (1997) prequential (Waudby-Smith and Ramdas, 2021) likelihoods rather than full likelihoods — which then however may be combined with priors (on $\mathcal{H}_1$) after all. Even though we emphasize Type-I error safety throughout, because of this generic freedom in using priors on $\mathcal{H}_1$ the link to Bayesian methods remains close.

The Dream With the massive criticisms of P-values in recent years, there seems to be growing consensus that, in the context of hypothesis testing, P-values should either not be used at all, or at least, with utter care (Wasserstein et al., 2016, Benjamin et al., 2018). Yet otherwise, the disputes among adherents of the three schools continue. For example, some highly accomplished statisticians reject the idea of testing without a clear alternative outright; others say that such goodness-of-fit tests are an essential part of data analysis. Some insist that significance testing (with binary decisions) should be abolished altogether (Amrhein et al., 2019), others (perhaps slightly cynically) acknowledge that significance may be silly in principle, yet maintain that journals and conferences will always require a significance-style 'bar' in practice and that therefore such bars should be made as meaningful as possible. Finally, within the Bayesian community, the Bayes factor is sometimes presented as a panacea for most testing ills, while others warn against its use, and protest, for example, against the claim that Bayes factors can 'handle optional stopping' (Hendriksen et al., 2021).

Wouldn't it be nice if all these accomplished but disagreeing people could continue to go their way, yet would have a common language or 'currency' to express amounts of evidence, and would be able to combine their results in a meaningful way? This is what E-variables can provide: consider three tests with the same null hypothesis $\mathcal{H}_0$, based on samples $\mathbf{Y}_{(1)}$, $\mathbf{Y}_{(2)}$ and $\mathbf{Y}_{(3)}$ respectively. The results of a δ-based E-variable test aimed to optimize power on sample $\mathbf{Y}_{(1)}$, an E-variable test for sample $\mathbf{Y}_{(2)}$ based on a Bayesian prior W_1 on $\mathcal{H}_1$ and a Fisherian E-variable test in which the alternative $\mathcal{H}_1$ is not explicitly formulated, can all be multiplied — and the result will be meaningful — both in terms of monetary gain and in terms of error probability.

Acknowledgments Thanks to A. Barron, J. Berger, R. Frongillo, P. Harremoës, A. Hendriksen, A. Henzi, A. Ly, R. Meester, M.F. Perez-Ortiz, A. Ramdas, J. ter Schure, G. Shafer, R. Turner, V. Vovk, B. Wagonner, E.J. Wagenmakers and J. Ziegel for helpful conversations.

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Appendix A Theorem 1, Corollaries and Conditions

Here we prove Theorem 1 and its Corollary 2. We also, in Appendix A.3, discuss the required regularity conditions for Theorem 1 and we prove that they are applicable in all our examples.

A.1 Proof of Theorem 1, Simplest Version, and Corollary 2

The proof of the first version of Theorem 1 relies on quite technical results from Li (1999), but if the minimum in (11) is achieved by some prior W_0^*, and under the further condition that we can exchange differentiation and expectation, then the partial and crucial result that (10) is an E-variable has a very simple proof: evaluate the derivative $f(\alpha) = (d/\alpha)D(Q\|(1-\alpha)P_{W_0^*} + \alpha P_\theta)$ at $\alpha = 0$ for arbitrary $\theta \in \Theta_0$ and note that it is ≥ 0 iff $\mathbf{E}_{\mathbf{Y}\sim\mathbf{Q}}[p_\theta(\mathbf{Y})/p_{W_0^*}(\mathbf{Y})] = \mathbf{E}_{\mathbf{Y}\sim\mathbf{P}_\theta}[q(\mathbf{Y})/p_{W_0^*}(\mathbf{Y})] \leq 1$. Differentiating again gives that $f(\alpha)$ is convex, and the result follows from convexity of $\{P_W : W \in \mathcal{W}(\Theta_0)\}$.

We proceed to give the complete and fully general proof. Let $\mathcal{P}_0 = \{P_W : W \in \mathcal{W}(\Theta_0)\}$ and note that $\mathcal{P}_0$ is convex, and (by assumption of the theorem) every distribution in $\mathcal{P}_0$ as well as Q has a density relative to μ and $\inf_{P\in\mathcal{P}_0} D(Q\|P) < \infty$. These three givens allow us to use a range of results about the reverse information projection (RIPr) established in the Ph.D. thesis (Li, 1999) (additional proofs of (extensions of) all of Li's results we need below can be found in the refereed paper Grünwald and Mehta (2020)).

First, the existence and uniqueness of a measure P_0^* (not necessarily a probability measure) with density p_0^* that satisfies $D(Q\|P_0^*) = \inf_{P\in\mathcal{P}_0} D(Q\|P)$ (i.e. it is the RIPr), and furthermore has the property

$$\text{for all } p \text{ that are densities of some } P \in \mathcal{P}_0\text{:}\ \ \mathbf{E}_{\mathbf{Y}\sim Q}\left[\frac{p(\mathbf{Y})}{p_0^*(\mathbf{Y})}\right] \leq 1, \tag{37}$$

follows directly from (Li, 1999, Theorem 4.3). But by writing out the integral in the expectation explicitly we immediately see that we can rewrite (37) as:

$$\text{for all } P \in \mathcal{P}_0\text{:}\ \ \mathbf{E}_{\mathbf{Y}\sim P}\left[\frac{q(\mathbf{Y})}{p_0^*(\mathbf{Y})}\right] \leq 1.$$

Li's Theorem 4.3 still allows for the possibility that $\int p_0^*(y)\,\mathrm{d}\mu(y) > 1$. To see that in fact this is impossible, i.e. p_0^* defines a (sub-) probability density, use Lemma 4.5 of Li (1999). This shows that $E^* = q(\mathbf{Y})/p_0^*(\mathbf{Y})$ is an E-variable, and (using that P_0^* is the RIPr) the second and third equality of (11). The final line of the result ('if ... then $P_0^* = P_{W_0^*}$') follows directly from Lemma 4.1 of Li (1999).

It remains to show the first equality of (11) and essential uniqueness of E^*. For the former, it is sufficient to show that for all E-variables, i.e. all $E \in \mathcal{E}(\Theta_0)$,

$$\mathbf{E}_{\mathbf{Y}\sim Q}\left[\log E\right] \leq \mathbf{E}_Q\left[\log E^*\right]. \tag{38}$$

To show this, fix any E-variable $E = e(\mathbf{Y})$ in $\mathcal{E}(\Theta_0)$. Now further fix $\epsilon > 0$ and fix a $W_{(\epsilon)} \in \mathcal{W}(\Theta_0)$ with $D(Q\|P_{W_{(\epsilon)}}) \leq \inf_{W_0\in\mathcal{W}(\Theta_0)} D(Q\|P_{W_0}) + \epsilon$. We must have, with $q'(y) \coloneqq e(y)p_{W_{(\epsilon)}}(y)$, that $\int q'(y)\,\mathrm{d}\mu = \mathbf{E}_{\mathbf{Y}\sim P_{W_{(\epsilon)}}}[E] \leq 1$, so q' is a sub-probability density, and by properness of the log scoring rule,

$$\mathbf{E}_Q[\log E] = \mathbf{E}_Q\left[\log \frac{q'(\mathbf{Y})}{p_{W_{(\epsilon)}}(\mathbf{Y})}\right] \leq$$
$$\mathbf{E}_Q\left[\log \frac{q(\mathbf{Y})}{p_{W_{(\epsilon)}}(\mathbf{Y})}\right] = D(Q\|P_{W_{(\epsilon)}}) \leq \inf_{W_0\in\mathcal{W}(\Theta_0)} D(Q\|P_{W_0}) + \epsilon.$$

Since we can take ϵ to be arbitrarily close to 0, (38) follows.

To show essential uniqueness of E^*, let E be any E-variable with $\mathbf{E}_Q[\log E] = \mathbf{E}_Q[\log E^*]$. By linearity of expectation, $E' = (1/2)E^* + (1/2)E$ is then also an E-variable, and by Jensen's inequality applied to the logarithm we must have $\mathbf{E}_Q[\log E'] > \mathbf{E}_Q[\log E^*]$ unless $Q(E = E^*) = 1$. Since we have already shown that for any E-variable E', $\mathbf{E}_Q[\log E'] \leq \mathbf{E}_Q[\log E^*]$, it follows that $Q(E \neq E^*) = 0$. But then, by our assumption that Q has full support, i.e. $q(\mathbf{Y}) > 0$ hold μ-almost everywhere, we must have that $P_\theta(E \neq E^*) = 0$ for all $\theta \in \Theta$, so E^* is essentially unique.

Proof of Corollary 2 Let W_0 be as in the corollary statement. By definition of E^* as in Theorem 1, simplest version, and then using strict convexity of the KL divergence in its second argument (Cover and Thomas, 1991) and the fact that $D(Q\|P)$ is minimized, over $P \in \{P_W : W \in \mathcal{W}(\Theta_0)\}$, we have:

$$\mathbf{E}_Q\left[\log E^*\right] = D(Q\|P_0^*) < D(Q\|P_{W_0}) = \mathbf{E}_Q\left[\log \frac{q}{p_{W_0}}\right]$$

so that, if $q/p_{W_0'}$ were an E-variable, we would have a contradiction with the first equality in (11).

A.2 Proof of full version of Theorem 1

The proof consists of two sub-parts, Part (a) relying on the simple version of the theorem presented in Section 2 and proven above (henceforth called 'the simple theorem'), and Part (b) relying on a nonstandard minimax/saddle-point theorem from Grünwald and Dawid (2004) (GD from now on), itself relying heavily on an earlier result from Topsøe (1979).

Part (a). We first show that E^f as in (26) is an E-variable. This follows by the simple theorem, with $\mathbf{V}$ in the role of $\mathbf{Y}$, Q in the theorem statement substituted by $P^{[\mathbf{V}]}_{W_1^*}$ and P_W for $W \in \mathcal{W}(\Theta_0)$ replaced by $P^{[\mathbf{V}]}_W$ and using that (25) implies that $\inf_{W_0 \in \mathcal{W}_0(\Theta_0)} D(P^{[\mathbf{V}]}_{W_1^*} \| P^{[\mathbf{V}]}_{W_0}) < \infty$. Next, we show that *if* (27) holds for E^f as in (26), then all E-variables E for which it holds must be essentially equal to E^f. To see this, suppose that E is another E-variable satisfying (27). Then we have

$$\inf_{W \in \mathcal{W}_1} \{ \mathbf{E}_{\mathbf{Y} \sim P_W}[\log E] - f(W) \} = D(P^{[\mathbf{V}]}_{W_1^*} \| P_0^{*[\mathbf{V}]}) - f(W_1^*) = \mathbf{E}_{\mathbf{Y} \sim P_{W_1^*}} \left[\log E^f \right] - f(W_1^*)$$

as follows by writing out the definition of $D(\cdot\|\cdot)$. On the other hand, using the definition of inf, we must have

$$\inf_{W \in \mathcal{W}_1} \{ \mathbf{E}_{\mathbf{Y} \sim P_W}[\log E] - f(W) \} \leq \mathbf{E}_{\mathbf{Y} \sim P_{W_1^*}}[\log E] - f(W_1^*).$$

The only way these two displays can be reconciled is if $\mathbf{E}_{\mathbf{Y} \sim P_{W_1^*}}[\log E] \geq \mathbf{E}_{\mathbf{Y} \sim P_{W_1^*}} \left[\log E^f \right]$. But since E^f is the RIPr of $P^{[\mathbf{V}]}_{W_1^*}$, we can use the simple theorem again, applied with $P_{W_1^*}$ in the role of Q and $\mathcal{H}_1 = \{P_{W_1^*}\}$, to conclude that E^f must be essentially equal to E. The final line of the fully general Theorem 1 follows again by reduction to the analogous statement of the simple theorem.

Finally, we will use the simple theorem to show the following one-sided version of (27):

$$\inf_{W \in \mathcal{W}_1} \{ \mathbf{E}_{\mathbf{Y} \sim P_W}[\log E^f] - f(W) \} \leq \sup_{E \in \mathcal{E}(\Theta_0)} \inf_{W \in \mathcal{W}_1} \{ \mathbf{E}_{\mathbf{Y} \sim P_W}[\log E] - f(W) \} \leq D(P^{[\mathbf{V}]}_{W_1^*} \| P_0^{*[\mathbf{V}]}) - f(W_1^*). \quad (39)$$

The first inequality is trivial since $E^f \in \mathcal{E}(\Theta_0)$. The second follows if we can show that

$$\sup_{E \in \mathcal{E}(\Theta_0)} \inf_{W \in \mathcal{W}_1} \mathbf{E}_{P_W}[\log E - f(W)] \leq \inf_{W \in \mathcal{W}_1} \inf_{W_0 \in \mathcal{W}_0} \{ D(P_W \| P_{W_0}) - f(W_1) \} \quad (40)$$

and recognizing that by assumptions of the theorem, the right-hand side of (40) coincides with the right-hand side of (39). To prove (40), note that by the simple version of the theorem we already have for each fixed $W_1 \in \mathcal{W}_1$ that

$$\inf_{W_0 \in \mathcal{W}_0} D(P_{W_1} \| P_{W_0}) = \sup_{E \in \mathcal{E}(\Theta_0)} \mathbf{E}_{P_{W_1}}[\log E]$$

and this directly implies the inequality by adding $-f(W_1)$ to both sides and using a standard "inf sup $\geq$ sup inf" argument (the trivial side of the minimax theorem). The equality follows by assumption of the Theorem.

Part (b). Taking stock, we see that the only thing that is left to prove is (39) with the reversed inequalities. For this, it suffices to show that

$$D(P^{[\mathbf{V}]}_{W_1^*} \| P_0^{*[\mathbf{V}]}) - f(W_1^*) \leq \inf_{W \in \mathcal{W}_1} \mathbf{E}_{P_W}[\log E^f - f(W)]. \quad (41)$$

Since all distributions occurring in (41) are marginals on $\mathbf{V}$, and E^f can be written as a function of $\mathbf{V}$, we will from now on simply refer to the marginal densities on $\mathbf{V}$ corresponding to P_W as p_W (rather than p'_W as in the main text), and we will omit the superscripts $[\mathbf{V}]$ from P; thus we take as our basic outcome now $\mathbf{V}$ rather than $\mathbf{Y}$.

We will show the stronger statement that (41) holds with equality, by using a minimax/saddle point result that holds for general functions $L : \Theta_1 \times \mathcal{W}_1 \to \mathbb{R} \cup \{\infty\}$ such that $L(W_{\text{N}}, W_{\text{U}}) := \mathbf{E}_{\theta \sim W_{\text{N}}}[L(\theta, W_{\text{U}})]$ is well-defined for all $W_{\text{N}} \in \mathcal{W}_1$ (the condition 'well-defined' is necessary since the expectation is over a function that may neither be bounded from below nor from above; see for example Section 3.1 of GD for the (standard) definitions). These L are interpreted as loss functions, with $\theta_1 \in \Theta_1$ denoting a state of nature and $\mathcal{W}_1$ an arbitrary convex set of distributions on Θ_1, each $W \in \mathcal{W}_1$ being interpreted as an action. Following GD, we can associate a *decision-theoretic entropy* $H(W_{\text{N}}) := \inf_{W_{\text{U}} \in \mathcal{W}} L_0(W_{\text{N}}, W_{\text{U}}) = L(W_{\text{N}}, W_{\text{N}})$ with any such L. The following result holds for all Θ_1, $\mathcal{W}_1$ and L as defined above but we will apply it to the instantiation of Θ_1 and $\mathcal{W}_1$ in Theorem 1.

GD's Theorem 6.3 Assume that (a) L is a proper scoring rule, i.e. for all $W_{\text{N}} \in \mathcal{W}_1$, $H(W_{\text{N}}) = L(W_{\text{N}}, W_{\text{N}})$. Suppose that (b) $W_1^* \in \mathcal{W}_1$ is 'maximum entropy' i.e. $\sup_{W_{\text{N}} \in \mathcal{W}_1} H(W_{\text{N}}) = H(W_1^*) < \infty$ and (c) the lower semi-continuity condition below holds. Then (W_1^*, W_1^*) is a saddle-point relative to L, i.e.

$$H(W_1^*) = L(W_1^*, W_1^*) = \sup_{W \in \mathcal{W}_1} L(W, W_1^*). \tag{42}$$

Lower Semicontinuity Condition (GD's Condition 6.1) Let (W_n) be a sequence of distributions in $\mathcal{W}_1$ such that $H(W_n)$ is bounded below and such that (W_n) converges weakly to some distribution W° on Θ_1. Then $L(W^\circ, W_{\text{U}})$ is well-defined for all $W_{\text{U}} \in \mathcal{W}$ and for all $W' \in \mathcal{W}$, $L(W', W^\circ) \leq \liminf_{n \to \infty} L(W', W_n)$.

We now define the specific loss function to which we will apply the above theorem. $\mathcal{W}_1$ and Θ_1 are defined as in the statement of Theorem 1. (25) implies that $D(P_{W_1^*} \| P_0^*) < \infty$ for some P_0^* with density p_0^*. Similarly, P_W must have some density p_W under all $W \in \mathcal{W}_1$. We can therefore define, using these densities,

$$L(\theta, W_{\text{U}}) = \mathbf{E}_{\mathbf{V} \sim P_\theta} \left[-\log \frac{p_{W_{\text{U}}}(\mathbf{V})}{p_0^*(\mathbf{V})} - f(\theta) \right]$$
$$L(W_{\text{N}}, W_{\text{U}}) = \mathbf{E}_{\theta \sim W_{\text{N}}} \mathbf{E}_{\mathbf{V} \sim P_\theta} \left[-\log \frac{p_{W_{\text{U}}}(\mathbf{V})}{p_0^*(\mathbf{V})} - f(\theta) \right].$$

Since P_θ has full support for all $\theta \in \Theta_1$, $P_{W_1^*}$ has full support and so $p_0^*(\mathbf{V}) > 0$ a.s. under $P_{W_1^*}$ and hence under all P_θ with $\theta \in \Theta_1$. Similarly $p_W(\mathbf{V}) > 0$ a.s. under all P_θ with $\theta \in \Theta_1$. Thus, the quantity inside the expectation is almost surely well-defined. To see that the expectations are themselves well-defined (using standard definitions, see again Section 3.1 of GD), note that we can write

$$\begin{aligned} L(W_{\text{N}}, W_{\text{U}}) = &+ \mathbf{E}_{W_{\text{N}}} \mathbf{E}_{P_\theta} \left[-\log \frac{p_{W_{\text{U}}}}{p_{W_{\text{N}}}} \right]_+ + \mathbf{E}_{W_{\text{N}}} \mathbf{E}_{P_\theta} \left[-\log \frac{p_{W_{\text{N}}}}{p_0^*} \right]_+ - \mathbf{E}_{W_{\text{N}}} \mathbf{E}_{P_\theta} \left[f(\theta) \right]_+ \\ &- \mathbf{E}_{W_{\text{N}}} \mathbf{E}_{P_\theta} \left[-\log \frac{p_{W_{\text{U}}}}{p_{W_{\text{N}}}} \right]_- - \mathbf{E}_{W_{\text{N}}} \mathbf{E}_{P_\theta} \left[-\log \frac{p_{W_{\text{N}}}}{p_0^*} \right]_- + \mathbf{E}_{W_{\text{N}}} \mathbf{E}_{P_\theta} \left[f(\theta) \right]_- \end{aligned}$$

with $[x]_+ := \max\{x, 0\}$ and $[x]_- = \max\{-x, 0\}$. The expectation would be undefined iff there is both a term equal to ∞ and a term equal to $-\infty$ on the right. We will show that this is not the case. We assume $f(\theta)$ bounded, and, under our finite KL condition, $D(P_{W_N}\|P_{W_U}) = \mathbf{E}_{W_N}\mathbf{E}_{P_\theta}\left[-\log \frac{p_{W_U}}{p_{W_N}}\right]_+ - \mathbf{E}_{W_N}\mathbf{E}_{P_\theta}\left[-\log \frac{p_{W_U}}{p_{W_N}}\right]_- < \infty$, so we only need to worry about the terms involving p_0^*. But these are also the positive and negative parts of a minus KL divergence, so the full expectation is well-defined as a number in $\mathbb{R} \cup \{-\infty\}$. The expectations are therefore welldefined and we can write

$$L(W_N, W_U) = D(P_{W_N}\|P_{W_U}) - D(P_{W_N}\|P_0^*) - \mathbf{E}_{\theta\sim W_N}[f(\theta)] \tag{43}$$

and analogously for $L(\theta, W_U)$.

Applying GD's theorem to L We apply GD's theorem to the loss function L above with W_1^* as in the statement of the theorem. From (43) we see that $L(W_1^*, W)$ is minimized, over $\mathcal{W}_1$, by $W = W_1^*$ and then finite, so that GD's requirements (a) and (b) hold for loss function L. We can now reason as follows. If the lower semicontinuity condition (c) also holds, then the theorem applies and (42) implies, taking minus on both sides,

$$-L(W_1^*, W_1^*) = \inf_{W\in\mathcal{W}_1} -L(W, W_1^*).$$

which, rewriting the left-hand side using (43) and the right-hand side using definition of L, is in turn seen to be equivalent to (41), and the desired result follows.

It thus only remains to show that the lower semicontinuity condition holds. Using (43) we can write $H(W_N) = -D(P_{W_N}\|P_0^*) + f(W_N)$ for all $W_N \in \mathcal{W}_1$. Take a sequence $(W_n)_n$ and W° as in the condition. Then $(P_{W_n})_n$ converges weakly to P_{W° (this is easy to see but see the proof of Lemma 9.2. of GD for an explicit proof). Since also f is bounded, $f(W_n)$ converges to $f(W^\circ)$. Also, for some $K \in \mathbb{R}$, for all n, we have $H(W_n) \geq K$ so for some $K' \in \mathbb{R}$, by boundedness of f, we have, for all n, $D(P_{W_n}\|P_0^*) \leq K' < \infty$. By Posner's (1975) theorem, $D(P_W\|P_0^*)$ is lower semi-continuous in its first argument. Posner only proves the result for P_0^* a probability measure; but it still holds even if P_0^* is a strict sub-probability measure, since then $p'(\mathbf{Y}) = p_0^*(\mathbf{Y})/\int p_0^*(\mathbf{Y})d\mu = 1$ represents a distribution P' and the result follows by applying Posner's result to P' and noting that $D(P_W\|P_0^*) = D(P_W\|P') + C$ for some constant C not depending on W.

The lower-semicontinuity in the first argument implies $\mathrm{t}D(P_{W^\circ}\|P_0^*) \leq \liminf_{n\to\infty} D(P_{W_n}\|P_0^*) \leq K' < \infty$. Following an argument exactly parallel to the proof of well-definedness for $L(W_N, W_U)$ given above, it now follows that $L(W^\circ, W_U)$ is well-defined for all $W_U \in \mathcal{W}_1$ as required. Next, using (43), we see that it is sufficient to show that for all $W \in \mathcal{W}_1$:

$$D(P_W\|P_{W^\circ}) \leq \lim\inf_{n\to\infty} D(P_W\|P_{W_n}).$$

But this again follows directly from Posner's theorem, which also says that KL divergence is lower semi-continuous in its *second* argument.

A.3 Remarks on and Checking of Conditions for Theorem 1

The Full Support and Finite KL Condition Requiring full support in the simplest version of the theorem ensures that E^* is a.s. well-defined: without it, there may be an

outcome $\mathbf{y}$ such that for some $\theta \in \Theta_0$, $P_\theta(\mathbf{y}) > 0$ whereas $P_{W_0}(\mathbf{y}) = Q(\mathbf{y}) = 0$. Then E^* is undefined with positive probability under this θ. The finite KL condition $D(P_\theta \| P_{\theta'}) < \infty$ imposed in the generalized versions of the theorem is just slightly stronger than the full support condition. It is required to make sure that all expectations in the proof are well-defined.

For standard parametric models in standard parameterizations (e.g. all multivariate exponential families in their mean-value parameterization), both conditions will hold automatically as long as one excludes points at the boundary of the parameter space, if those exist. For example, in the 2×2 setting without a pre-specified effect size we restrict Θ_1 to $(0,1)^2$, requiring the Bernoulli probabilities $\mu_{1|a}$ and $\mu_{1|b}$ to be non-degenerate. But, since the condition only refers to Θ_1, it is o.k. to set $\Theta_0 = [0,1]$ to include the boundary points in the null.

Additional Condition: Existence of W_1^* The requirement for composite $\mathcal{H}_1$ that a W_1^* exists achieving the minimum in (27) is strong in general, but it holds in all our examples with composite $\mathcal{H}_1$: Example 4 (W_1^* is shown to be a point prior in the example), Example 5 (since there we restrict Θ_1 to be compact) and Example 6 and 7 (here verifying the condition requires some work, see below). It also holds in the t-test setting underneath Theorem 2 with effect sizes δ^+ and δ^- (W_1^* reduces to a point prior on δ^+). By allowing E-variables to be functions of $\mathbf{V}$ that are a coarsening of $\mathbf{Y}$ as in the latter example, we make the condition considerably weaker.

Applicability of Theorem 1 and existence of minimizing W_1^* and W_0^* in Example 6 and 7 We have $\mathcal{H}_1 = \{P_{\mu_{1|a},\mu_{1|b}} : (\mu_{1|a}, \mu_{1|b}) \in \Theta_1\}$ and $\mathcal{H}_0 = \{P_\mu : \mu \in \Theta_0\}$, $\Theta_0 = [0,1]$ with definitions as in Example 2. In Example 6, we take $\Theta_1 = (0,1)^2$ and we can take $\Theta_0 = [0,1]$ or $\Theta_0 = (0,1)$ (the same minima will be achieved in both cases). In Example 7 we take $\Theta_1 = \{(\mu_{1|a}, \mu_{1|a} + \delta) : 0 < \mu_{1,a} < 1 - \delta\}$ and $\Theta_0 = (0,1)$.

We only give the proof for Example 6; the proof for Example 7 is entirely analogous.

The requirement for applying Theorem 1 that $P_\theta, P_{\theta'}$ with $\theta, \theta' \in \Theta_1$ satisfy $D(P_\theta \| P_{\theta'}) < \infty$, and have full support trivially holds by our exclusion of the boundary points in Θ_1. The only remaining condition for applying Theorem 1 is the existence of a KL minimizing prior W_1^*. We will show the stronger result that there exists a pair of minimizing priors (W_1^*, W_0^*) with $W_1^* \in \mathcal{W}(\Theta_1)$ and $W_0^* \in \mathcal{W}(\Theta_0)$ such that

$$\inf_{W_1 \in \mathcal{W}(\Theta_1), W_0 \in \mathcal{W}(\Theta_0)} (D(P_{W_1} \| P_{W_0}) - f(W_1)) = D(P_{W_1^*} \| P_{W_0^*}) - f(W_1^*) < \infty, \qquad (44)$$

with $f(W) = \mathbf{E}_{(\mu_{1|a},\mu_{1|b}) \sim W}[f(\mu_{1|a}, \mu_{1|b})]$ and $f(\mu_{1|a}, \mu_{1|b}) = D(P_{\mu_{1|a},\mu_{1|b}} \| P_{\mu^\circ})$ with μ° as in (14). We do this by first, in Part (a), showing that there exists such a pair with $W_1^* \in [0,1]^2$, i.e. with Θ_1 extended to include its boundary points. We then, in Part (b), show that the resulting W_1^* puts no mass on these boundary points, so that it also achieves the minimum on $\mathcal{W}(\Theta_1)$.

Part (a) The sets $\mathcal{W}([0,1]^2)$ and $\mathcal{W}(\Theta_0)$ are convex and compact in the weak topology; by Posner's (1975) theorem, $D(P_{W_1} \| P_{W_0})$ is lower-semicontinuous in its second argument in the weak topology on $\{P_{W_0} : W_0 \in \mathcal{W}(\Theta_0)\}$ and hence on $\mathcal{W}(\Theta_0)$ itself (see Section 9 of GD) and $f(W)$ is linear and bounded on $\mathcal{W}(\Theta_0)$; this shows that for each W_1, the corresponding minimizing W_0^* is achieved; since $D(P_{W_1} \| P_{W_0^*}) \leq D(P_{W_1} \| P_{1/2}) < \infty$ (with $P_{1/2} \in \mathcal{H}_0$ representing Bernoulli(1/2)) and f is bounded, the finiteness in (44) is guaranteed

as well. To see that the minimum W_1 is achieved as well, note that, again by Posner's theorem, $D(P_{W_1}\|P_{W_0})$ is also lower-semicontinuous in its first argument in the weak topology on $\{P_{W_1} : W_1 \in \mathcal{W}([0,1]^2)\}$. The same argument as before now gives that the minimum W_1^* is achieved.

Part (b) It now suffices to show that $P_{W_1^*}$ has full support, for this implies that W_1^* assigns mass 1 on $\Theta_1 = (0,1)^2$ and then W_0^* must assign mass 1 on $(0,1)$ (otherwise the KL divergence in (44) would be infinite, and we already established it is not). To show full support of $P_{W_1^*}$, note that by symmetry considerations, we must have that, with our choice of f,

$$M := D(P_{W_1^*}\|P_{W_0^*}) - f(W_1^*) = D(P_{W_1^\circ}\|P_{W_0^\circ}) - f(W_1^\circ) \tag{45}$$

for a prior W_1° such that, for all $\mathbf{y} \in \{0,1\}^n$, $p_{W_1^\circ}(\mathbf{y} \mid \mathbf{x}) = p_{W_1^*}(\mathbf{y}' \mid \mathbf{x})$ with $\mathbf{y}'$ is the modification of $\mathbf{y}$ with all 0s and 1s interchanged, and similarly for W_0°. Now if $P_{W_1^*}$ would not have full support, we have $P_{W_1^*}(Y_1 = y_1, Y_2 = y_2 \mid X_1 = a, X_2 = b) = 0$ for some $(y_1, y_2) \in \{0,1\}^2$. Then $P_{W_1^\circ}(Y_1 = \bar{y}_1, Y_2 = \bar{y}_2 \mid X_1 = a, X_2 = b) = 0$ for $\bar{y}_j = 1 - y_j$. But $\inf_{W_0 \in \mathcal{W}(\Theta_0)} D(P_{W_1^*}\|P_{W_0}) - f(W_1^*)$ as a function of W_1^* is easily checked to be strictly convex on $\mathcal{W}(\Theta_1)$, so by (45) we must have that, for $W' = (1/2)W_1^* + (1/2)W_1^\circ$, it holds that $\inf_{W_0 \in \mathcal{W}(\Theta_0)} D(P_{W'}\|P_{W_0}) + f(W') < M$. But this contradicts that M is the minimum. This shows that $P_{W_1^*}$ has full support.

Finiteness of Support in Example 6 and 7 We claimed that the supports of the priors W_1^* and W_0^* in Example 7 (restricted Θ_1) are finite. In fact they are finite also in Example 6 (unrestricted Θ_1). We verify this for W_1^*, the case for W_0^* is analogous. Note that for given sample size n, the probability distribution P_W is completely determined by the probabilities assigned to the sufficient statistics $N_{1|a}, N_{1|b}$. This means that for each prior $W \in \mathcal{W}(\Theta_1)$, the Bayes marginal P_W can be identified with a vector of $M_n := (n_a + 1) \cdot (n_b + 1)$ real-valued components. Every such P_W can also be written as a mixture of P_θ's for $\theta = (\mu_{a|1}, \mu_{b|1}) \in \Theta_1$, a convex set. By Carathéodory's theorem we need at most M_n mixture components to describe an arbitrary P_W as a mixture of the P_θ's; this proves the claim.

Appendix B Additional Clarifications and Proofs

B.1 Section 1.3

We first discuss several aspects of the filtration to be used in Definition 1. For concreteness and simplicity we consider the sequential set-up of Example 1 and Section 5 in which there is an underlying data stream $Y_1, Y_2, \ldots$.

Filtration: Subtleties The natural filtration to use if there is no additional nonstochastic side-information, is $(\sigma(Y^{(j)}))_{(j)}$, i.e. when specifying a new E-variable $E_{(j+1)}$ we are allowed to make use of all data we observed in the past. However, the E-variable assignment based on the t-test leads to a filtration that is coarser than $\sigma(Y^{T_{(j)}})$, as can be seen from (33) in Section 5. A similar phenomenon may ocur in other settings with nuisance parameters expressing group invariance. To illustrate, consider a fully sequential assignment as defined in that section. Then $U_i = V_i = Y_i/|Y_1|$, so if one knows all the U_i one still does not know the Y_i. The only repercussion of this is that the stopping times $T_{(j)}$ are now not allowed to directly depend

directly on the Y_i, since this is not measurable. Under standard (e.g. fixed, or stopping when the likelihood overshoots some threshold — the likelihood can be written as a function of the V_i and thus *is* measurable) stopping times all $E_{(j)}$ will still be E-variables and Proposition 2 can still be applied and Ville-Robbins still holds — so the restriction is quite harmless.

Filtration: Conditional Distributions In the 2×2 setting the Θ_1 represent conditional distributions of $Y \mid X$. While neither Section 1.3 nor Section 5 formally allowed for that setting, the extension is straightforward. We simply assume the underlying stream is of the form $(X_1, Y_1), (X_2, Y_2), \ldots$ (with, in the 2×2 example, $X_i \in \{a, b\}$). The distributions in $\mathcal{H}_1$ are now extended to define a random process of independent outcomes with the same conditional distribution, i.e. for all $\theta_1 \in \Theta_1$, we set $p_{\theta_1}(y^n \mid x^n) := \prod_{i=1}^n p_{\theta_1}(y_i \mid x_i)$. We then only need to extend Definition 1 of conditional E-variables to deal with this extension. This is achieved by setting $\mathcal{F}_{(m-1)}$ in the definition to $\sigma(\mathbf{Y}^{(m-1)}, \mathbf{X}^{(m)})$.

Filtration: Side Information Now we consider how the set-up can be extended to deal with side-information that may be used e.g. after j studies to decide whether to start a new, $j+1$st study at all, and if so, what the sample size of that study will be. For this we again need to extend $\mathcal{H}_0$ and $\mathcal{H}_1$ so that its elements define a conditional random process, with at the time that the j-th study has just been observed, also the additional variables $\mathbf{R}_{(1)}, \ldots, \mathbf{R}_{(j)}$ observed. Even if there is an underlying stream of data $Y_1, Y_2, \ldots$ so that the $\mathbf{Y}_{(j)}$ have an internal structure, the $\mathbf{R}_{(j)}$ are not required to have such a structure. To make all desired probabilities well-defined and at the same time make sure that the side-information is really external, we impose a conditional independence on the underlying stream: under each $P \in \mathcal{H}_0$, for each $j \in \mathbb{N}$, each i with $T_{(j-1)} < i \leq T_{(j)}$, the conditional distribution of Y_i given Y^{i-1} and $\mathbf{R}^{(j-1)}$ is defined to be the same as the distribution under P of Y_i given Y^{i-1}, which is already well-defined once $\mathcal{H}_0$ is specified. With this definition, we can set $\mathcal{F}_{(j)} = \sigma(Y^{(j)}, \mathbf{R}^{(j)})$ (or $\mathcal{F}_{(j)} = \sigma(\mathbf{U}^{(j)}, \mathbf{R}^{(j)}, \mathbf{X}^{(j+1)})$ if the $\mathcal{H}_j$ already contain conditional distributions, or $\mathcal{F}_{(j)} = \sigma(\mathbf{V}^{(j)}, \mathbf{R}^{(j)}, \mathbf{X}^{(j+1)})$ with $\mathbf{U}^{(j)}$ a coarsening of $\mathbf{Y}^{(j)}$ if required, such as in the t-test setting). Focusing on the simplest case with $\mathcal{F}_{(j)} = \sigma(Y^{(j)}, \mathbf{R}^{(j)})$, the construction ensures that $\mathbf{E}_P[E_{(j)} \mid \mathbf{Y}^{(j-1)}, \mathbf{R}^{(j-1)}] \leq 1$ (and hence we have safety under OC) if, just as in Proposition 4, we either use a valid sequential assignment to define the $E_{(j)}$ (the S-variable specification is sequentially decomposable as in Section 5) or data are i.i.d. and we use the local assignment to define $E_{(j)}$. Note that with this construction, the $\mathbf{R}^{(j)}$ are allowed to depend on $\mathbf{Y}_{(1)}, \ldots, \mathbf{Y}_{(j)}$ in unspecified ways. This is unproblematic because (other than in the case with conditional distributions and $\mathbf{X}^{(j+1)}$) the $E_{(j+1)}$ cannot depend on $\mathbf{R}^{(j+1)}$. For example, based on what she sees in the $\mathbf{Y}_{(j)}$, your boss may decide to announce 'we have money to do an additional study with 100 patients', which can be encoded as a particular outcome of $\mathbf{R}_{(j)}$. This may then be used to decide to continue (i.e. set τ in Proposition 2 to be larger than j) and set $T_{(j+1)}$ to be 100.

B.2 Section 2

Proof of (9) To see this, note that for every E-variable $E = e(\mathbf{Y})$ we can define $q(\mathbf{Y}) := e(\mathbf{Y}) \cdot p_0(\mathbf{Y})$ and then $\int q d\mu = \int p_0(\mathbf{Y}) e(\mathbf{Y}) d\mu \leq 1$; conversely every sub-density q defines an E-variable by setting $E = q(\mathbf{Y})/p_0(\mathbf{Y})$ which obviously gives $\mathbf{E}_{P_0}[E] \leq 1$.

B.3 Section 3

Proof of Proposition 3 The monotone likelihood ratio property implies stochastic dominance (Lehmann et al., 2005), i.e. with $P[T]$ denoting the distribution of the statistic T, we must have $\mathbf{E}_{P_\delta[T]}[f(T)] \geq \mathbf{E}_{P_{\delta'}}[f(T)]$ for $\delta \geq \delta'$ and every increasing function f. This implies that

$$D(P_{\delta^+} \| P_{\delta^-}) = \mathbf{E}_{\mathbf{Y}\sim P_{\delta^+}}\left[\log \frac{p_{\delta^+}(\mathbf{Y})}{p_{\delta^-}(\mathbf{Y})}\right] = \inf_{\delta \geq \delta^+} \mathbf{E}_{\mathbf{Y}\sim P_\delta}\left[\log \frac{p_{\delta^+}(\mathbf{Y})}{p_{\delta^-}(\mathbf{Y})}\right]. \tag{46}$$

We also have, by the same stochastic dominance result, for $\delta \leq \delta^-$,

$$\mathbf{E}_{\mathbf{Y}\sim P_\delta}\left[\frac{p_{\delta^+}(\mathbf{Y})}{p_{\delta^-}(\mathbf{Y})}\right] \leq \mathbf{E}_{\mathbf{Y}\sim P_{\delta^-}}\left[\frac{p_{\delta^+}(\mathbf{Y})}{p_{\delta^-}(\mathbf{Y})}\right] = 1,$$

so that $E^* = p_{\delta^+}(\mathbf{Y})/p_{\delta^-}(\mathbf{Y})$ is an E-variable, which directly leads to the first inequality in the chain of (in)equalities (47) below. The first equality follows by (46), the second because, since E^* is of form p_{δ^+}/p_{W_0}, with $W_0 \in \mathcal{E}(\Theta_0)$ (namely, W_0 puts all mass on δ^-), it must, by Corollary 2, be the GRO-E-variable for testing between a modified $\Theta_1' = \{\delta^+\}$ and Θ_0. The final two inequalities are immediate:

$$\begin{aligned}
&\sup_{E\in\mathcal{E}(\Theta_0)} \inf_{\delta\geq\delta^+} \mathbf{E}_{\mathbf{Y}\sim P_\delta}[\log E] \geq \inf_{\delta\geq\delta^+} \mathbf{E}_{\mathbf{Y}\sim P_\delta}\left[\log \frac{p_{\delta^+}(\mathbf{Y})}{p_{\delta^-}(\mathbf{Y})}\right] = \mathbf{E}_{\mathbf{Y}\sim P_{\delta^+}}\left[\log \frac{p_{\delta^+}(\mathbf{Y})}{p_{\delta^-}(\mathbf{Y})}\right] = \\
&\sup_{E\in\mathcal{E}(\Theta_0)} \mathbf{E}_{\mathbf{Y}\sim P_{\delta^+}}[\log E] \geq \inf_{\delta\geq\delta^+} \sup_{E\in\mathcal{E}(\Theta_0)} \mathbf{E}_{\mathbf{Y}\sim P_\delta}[\log E] \geq \sup_{E\in\mathcal{E}(\Theta_0)} \inf_{\delta\geq\delta^+} \mathbf{E}_{\mathbf{Y}\sim P_\delta}[\log E].
\end{aligned} \tag{47}$$

This chain of inequalities implying that all its parts are equal, the result follows.

B.4 Section 4.3

Proof of (29) (29) follows from (Lai, 1976, page 273) or as special case of Theorem 2.1. of Berger et al. (1998), but the first proof leaves out details and the second is very abstract, so for convenience we give a direct proof. For simplicity restrict to the case with W putting all its mass on a particular δ. Fix arbitrary $\sigma > 0$ and $n \geq 2$ and note that $V_1 \in \{-1, 1\}$ and $P_\delta(V_1 = 1) = P_{\delta,\sigma}(Y_1 > 0)$ (note that $p'_{W[\delta]}(V_1)$ is a probability mass function, whereas $p'_{W[\delta]}(V_i \mid V^{i-1})$ is defined as density relative to Lebesgue measure for $i > 1$). We must then have:

$$\begin{aligned}
&P_\delta(V_1 = 1) \cdot p'_\delta(v_2, \ldots, v_n \mid V_1 = 1) \\
&= P_\delta(V_1 = 1) \cdot \int_0^\infty p_{\delta/|y_1|,\sigma/|y_1|}(v^n \mid Y_1 = y_1) p_{\delta,\sigma}(y_1 \mid Y_1 > 0) dy_1 \\
&= \int_0^\infty \prod_{i=2}^n \left(\frac{1}{\sqrt{2\pi}} \cdot \frac{|y_1|}{\sigma} \cdot e^{-\frac{1}{2}\left(\frac{v_i}{\sigma/|y_1|} - \delta\right)^2}\right) \cdot \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{y_1}{\sigma} - \delta\right)^2}\right) dy_1 \\
&= \int_0^\infty \prod_{i=1}^n \left(\frac{1}{\sqrt{2\pi}} \cdot \frac{|y_1|}{\sigma} \cdot e^{-\frac{1}{2}\left(\frac{v_i}{\sigma/|y_1|} - \delta\right)^2}\right) \cdot \frac{1}{|y_1|} dy_1 \\
&= \int_0^\infty \prod_{i=1}^n \left(\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\tau} \cdot e^{-\frac{1}{2}\left(\frac{v_i}{\tau} - \delta\right)^2}\right) \cdot \frac{\tau}{\sigma} \left|\frac{dy_1}{d\tau}\right| d\tau = \int_0^\infty \prod_{i=1}^n \left(\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\tau} \cdot e^{-\frac{1}{2}\left(\frac{v_i}{\tau} - \delta\right)^2}\right) \cdot \frac{1}{\tau} d\tau
\end{aligned}$$

Here in the first equality we used that, given $Y_1 = y_1$, the V_i are independent Gaussian, with variance $\sigma/|y_1|$ and mean $\delta\sigma/|y_1|$ hence effect size $\delta/|y_1|$ and hence density $p_{\delta/|y_1|,\sigma/|y_1|}$. The second equality replaces the conditional density of Y_1 by the marginal (so that $P_\delta(V_1 = 1)$ cancels) and exploits that $y_1 = v_1|y_1|$, the third is a standard change-of-variable from $\sigma/|y_1|$ to τ using the Jacobian transformation and the fourth is immediate.

This shows the desired result if $V_1 = 1$. The case for $V_1 = -1$ is analogous.

Applicability of Proposition 3 to the t-test Setting (Section 4.3 underneath Theorem 2) A simple calculation gives that $p'_{\delta^+}(\mathbf{V})/p'_{\delta^-}(\mathbf{V})$ can be re-expressed as a density ratio of the t-statistic $T = t(\mathbf{Y})$, i.e. $p'_{\delta^+}(\mathbf{V})/p'_{\delta^-}(\mathbf{V}) = p''_{\delta^+}(t(\mathbf{Y}))/p'_{\delta^-}(t(\mathbf{Y}))$, where p''_δ is the density of a noncentral t-distribution with $\nu := n-1$ degrees of freedom and noncentrality parameter $\mu = \sqrt{n}\delta$. But these densities are well-known to form a monotone likelihood ratio family in the T statistic, so that we can apply Proposition 3 to $\{p''_\delta : \delta \in \Delta\}$.

B.5 Section 5

Proof of Proposition 4 For Part 1, note that, for each $P \in \mathcal{H}_0$,

$$\mathbf{E}_{P|\sigma(U^{T_{(j-1)}})}\left[\prod_{i=T_{(j-1)+1}}^{T_{(j)}} E_i\right] =$$

$$\mathbf{E}_{U_{T_{(j-1)}+1}\sim P|\sigma(U^{(T_{(j-1)})})}\left[\ldots\left[\mathbf{E}_{U_{T_{(j)}-1}\sim P|\sigma(U^{T_{(j)-2}})}\left[E_{T_{(j)}-1}\right]\cdot\mathbf{E}_{U_{T_{(j)}}\sim P|\sigma(U^{T_{(j)-1}})}\left[E_{T_{(j)}}\right]\right]\right] \leq 1,$$

with the inequality following by successively replacing the expectations from right to left by their upper bound of 1, which holds by sequential compatibility. Part 2 is immediate from the fact that we assume the data are i.i.d. Part 3 is immediate from combining Part 1 and 2. For Part 4, note that for each j, for the outcomes Y_i with $i \in \{T_{(j-1)+1}, \ldots, T_{(j)}\}$ in the j-th study, we have $\mathbf{E}[E_i \mid \sigma(U^{i-1})] \leq 1$ if the sequential assignment was used in the j-th study, by (31); but we also have $\mathbf{E}[E_i \mid \sigma(Y^{T^{(j-1)}}, U_{T_{(j-1)}+1}, \ldots, U_{T_{(j-1)}+i})] = \mathbf{E}[E_i \mid \sigma(U_{T_{(j-1)}+1}, \ldots, U_{T_{(j-1)}+i})] \leq 1$ if the local assignment was used, since data are i.i.d. and using again (31).

B.6 Section 6

Determining Sample Size for a Desired Power Consider a 1-sided test for the normal location family with variance 1 which rejects if $\hat{\mu} \geq f(n)/\sqrt{n}$ where $\hat{\mu}$ is the MLE at sample size n and f is some increasing function of n. We want to find the smallest n at which we achieve power $1-\beta$ under mean δ, i.e. such that

$$P_\delta(\sqrt{n}(\hat{\mu} - \delta) \geq f(n) - \delta\sqrt{n}) \geq 1 - \beta,$$

where under P_δ, the $Y_1, \ldots, Y_n$ are i.i.d. $N(\delta, 1)$. This is the smallest n at which $f(n) - \delta\sqrt{n} \geq -z_\beta$, i.e. $\sqrt{n} \geq (z_\beta + f(n))/\delta$. The standard result for n_{NP} now follows by taking $f(n) = z_\alpha$. For the Bayesian test, to very good approximation, $f(n) = \sqrt{6 + \log n}$ (Example 3). Since the smallest n for the Bayesian test must be larger than n_{NP}, the minimal n_{BAYES} needed satisfies $n_{\text{BAYES}}/n_{\text{NP}} \geq (z_\beta + \sqrt{6 + \log n_{\text{NP}}})^2/(z_\beta + z_\alpha)^2$, giving a logarithmic factor as claimed.

Brownian Motion Fix $a > 0$ and, for a standard Brownian motion $X_t = B_t + ct$ with drift c, define $S = \min\{t > 0 : X_t \geq a\}$. The distribution of S is well-known and has density given by

$$f_{a,c}(s) = \frac{a \exp\left(-\frac{(a-cs)^2}{2s}\right)}{\sqrt{2\pi s^3}}. \tag{48}$$

Fix δ and let $n_t = t/\delta^2$ and $\mathcal{T} = \{\delta^2, 2\delta^2, 3\delta^2, \ldots\}$. Consider, for $t \in \mathcal{T}$, the discrete time process

$$W_t := \log \frac{p_\delta(Y^{n_t})}{p_0(Y^{n_t})}$$

where p_δ is the density of Y^n under $N(\delta, 1)$. Writing out the definition and re-arranging, we find that if $Y_1, Y_2, \ldots$ are i.i.d. $\sim N(\delta, 1)$, then for all $\mathcal{T}' \subset \mathcal{T}$ the conditional distribution of W_t given $\{W_t : t \in \mathcal{T}'\}$ (in particular, this includes the marginal distribution of W_t if we take $\mathcal{T}'$ empty) agrees with the conditional distribution of $X_t = B_t + (1/2)t$ and we can thus approximate the distribution of the first time when W_t exceeds $-\log\alpha$ by the distribution with density (48) with $c = 1/2$ and $a = -\log\alpha$ — the distribution of the first hitting time of B_t will be shifted slightly to the left, because when stopping W_t we are only checking the process at intervals of size δ^2. Intuitively, as $\delta \downarrow 0$, we expect the distribution functions to converge. To make this concrete, let q_β be the β-quantile of the distribution given by $f_{-\log\alpha,1/2}(s)$. We want to calculate $n_{\max}(\beta)$, the smallest n such that $P_\delta(\tau_1 \leq n) > 1-\beta$, i.e. we want to find the smallest n such that, with $t^* = \delta^2 n$, we have $P_\delta(\delta^2\tau_1 > t^*) < \beta$. The correspondence between W_t and X_t suggests that in the limit for $\delta \downarrow 0$, t^* converges to q_β, giving that $n_{\max}(\beta) \sim C_{\beta,0}/\delta^2$ with $C_{\beta,0} = q_\beta$ and $\mathbf{E}_{P_\delta}[\tau_\beta] \sim C'_{\beta,0}/\delta^2$ with $C'_{\beta,0} = \int_0^{q(\beta)} tf(t)dt + \beta q_\beta$.