Slow and Easy: a Theory of Browsing*

Job Market Paper

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Abstract

An agent needs to choose the best alternative drawn randomly with replacement from a menu of unknown composition. The agent is boundedly rational and employs an automaton decision rule: she has finitely many memory states, and, in each, she can inquire about some attribute of the currently drawn alternative and transition (possibly stochastically) either to another state or to a decision. Defining the complexity of a decision rule by the number of transitions, I study the minimal complexity of a decision rule that allows the agent to choose the best alternative from any menu with probability arbitrarily close to one. Agents in my model differ in their *languages*—collections of binary attributes used to describe alternatives. My first result shows that the tight lower bound on complexity among all languages is $3\lceil \log_2(m) \rceil$, where m is the number of alternatives valued distinctly. My second result provides a linear upper bound. Finally, I call *adaptive* a language that facilitates additive utility representation with the smallest number of attributes. My third result shows that an adaptive language is always *simple*: it admits the least complex decision rule that solves the choice problem. When $(3/4) \cdot 2^n < m \leq 2^n$ for a natural n, a language is simple if and only if it is adaptive.

Keywords: Bounded Rationality, Search, Automata, Complexity, Attributes

JEL: D01, D11, D83

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1 Introduction

Window shopping and its online cousin, browsing, are commonly observed behavioral patterns among consumers. Their defining features are a lack of urgency and a lack of direction. Consumers who browse are in no hurry to make a purchase and their attention jumps from item to item at random. In this paper, I interpret a browsing consumer as someone who sacrifices fast decision making to save on cognitive resources. Specifically, I show how consumers can achieve near optimality with very limited cognitive resources and no direction to their search as long as they take their time.

Consider, for example, the search for an appliance, for a toy or for an extra-curricular activity for a child. Most consumers are not regular participants in these markets and, therefore, do not know what options are available. Furthermore, the choice objects are difficult to compare because they are differentiated along many attributes, and many of them are difficult to quantify. Finally, the decision is not urgent, that is, the decision can be delayed by days, weeks, or even longer. Because of this last feature, the decision maker can adopt slow procedures and is willing to trade-off decision speed in favor of economizing on cognitive resources. My main results characterize the boundedly rational procedures that achieve near optimal decisions with very limited cognitive resources when time is not of the essence.

My model has two components, a *language* and an *automaton strategy*. The language is a list of yes-no questions such as *"Is this appliance rechargeable?"*; thus, the language encapsulates how the agent describes the alternatives. The automaton strategy consists of memory states, which play the same role here as in other automaton models, an interrogation rule that specifies which questions to ask, and a stochastic transition rule that describes how to move between the memory states and when to choose.

The economic environment is one of undirected search. The decision maker's attention is randomly drawn to one of the available objects which she proceeds to examine. This examination takes the form of "asking questions" and transitions between memory states upon hearing the answers. The examination stops when the decision maker selects the item for consumption or when a new alternative is brought to her attention. The search process assumes that the decision maker has no control over which alternatives captures her attention. More realistic search procedures will have aspects of undirected search but also feature some purposeful directed search. I choose the extreme case of undirected search to establish a benchmark. As my main results show, the agent can do remarkably well with very limited cognitive resources even in this extreme benchmark.

The decision maker's objective is to choose optimally while economizing on cognitive resources, that is, to find the least complex optimal strategy. As a measure of complexity, I consider *transitional complexity*. As the name suggests, transitional complexity counts the number of state-action pairs the decision maker employs. This notion of complexity measures the difficulty of executing the strategy. Banks and Sundaram (1990) discuss this notion of complexity in the context of repeated games played by finite automata. An example below illustrates the complexity and introduces my solution concept.

Suppose that the universe of items is the four element set $\{(1,1),(1,0),(0,1),(0,0)\}$, where each vector specifies the two attribute values of the item. The decision maker must search from an unknown (non-empty) subset of the four possible objects. Specifically, her task is to find the alternative that maximizes her preference $(1,1) \succ (0,1) \succ (1,0) \succ (0,0)$.

Consider a decision maker who has two memory states and employs the following strategy. In state 1, the strategy inquires about the first attribute: if the object has attribute 1, then the agent transitions to state 2—this counts as one transition—whereas if it does not have attribute 1, then the transition to state 2 occurs only with probability ϵ; with probability $1-\epsilon$ the agent remains in (transitions to) state 1—this counts for two more transitions. In state 2, the algorithm inquires about the second attribute: if the object has the second attribute, the agent chooses the object with probability 1—this counts for one transition—and if the object does not have the second attribute, the agent chooses it with probability ϵ^2 and remains in state 2 with probability $1-\epsilon^2$—this counts for two transitions. Thus, the complexity of the described above automaton strategy is $1+2+1+2=6$.

If the agent examines item $(1,1)$, then she will choose it first time she starts to investigate it, provided her attention is not drawn to a new alternative before that. When $\epsilon > 0$ is small, item $(0,1)$, on average, requires $\eta/((1-\eta)\epsilon)$ draws from the menu to be chosen, where η is the probability to draw a new alternative. For item $(1,0)$, this number is $\eta/((1-\eta)\epsilon^2)$ and for item $(0,0)$, it is $\eta^2/((1-\eta)\epsilon^3)$. If ϵ is small and the menu does not contain item $(1,1)$, then the decision maker is likely to cycle multiple times through all her available choices before settling on one of the items. Moreover, in that case, the decision maker will most likely settle on the best available choice. In the limit, as ϵ converges to zero, the probability of picking the best available item from any conceivable menu converges to one. I will refer to this property of a strategy as *solving* the decision problem.

In the example above, there are only 4 alternatives that can comprise a potential menu, and the strategy used to solve the choice problem employs 6 transitions. What happens when there are a lot of alternatives and they are differentiated along many attributes? My first main result, Theorem 1, shows that if the agent's language allows her to learn a convenient subset of attributes, she can solve the decision problem using as few as $3\lceil \log_2(m) \rceil$ transitions, where m is the number of indifference classes of items that the agent distinguishes, and $\lceil x \rceil$ denotes the ceiling of number x—the smallest integer weakly greater than x. The example above illustrates one of these "simple" solutions, since $6 = 3 \cdot \log_2(4)$. Theorem 1 also shows that the total number of transitions required to solve a decision problem for an agent with *any* language is at least $3\lceil \log_2(m) \rceil$; thus, $3\lceil \log_2(m) \rceil$ is a tight lower bound on complexity of a strategy that can solve the choice problem.

My next main result, Theorem 2, shows that some languages provide much less convenient description of items for the decision maker: the minimum number of transitions required to solve the choice problem can be more than $2M - 2$, where M is the total number of possible items. Theorem 2 also provides an upper bound on complexity: any language that can correctly distinguish items from distinct indifference classes allows for a solution of the choice problem by using at most $3M - 3$ transitions.

Theorems 1 and 2 show that language—that is, the way how the agent describes the alternatives—plays an important role in the demand for cognitive resources, which ranges from logarithmic to linear in the size of the choice problem. Thus, I study a particular class of languages that I call *adaptive* for the agent's preference relation. An adaptive language contains a small subset of questions—of the size of $\lceil \log_2(m) \rceil$—such that there exists a utility function additive with respect to the corresponding subset of attributes that correctly represents the strict part of the agent's preference relation. In the example above, the language that the agent uses to describe the items is adaptive because $m = 4$, and a two-attribute utility function $u(a) = a_1 + 2a_2$ represents the agent's preference relation.

My final main result, Theorem 3, shows that an agent with an adaptive language can always solve the choice problem by using the simplest possible strategy—a strategy that contains $3\lceil \log_2(m) \rceil$ transitions. Moreover, if the number of indifference classes m satisfies $(3/4) \cdot 2^n < m \leq 2^n$ for some natural n, then agent who can solve the choice problem by using the simplest possible strategy must have an adaptive language. Thus, for this range of m, Theorem 3 fully characterizes all *simple* languages—the ones that allow for the simplest possible solution of the choice problem—as being adaptive.

1.1 Related Literature

This paper contributes to the literature on the search for a multi-attribute alternative, on the decision making with limited memory and information processing constraints, and, more broadly, on the complexity of the decision problems.

There is an extensive literature discussing the optimal search, including Weitzman (1979), Kohn and Shavell (1974), Morgan and Manning (1985). In Klabjan, Olszewski, and Wolinsky (2014), the authors are concerned with the optimality of the search for a multi-attribute alternative. In their paper, the agent is fully rational, but the information acquisition is costly.

Sanjurjo (2014) builds a theoretical model of the memory-constrained search for a multi-attribute alternative. In his model, the agent memorises both the values of the discovered attributes and indexes of the searched alternative/attribute pairs. His paper is mainly focused on discussing a class of the exhaustive search procedures that deliver the least memory load, without emphasis on the optimality. Related papers Sanjurjo (2015), Sanjurjo (2017) study a multi-attribute search with values of items, additive with respect to the attributes, in the laboratory setting.

In Dow (1991), the author considers an agent with one bit of memory who can use it to solve two separate decision problems. This paper shows that it is optimal to remember information relevant to one of the problems, but not to both problems jointly.

In a series of papers Cover (1969), Cover and Hellman (1970), Hellman and Cover (1970), Hellman and Cover (1971), authors study hypothesis testing and learning with finite memory. They show that allowing for randomization improves the effectiveness of automata designed to deliver the best possible long-run payoff subject to the m-state memory constraint. They find upper bounds on optimality and construct "ϵ-optimal" sequences of automata that yield payoff, arbitrary close to the upper bound. My paper studies a different decision problem from theirs in many aspects, but also relies on near-optimal sequences of the stochastic finite-state automata.

In Economics literature, the automata are used to model bounded rationality in the game theory context in Abreu and Rubinstein (1988), Kalai and Stanford (1988), Banks and Sundaram (1990), and in the context of information processing in Börgers and Morales (2004), Wilson (2014), Kocer (2010). The benefits of using randomization in simple algorithms is also discussed in Kalai and Solan (2003) in the context of dynamic decision making.

2 Model

2.1 Preference

The set of alternatives, A, is finite with generic element $a \in A$. A complete and transitive binary relation $\succeq$ describes the agent's preferences over alternatives. The symbols $\sim$ and $\succ$ denote the symmetric and asymmetric parts of $\succeq$. Let $C(a) = \{a' \in A | a' \sim a\}$ be an indifference class and let m be the number of distinct indifference classes, that is, m is the cardinality of the set $\{C(a) | a \in A\}$. I assume that the choice problem is not trivial: $m > 1$.

2.2 Undirected search

From the finite set of alternatives, A, nature chooses a non-empty menu $B \subseteq A$, of available alternatives. The particular modality of nature's choice of a menu is unimportant for the subsequent analysis except for the fact that every non-empty subset B of A is a possible menu.

As long as the search continues, there is a fixed probability $\eta \in (0,1)$ that, in any period, a new alternative draws the agent's attention. This alternative is chosen by the nature randomly with replacement from the set B according to the full-support probability[1] $\rho^B(a) = \rho(a)/\sum_{b \in B} \rho(b)$, where $\rho(a)$ is a sampling likelihood of alternative a. If an alternative $a \in B$ has the agent's attention, she examines it by sequentially asking questions about it in the manner described below and may choose it. The search ends if the agent chooses an alternative. There is no cost of delay and, thus, the agent's objective is to find the best alternative in the menu.

2.3 Language

A *language* is a family of non-trivial binary partitions $Q = \{Q_{i0}, Q_{i1}\}_{i=1}^n$ of A. I interpret Q_i as the question "does a have property i?" It does if $a \in Q_{i1}$ and it does not if $a \in Q_{i0}$. Let $N = \{1, \ldots, n\}$ be the index set of questions.

For example, let $A = \{b, c, d\}$ be three cars where b is a red car from 2010 with manual transmission, c is a red car from 2011 with automatic transmission, and d is a green car from 2010 with automatic transmission. Then, $Q = \{Q_{i0}, Q_{i1}\}_{i=1}^3$ such that $Q_1 = \{\{b\}, \{c, d\}\}$, $Q_2 = \{\{c\}, \{b, d\}\}, Q_3 = \{\{d\}, \{b, c\}\}$ is a language that describes the three cars. In this exam-

[1] My main results in Sections 3,4 hold for an arbitrary family of full-support distributions $\{\rho^B\}_{\emptyset \neq B \subseteq A}$.

ple, question 1 asks whether the car has manual transmission, question 2 asks if it is from 2011, and question 3 asks if it is green.

Every language corresponds to a description of alternatives as n-dimensional vectors with coordinates taking values 1 (if the answer is "yes") and 0 (if the answer is "no"). In the example above, a description of items in language Q is $b = (1,0,0)$, $c = (0,1,0)$, $d = (0,0,1)$. I denote by $Q_i(a) = 1$ if $a_i = 1$ and $Q_i(a) = 0$ if $a_i = 0$ the value of the i-th attribute of alternative a according to the language Q.

In the example above, an alternative language might consist of question 3 alone; such a language is adequate for an agent who is indifferent between the two red cars b, c, but is not adequate for agents who are not indifferent between b and c. Formally, a language Q is *adequate* for $\succeq$ if for any $a, b \in A$ such that $a \succ b$ there exists $i \in N$ such that $a_i \neq b_i$. Thus, adequate languages facilitate distinction of any pair of non-indifferent alternatives. Henceforth, I assume that the language is adequate for $\succeq$. I refer to the pair $(Q, \succeq)$ as the choice problem.

2.4 Automaton

To solve the choice problem, the agent employs an automaton with state space $S = S^o \cup \{\diamond\}$, where $S^o = \{1, ..., k\}$ contains the flexible memory states, and $\diamond$ is the decision state. Upon reaching $\diamond$, the agent chooses the alternative and the search ends. Each time a new alternative draws the agent's attention, the state initializes at $s = 1$; we adopt a convention that the decision to choose an alternative is executed before the new alternative is drawn[2].

An interrogation rule $\iota : S^o \to N$ specifies which question the agent asks in a memory state s. A more general model would allow the agent to ask no question at a particular state. None of the subsequent results would change if we allowed this possibility; we omit it to simplify the notation. A stochastic transition rule $\tau : S^o \times \{0,1\} \to \triangle(S)$ specifies the probability that the agent transitions from state $v \in S^o$ to state $s \in S$ as a function of the answer received in state v. Note that, if alternative a is under consideration, then this transition occurs with probability $\tau\Big(v, s, Q_{\iota(s)}(a)\Big)$.

Thus, an automaton $\sigma = (S, \iota, \tau)$ consists of the state space $S = S^o \cup \{\diamond\}$, the interrogation rule ι, and the stochastic transition rule τ. I denote by Σ the set of all automatons with a finite state space.

[2] Alternatively, if the agent decides to choose the current item, i.e. to move to the state $\diamond$ in the next period, but her attention switches to a new item, she would investigate the new item instead of choosing the previous one. My results remain the same in this case.

2.5 Stochastic choice

The realized menu $B \subseteq A$ and the sampling probability ρ^B from B determine the economic environment. The language Q and the automaton $\sigma = (S, \iota, \tau) \in \Sigma$ determines the agent's decision procedure. Together, the economic environment and the decision procedure, induce a dynamic random choice in a straightforward way.[3] In this paper, I am concerned with the total probability that the agent chooses alternative a from menu B:

$$p^B(a) := Pr\Big(\text{alternative } a \text{ is chosen from menu } B \text{ in some period}\Big) \tag{1}$$

When necessary, I will use the subscript σ to indicate the dependence of $p^B(a)$ on σ.

As an illustration, consider the car example, described above. The automaton $\sigma \in \Sigma$, described in Figure 1 below, has a single state which asks the question, "does the car have

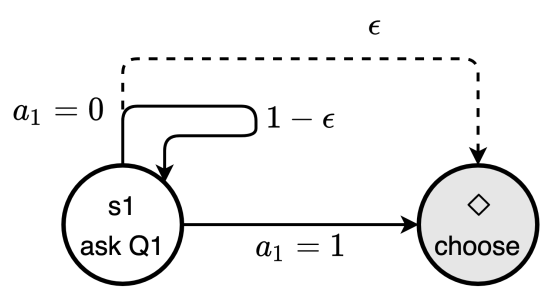

Figure 1

manual transmission?" If the answer is yes, then the agent chooses the car immediately; if the answer is no, then the agent chooses the car with probability $\epsilon \in (0,1)$, and, otherwise, remains in state 1.

Suppose the menu $B = \{b, c\}$ consists of car b with manual transition and car c with automatic transmission. If car c is drawn from the menu in some period T, then with probability ϵ, it is chosen. With probability $(1-\epsilon)\cdot\eta$, it is not chosen and a new car catches the agent's attention in period $T+1$, which ends the investigation of c.[4] Finally, with probability $(1-\epsilon)\cdot(1-\eta)$, the agent continues to investigate car c and the process repeats. Summing the probabilities of choosing c in periods $T, T+1, \ldots$, we get $q(c) = \epsilon/(1-(1-\eta)(1-\epsilon))$ for the probability of choosing c during a single uninterrupted investigation process. Similarly, $q(b) = 1$ is the probability of choosing car b during a single uninterrupted investigation process.

[3]The straightforward, but somewhat bulky formal description can be found in the Appendix B.7

[4]Note that the newly drawn car can be car c as well.

To calculate the total probability $p^B(c)$ that c is chosen, note that c is drawn from the menu with probability $\rho^B(c) = \rho(c)/(\rho(b)+\rho(c))$. Conditional on being drawn, c is chosen with probability $q(c)$. If c is not chosen then either b is drawn or c is drawn again. The former event occurs with probability $(1-q(c))\cdot\rho^B(b)$ and the latter with probability $(1-q(c))\cdot\rho^B(c)$. Summing the probabilities of choosing c after a single investigation, after two investigations, et cetera, we obtain $p^B(c) = q(c)\rho(c)/(q(c)\rho(c)+\rho(b))$.

A Luce rule (Luce (1959)) is a random choice rule in which each alternative a has a weight $V(a)$ and the probability of choosing a from menu B is $V(a)/\sum_B V(a')$. Calculating the probabilities $p^B(a)$ for all possible menus B and all possible alternatives reveals that the resulting random choice rule is a Luce choice rule with weight $\rho(a)\cdot q(a)$ assigned to car $a \in \{b,c,d\}$. Lemma 1, below, notes that this observation holds generally.

Lemma 1. *The family of choice probabilities $\rho^B(a)$ constitutes a Luce choice rule with the convention that if all weights of alternatives in menu B are zero, then $\rho^B(a) = 0$ for all $a \in B$.*

2.6 Near optimal decision procedures

The random choice rule described in Lemma 1 makes mistakes, that is, it may choose inferior choices from a menu. My notion of optimality allows such mistakes but requires that they can be made arbitrarily small. Specifically, I study sequences of automata for which the probability of choosing optimally converges to one. For $\sigma \in \Sigma$, let $\mathcal{T}_\sigma$ denote the *transitions of* σ, that is, the transitions that occur with positive probability:

$$\mathcal{T}_\sigma := \Big\{(v,s,j) \in S^o \times S \times \{0,1\} \,\Big|\, \tau(v,s,j) > 0\Big\} \tag{2}$$

Let $\mathbb{N}$ be the set of natural numbers. A decision rule $\psi_{r\in\mathbb{N}}$ is a sequence of automata $\{\sigma_r\}$ such that all elements of the sequence share the same state space $S = S_\psi = S^o_\psi \cup \{\diamond\}$, the same interrogation rule $\iota = \iota_\psi$, and the same set of transitions $\mathcal{T} = \mathcal{T}_\psi$. I denote with Ψ the set of all decision rules; thus,

$$\Psi := \Big\{\{\sigma_r\}_{r\in\mathbb{N}} \subset \Sigma^{\mathbb{N}} \,\Big|\, (S_{\sigma_r}, g_{\sigma_r}, \mathcal{T}_{\sigma_r}) \;=\; (S_{\sigma_t}, g_{\sigma_t}, \mathcal{T}_{\sigma_t})\ \forall r,t \in \mathbb{N}\Big\} \tag{3}$$

When there is some parameter, $\epsilon = \epsilon_r \longrightarrow 0$, such that the transition probabilities τ_r depend on r only via ϵ_r, I denote the decision rule by $\psi = \{\sigma_\epsilon\}_{\epsilon\to 0}$ or simply $\psi = \sigma_\epsilon$ instead of $\psi = \{\sigma_r\}_{r=\mathbb{N}}$ to ease notation.

A decision rule $\psi = \{\sigma_r\}_{r\in\mathbb{N}} \in \Psi$ *solves* the choice problem $(Q,\succeq)$ if

$$\lim_r \sum_{a\in B:\ a\succeq b\ \forall b\in B} p^B_{\sigma_r}(a) \;=\; 1 \qquad\qquad \forall B:\ \varnothing \neq B \subseteq A \tag{4}$$

Thus, a decision rule solves the choice problem if, for every menu, the probability of choosing optimally converges to one.

To illustrate the solution concept, let us return to the car example, discussed above. Suppose the agent prefers manual transmission but is otherwise indifferent between the features so that the agent's preference relation is $b \succ c \sim d$. Consider the decision rule $\psi^* = \sigma_\epsilon$, where $\epsilon \longrightarrow 0$, and for a fixed $\epsilon > 0$, $\sigma_\epsilon = \sigma$ is the automaton, shown in Figure 1, and discussed above. To see why the rule σ_ϵ solves the choice problem, consider menu $B = \{b, c\}$ and recall our calculations, performed above, to get

$$p^B(c) \;=\; \frac{q(c)\rho(c)}{q(c)\rho(c) + \rho(b)} \;=\; \frac{\epsilon\rho(c)}{\epsilon\rho(c) + (\eta + \epsilon - \eta\epsilon)\rho(b)} \;\longrightarrow\; 0$$

Thus, the agent chooses car b from menu $B = \{b, c\}$ with probability, converging to one in the limit. Similar calculations show that, in the limit $\epsilon \longrightarrow 0$, car b is always chosen with probability one from any menu that contains this car. If the choice set contains no car with automatic transmission then every choice is optimal and, thus, it suffices to note that the agent will eventually choose one of the cars.

Every choice problem with adequate language has a solution:

Lemma 2. *There exists a decision rule $\psi \in \Psi$ that solves the choice problem $(Q, \succeq)$.*

3 Complexity

A standard measure of complexity of an automaton is the cardinality of its state space[5]. We can interpret it as measuring the size of the "operational" memory that the decision maker requires to implement the choice procedure. However, two automata with the same state space may differ in terms of the complexity of their transition structures, as has been pointed out in Banks and Sundaram (1990). The importance of transition structures for perceived complexity has been shown experimentally in Oprea (2020). I define the complexity, κ, of the automaton $\sigma = (S, g, \tau) \in \Sigma$ as the number of distinct transitions that are a possible:

$$\kappa(\sigma) := |\mathcal{T}_\sigma| \;=\; \left|\left\{(v, s, j) \in S^o \times S \times \{0,1\} \;\middle|\; \tau(v, s, j) > 0\right\}\right| \tag{5}$$

I interpret κ, in line with Banks and Sundaram (1990), to be the number of distinct events that the decision maker needs to distinguish to be able to use the automaton σ. Thus, it is a measure of the memory required to recall the decision procedure itself.

[5]See , for instance, Rubinstein (1986) in the context of repeated games

The complexity $\kappa(\psi)$ of the decision rule $\psi = \{\sigma_r\}_{r\in\mathbb{N}} \in \Psi$ is the complexity of the automaton σ_r for any r—recall that transitions $\mathcal{T}$ are the same along the sequence of automata that comprise a decision rule.

For example, the complexity of the decision rule ψ^*, depicted in Figure 1 is $\kappa(\psi^*) = |\mathcal{T}_{\psi^*}| = 3$, since $\mathcal{T}_{\psi^*} = \{(1,\diamond,1),(1,\diamond,0),(1,1,0)\}$; to calculate the complexity from the graphical representation of the automaton, we can simply count the number of arrows.

As the following example illustrates, some languages are more amenable to low complexity solutions than others. In the car example above, the agent might have the following language: $Q^{**} = \{Q_2, Q_3\}$, where $Q_2 = \{\{c\},\{b,d\}\}$ and $Q_3 = \{\{d\},\{b,c\}\}$. A decision rule $\psi^{**} = \sigma_\epsilon$ that solves the choice problem $(Q^{**},\succeq)$, is shown in Figure 2.

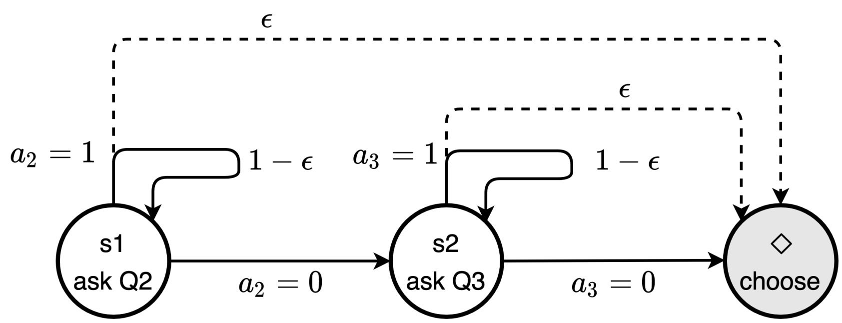

Figure 2

The set of transitions of ψ^{**} is $\mathcal{T}_{\psi^{**}} = \{(1,1,1),(1,\diamond,1),(1,2,0),(2,2,1),(2,\diamond,1),(2,\diamond,0)\}$ and hence, its complexity is $\kappa(\psi^{**}) = 6$. Note also that ψ^{**} uses $|S^o_{\psi^{**}}| = 2$ memory states. It can be shown that any decision rule that solves the choice problem $(Q^{**},\succeq)$ in our example should have complexity at least 6 and use at least 2 memory states.

For a given preference relation $\succeq$, I define the complexity $\kappa(Q)$ of an adequate language Q to be the minimal complexity of the decision rules that solve the choice problem $(Q,\succeq)$:

$$\kappa(Q) := \min_{\psi\in\Psi:\ \psi \text{ solves } (Q,\succeq)} \kappa(\psi)$$

My first main result provides a lower bound on complexity *for all languages* Q and, moreover, shows that there exists a language that achieves this lower bound. Recall that $\lceil x \rceil$ denotes the smallest integer weakly greater than x.

Theorem 1. *Let the preference relation $\succeq$ have m indifference classes. Then for any language Q, $\kappa(Q) \geq 3\lceil \log_2(m) \rceil$, and there exists a language Q such that $\kappa(Q) = 3\lceil \log_2(m) \rceil$.*

While Theorem 1 focuses on the transitional complexity, the size of the state space required to solve the choice problem is also an important measure of its complexity. For a given a preference relation $\succeq$, I define a memory load $\mathcal{M}(Q)$ of an adequate language Q to be the smallest size of the memory state space of a decision rule that solves the choice problem $(Q, \succeq)$:

$$\mathcal{M}(Q) := \min_{\psi \in \Psi:\ \psi \text{ solves } (Q,\succeq)} |S_\psi|$$

My next result gives the tight lower bound on the memory load of a language and says that decision rules that deliver the smallest complexity must also achieve the minimal memory load.

Proposition 1. *Let the preference relation $\succeq$ have m indifference classes. Then for any language Q, $\mathcal{M}(Q) \geq \lceil \log_2(m) \rceil$, and there exists a language Q such that $\mathcal{M}(Q) = \lceil \log_2(m) \rceil$. Moreover, if a decision rule ψ solves the choice problem $(Q, \succeq)$, and $\kappa(\psi) = 3\lceil \log_2(m) \rceil$, then $|S_\psi| = \lceil \log_2(m) \rceil$.*

Theorem 1 says that the minimal required complexity grows as a logarithm of the number of indifference classes of $\succeq$. To interpret this result, it is useful to understand how complex a language can be; that is, what is the upper bound on complexity?

Suppose that $C^1, C^2 \subset A$ are two indifference classes of the preference relation $\succeq$. If C^1 or C^2 contains many alternatives, then there are languages that require a lot of questions to figure out if a particular alternative belongs to C^1 or C^2. All these questions should be asked in a decision rule that solves the corresponding choice problem. Thus, in contrast to the lower bound provided by Theorem 1, the number of the indifference classes of the preference relation does not limit the maximum complexity of a language.

Proposition 2. *For any natural number k, there is a set of items A, a preference relation $\succeq$ on A with two indifference classes and an adequate language Q such that $\kappa(Q) > k$.*

My next main result shows that the maximum possible complexity of an adequate language grows linearly in the total number of alternatives in A.

Theorem 2. *Let $|A|$ be the total number of alternatives. Then for any preference relation $\succeq$ and language Q, $\kappa(Q) \leq 3|A| - 3$. Moreover, for any $m \in \{2, ..., |A|\}$, there is a preference relation $\succeq$ with m indifference classes and a language Q such that $\kappa(Q) \geq 2|A| + \lceil \log_2(m) \rceil - 2$.*

Theorem 2 says that for any preference relation $\succeq$ on A, for any language Q, adequate with respect to $\succeq$, there is a decision rule with no more than $3|A|-3$ transitions that solves the choice problem $(Q,\succeq)$. Moreover, for any preference relation $\succeq$ there is an adequate language Q such that any decision rule that solves the choice problem $(Q,\succeq)$ must have at least $2|A|+\lceil\log_2(m)\rceil-2$ transitions.

An analogous result holds for the memory load; moreover, the upper bound is tight.

Proposition 3. *Let $|A|$ be the total number of alternatives. Then for any preference relation $\succeq$ and adequate language Q, $\mathcal{M}(Q) \leq |A|-1$. Moreover, for any $m \in \{2,\ldots,|A|\}$, there is a preference relation $\succeq$ with m indifference classes and a language Q such that $\mathcal{M}(Q) = |A|-1$.*

To compare the lower and upper bounds given by Theorems 1 and 2, suppose that set A contains 2^n alternatives, and there is no pair of indifferent alternatives; thus, $m = |A| = 2^n$. In this case, the tight lower bound on complexity of a language is $3 \cdot n$, while there are languages with complexity at least $2^{n+1}+n-2$.

4 Adaptive Languages

What languages Q allow for the least complex solutions of the choice problem $(Q,\succeq)$? I call such languages simple, meaning that the description of the problem in a language maximally eases the agent's choice problem. Formally, given a preference relation $\succeq$ with m indifference classes, a language Q is *simple*, if

$$\kappa(Q) \;=\; 3\lceil\log_2(m)\rceil$$

Thus, a language is simple if there is a decision rule that achieves the lower bound on complexity, given by Theorem 1, and solves the choice problem for that language.

Intuitively, if a language describes a certain choice problem conveniently for the decision maker, it should contain as few attributes as possible and, moreover, the relationship between the attributes and the value of the alternatives should be "simple."

Let the preference relation $\succeq$ with m indifference classes be given. In line with the intuition discussed above, I say that a language Q is *adaptive* if it contains a subset $Q' = \{Q_i\}_{i\in N'} \subseteq Q$ of $|N'| = \lceil\log_2(m)\rceil$ questions such that there exists a vector $\lambda \in \mathbb{R}^{N'}$ with $\lambda_i \neq 0$ for all $i \in N'$ and for all $a,b \in A$,

$$a \succ b \quad\Longrightarrow\quad u(a) > u(b), \qquad \text{where} \quad u(a) := \sum_{i\in N'} \lambda_i a_i \tag{6}$$

Thus, a language is adaptive if it facilitates the usage of an additive utility function u with few attributes. Note that the utility function u does not necessary represent the preference relation $\succeq$: I allow for the cases when $a \sim b$, but $u(a) > u(b)$. In other words, the utility function u correctly differentiates the non-indifferent alternatives, but can break ties of the preference relation $\succeq$. Note also that an adequate language should contain at least $\lceil \log_2(m) \rceil$ questions to be able to differentiate any pair of non-indifferent alternatives, hence I use this bound in the definition of an adaptive language.

For example, let $A = \{b, c, d, e\}$, and $b \succ c \succ d \succ e$. Then $m = 4$, $\lceil \log_2(m) \rceil = 2$, and the following language is adaptive: $Q = \{Q_1, Q_2\}$ with

$$Q_1 = \{\{b, c\}, \{d, e\}\}, \qquad Q_2 = \{\{b, d\}, \{c, e\}\}.$$

Thus, $b = (1,1)$, $c = (1,0)$, $d = (0,1)$, $e = (0,0)$ is a description of the alternatives in the language Q. Take $\lambda_1 = 2$ and $\lambda_2 = 1$, then $u(a) = 2a_1 + a_2$:

$$\begin{aligned} u(b) &= 2+1 = 3, & u(c) &= 2+0 = 2, \\ u(d) &= 0+1 = 1, & u(e) &= 0+0 = 0, \end{aligned}$$

and $u(b) > u(c) > u(d) > u(e)$, showing that Q is indeed adaptive.

Examples of the non-adaptive, but adequate languages for the preference relation $\succeq$ are $Q^* = \{Q_1^*, Q_2^*\}$ and $Q^{**} = \{Q_1^{**}, Q_2^{**}, Q_3^{**}, Q_4^{**}\}$, where

$$\begin{aligned} Q_1^* &= \{\{b, c\}, \{d, e\}\}, & Q_1^{**} &= \{\{b\}, \{c, d, e\}\}, \\ Q_2^* &= \{\{b, e\}, \{c, d\}\}, & Q_2^{**} &= \{\{c\}, \{b, d, e\}\}, \\ & & Q_3^{**} &= \{\{d\}, \{b, c, e\}\}, \\ & & Q_4^{**} &= \{\{e\}, \{b, c, d\}\}. \end{aligned}$$

A description of items in language Q^* is $b = (1,1)$, $c = (1,0)$, $d = (0,0)$ and $e = (0,1)$. Would Q^* be adaptive, there should exists an additive utility function u^*, defined on the corresponding vectors of attributes, satisfying $u^*(1,1) > u^*(1,0) > u^*(0,0) > u^*(0,1)$. However, the function u^* is not monotone with respect to the second attribute: $u^*(0,1) < u^*(0,0)$, but $u^*(1,1) > u^*(1,0)$, and hence, u^* cannot be additive.

The language Q^{**} describes alternatives as $b = (1,0,0,0)$, $c = (0,1,0,0)$, $d = (0,0,1,0)$ and $e = (0,0,0,1)$ and hence, allows for an additive utility representation $u^{**}(a) = 3a_1 + 2a_2 + a_1$. However, this representation uses too many attributes: $3 > 2 = \lceil \log_2(m) \rceil$. Any pair of questions of the language Q^{**} do not allow the agent to differentiate some pair of alternatives from A and thus, is not sufficient for utility representation of an antisymmetric preference relation $\succeq$. It follows that the language Q^{**} is not adaptive.

Adaptive languages always exist:

Proposition 4. *For any preference relation, there is an adaptive language.*

Note that if we add new questions to an adaptive language, the language remains adaptive. Thus the following Corollary is straightforward.

Corollary 1. *(i) The language that consists of all non-empty binary partitions of the set A is adaptive for every preference relation $\succeq$ on A; (ii) if language Q is adaptive for the preference relation $\succeq$, then language $Q' \supseteq Q$ is also adaptive for the preference relation $\succeq$.*

My main result in this section shows that all adaptive languages are simple; moreover, for a wide range of environments, adaptive languages are the only simple languages.

Theorem 3. *Let the preference relation $\succeq$ have m equivalence classes. If a language is adaptive, then it is simple. Moreover, if $(3/4) \cdot 2^n < m \leq 2^n$ for a natural n, then a language is simple if and only if it is adaptive.*

The first statement of Theorem 3 says that for any adaptive language, there is a decision rule ψ with $\kappa(\psi) = 3\lceil \log_2(m) \rceil$ that solves the choice problem $(Q, \succeq)$. As an illustration, recall the example from this section with $b \succ c \succ d \succ e$ and an adaptive language Q with $Q_1 = \{\{b,c\},\{d,e\}\}$, $Q_2 = \{\{b,d\},\{c,e\}\}$. A least-complex decision rule $\psi = \sigma_\epsilon$, $\epsilon \to 0$, that solves the choice problem $(Q, \succeq)$ is shown in Figure 3.

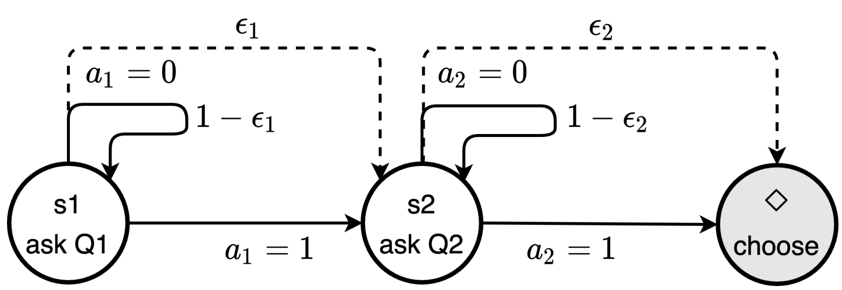

Figure 3

Recall that the utility function $u(a) = \lambda_1 a_1 + \lambda_2 a_2 = 2a_1 + a_2$ represents the preference relation $\succeq$ when alternatives are described in language Q. Consider $\epsilon_1 = \epsilon^{\lambda_1} = \epsilon^2$, and

$\epsilon_2 = \epsilon^{\lambda_2} = \epsilon$ with $\epsilon \longrightarrow 0$. Let $q(a)$ be the probability that alternative $a \in \{b, c, d, e\}$ is chosen during a single investigation process.

To calculate $q(d)$, for instance, let the investigation of d starts at period T. Since $d_1 = 0$, with probability $\epsilon_1 = \epsilon^2$, the automaton transitions to state $s = 2$, and with the remaining probability $1-\epsilon^2$, the state remains at $s = 1$, in which case, with total probability $(1-\epsilon^2)\cdot\eta$, the investigation ends, and with total probability $(1-\epsilon^2)\cdot(1-\eta)$, the agent continues the investigation process at state $s = 1$ in period $T+1$, and the analysis repeats. Summing up the probabilities of transition to state $s = 2$ in periods T, $T+1$,..., we obtain $\epsilon^2/(1-(1-\eta)(1-\epsilon^2))$ for the total probability to transition to state $s = 2$ during a single investigation process of d. Once the automaton is in state $s = 2$, with probability η, the investigation ends, and with probability $(1-\eta)$, the state transitions to $\diamond$, since $d_2 = 1$. Therefore,

$$q(d) \;=\; \frac{(1-\eta)\epsilon^2}{1-(1-\eta)(1-\epsilon^2)}$$

If we express quantities $q(a)$ for $a \in \{b, c, d, e\}$ in terms of the power expansion with respect to a vanishing parameter ϵ, we get the following leading terms:

$$\begin{array}{rclcrcl} q(b) & = & 1-\eta & \qquad & q(c) & \approx & ((1-\eta)/\eta)\cdot\epsilon \\ q(d) & \approx & ((1-\eta)/\eta)\cdot\epsilon^2 & \qquad & q(e) & \approx & ((1-\eta)/\eta^2)\cdot\epsilon^3 \end{array}$$

Note that for $a \in \{b, c, d, e\}$,

$$q(a) \;\approx\; constant(a)\cdot\epsilon^{u(b)-u(a)} \tag{7}$$

where the term $constant(a)$ does not depend on ϵ. Similarly to the car example discussed in Section 2.5, we can analyze the probability of choosing alternative $a \in \{b, c, d, e\}$ from menu $B \subseteq A$ after $1, 2, 3, \ldots$ investigation processes. The resulting random choice rule turns out to be a Luce rule with weights $V(a) = \rho(a)\cdot q(a)$. When $\epsilon \longrightarrow 0$, the ratio $q(a')/q(a)$ converges to zero for any pair of alternatives a, a' such that $a \succ a'$. Thus, in the limit, the agent chooses the best alternative with probability one from any menu.

In the general case, given an adaptive language Q, it is always possible to construct a decision rule that generalizes the rule ψ from the example above such that eq. (7) holds, where $u(b)$ is the utility of the best alternative in A. Let us describe a class of such decision rules.

For a given natural n and language Q with $|N| \geq n$ questions, consider a decision rule ψ^+ with the memory state space $S^o_\psi = \{1, \ldots, n\}$, a one-to-one interrogation function $\iota_\psi : S \to N$ and a set of transitions $\mathcal{T}_\psi$ defined as follows:

(a) $\tau(s, s+1, x_s) = 1$ with the convention that the state $|S|+1$ denotes $\diamond$;

(b) $\tau(s, s+1, 1-x_s) = \epsilon_s$ with the convention that the state $|S|+1$ denotes $\diamond$;

(c) There exists a unique $s' \in \{1, ..., s\}$ such that $\tau(s, s', 1-x_s) = 1-\epsilon_s$.

where $(x_1, ..., x_n) \in \{0,1\}^n$ and $(\epsilon_1, ..., \epsilon_n) \longrightarrow (0, ..., 0)$; and in particular, for all $i \in \{1, ..., n\}$, $\epsilon_i = (\epsilon_i)_r > 0$ for $r = 1, 2, ...$ is a sequence, converging to zero. A generic k-th state of the decision rule ψ^+ is represented in Figure 4, where the interrogation rule is assumed to be $\iota(k) = k$ for convenience.

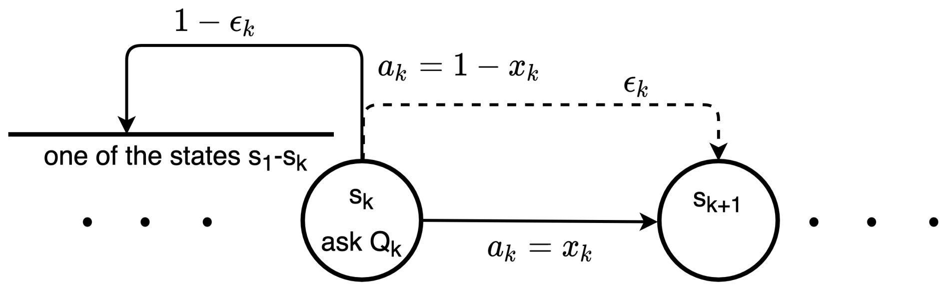

Figure 4

I denote by Ψ_n^+ the set of decision rules ψ such that there exists a permutation $\pi : \{1, .., n\} \to \{1, ..., n\}$ with $\pi(1) = 1$ of the indexes of the states $s \in S^0$ such that $\pi(\psi) = \psi^+$ is represented by the described above way, where the action of π on ψ is assumed to be re-enumeration of the states of S^0 and, correspondingly, the states in the interrogation and transition rules. Note that

$$\kappa(\psi^+) \; = \; 3n \qquad \forall \psi \in \Psi_n^+$$

The following result characterizes the adaptive languages as those that allow for a solution of the choice problem via some decision rule from the set Ψ_n^+ and also characterizes the simplest decision rules as the rules from the set Ψ_n^+ in case when $(3/4) \cdot 2^n < m \leq 2^n$.

Proposition 5. *Let the preference relation $\succeq$ have m equivalence classes. Then a language Q is adaptive if and only if there exists a decision rule $\psi \in \Psi_n^+$ that solves the choice problem $(Q, \succeq)$. Moreover, if $(3/4) \cdot 2^n < m \leq 2^n$ for a natural n, then a decision rule ψ such that $\kappa(\psi) = 3n$ solves the choice problem $(Q, \succeq)$ if and only if $\psi \in \Psi_n^+$.*

Note that for any $m \geq 2$, either $(1/2) \cdot 2^n < m \leq (3/4) \cdot 2^n$, or $(3/4) \cdot 2^n < m \leq 2^n$, where $n = \lceil \log_2(m) \rceil$—the two cases split the set of possible values of m such that $2^{n-1} < m \leq 2^n$

in half. Thus, the condition $(3/4)\cdot 2^n < m \leq 2^n$ serves as a richness assumption for the number of indifference classes in Theorem 3 and Proposition 5. Since my model operates with binary attributes, then powers of 2 play a special role in the analysis: note that the lower bound on complexity, given by Theorem 1, increases by 3 each time the number of indifference classes surpasses a power of 2.

Intuitively, a decision rule $\psi \in \Psi_n^+$ can be interpreted as a procedure that uses the probability to reach a certain memory state as a proxy for the alternative's utility based on the subset of attributes that have been already investigated. By encountering an alternative from the menu, the agent starts to learn its attributes sequentially; if the attribute's value is good, she jumps to the next memory state immediately and thus, the alternative endowed with all the good properties is quickly picked.

When the agent learns an undesirable property of the item, she becomes less enthusiastic and continues the investigation of other attributes only with a small probability. This penalty in the probability of transition to the next memory state applied upon learning the bad news reflects the relative importance of the corresponding attribute for the agent's preference which, by Proposition 5, has a representation via an additive utility function.[6]

Theorem 3 and Proposition 5 tell us that the agents who are able to describe alternatives using adaptive languages can save their cognitive resources by employing an intuitively simple decision procedure, described above. If the agent has a language that admits an additive utility representation of the agent's preference, but contains $n > \lceil \log_2(m) \rceil$ attributes, she can use a decision rule $\psi \in \Psi_n^+$ to solve the choice problem[7]. Each additional attribute increases the procedure's complexity by 3, but so long as the number of attributes that require investigation remains small, the agent's procedure is not too complex.

5 Sketches of the Proofs

In this section, I provide the key ideas behind the proofs of my main results. I first provide two lemmas that allow us to transform the analysis of solvability of a choice problem $(Q, \succeq)$ by a decision rule ψ to the analysis of directed graphs with vertex sets S_ψ, and the sets of directed edges (links) $\mathcal{T}_\psi$.

[6] Precisely, the strict part of the agent's preference relation should be represented by such function.

[7] Lemma 28 in the Appendix.

Let $q_\sigma(a)$ be the probability to choose alternative $a \in A$ during a single uninterrupted investigation process using the automaton strategy σ—this quantity has been introduced in Section 2.5 when discussing the car example, and discussed later in Sections 2.6 and 4.

Lemma 3. *A decision rule $\{\sigma_r\}_{r=1,2,...} \in \Psi$ solves choice problem $(Q,\succeq)$ if and only if the following two conditions hold:*
(i) $a \succ b$ implies $q_{\sigma_r}(b)/q_{\sigma_r}(a) \longrightarrow 0$ for all $a, b \in A$;
(ii) $q_{\sigma_r}(a) > 0$ for all $a \in A$.

Thus, Lemma 3 allows us to focus on analyzing the limiting properties of $q_{\sigma_r}(a)$ and abstract away from consideration of random choice from various menus. Consider a sequence of states of automaton $s_1...s_k$ that goes from the starting state $s_1 = 1$ to the decision state $s_k = \diamond$, and assume that this *path* contains each state at most once. For every alternative $a \in A$ and every such path $s_1...s_k$, we can calculate the probability of realization of $s_1...s_k$ during a process of investigation of a. Let $\omega^*_\sigma(a)$ be the largest of these probabilities among all paths for a fixed alternative $a \in A$.

Lemma 4. *A decisions rule $\{\sigma_r\}_{r=1,2,...} \in \Psi$ solves choice problem $(Q,\succeq)$ if and only if the following two conditions hold:*
(i) $a \succ b$ implies $\omega^*_{\sigma_r}(b)/\omega^*_{\sigma_r}(a) \longrightarrow 0$ for all $a, b \in A$;
(ii) $\omega^*_{\sigma_r}(a) > 0$ for all $a \in A$.

Thus, Lemma 4 allows us to consider only one highest-probability path[8] from the starting state $s = 1$ to the decision state $s = \diamond$ when analyzing the limiting properties of $q_{\sigma_r}(a)$.

Note that $\omega^*(a)$ is the product of transitional probabilities between states s_l, s_{l+1} in the highest-probability path $s_1...s_k$ when alternative a is under investigation. Given a decision rule ψ, I say that a link $(s, s', j) \in \mathcal{T}_\psi$ is *weak* if the corresponding probability $\tau(s, s', j)$ converges to zero. Otherwise, the link is *strong*.

5.1 Theorem 1—sketch of the proof

Assume that ψ solves $(Q,\succeq)$. I show that ψ should contain at least $2\cdot\lceil\log_2(m)\rceil$ strong links and at least $\lceil\log_2(m)\rceil$ weak links, proving statement (i) of the Theorem.

[8] The fact that it in my model, it is enough to analyze simple paths on the set of states to figure out the limiting properties of the dynamic system resembles the "Z-tree" technique from Kifer (1988) utilized in Kandori, Mailath, and Rob (1993) in the context of evolutionary game theory.

Indeed, if $Q' \subseteq Q$ is the set of questions that are used in some state according to the interrogation rule ι_ψ, then Q' should contain at least $\lceil \log_2(m) \rceil$ questions asked in different states of ψ to differentiate any pair of alternatives from distinct indifference classes of $\succeq$. There are at least two outgoing strong links from each of these states, providing the lower bound of $2 \cdot \lceil \log_2(m) \rceil$ on the number of strong links of ψ.

Next, if the agent is not indifferent between alternatives a and b, then Lemma 4 implies that the highest-probability paths for alternatives $a, b \in A$ discussed in the previous section should have different sets of weak links. If n_{weak} is the number of weak links of ψ, then it should be that $2^{n_{weak}} \geq m$, providing the lower bound of $\lceil \log_2(m) \rceil$ on the number of weak links of ψ.

To show that there exists a language with complexity $3\lceil \log_2(m) \rceil$, I first prove the existence of an adaptive language, that is, Proposition 4. For the proof of Proposition 4, it is without loss of generality to assume that $\succeq$ is antisymmetric and that $m = 2^n$ for some natural n. Let the first question of language Q be: "does alternative a belong to top 2^{n-1} alternatives in A?" The answer to this question divides A in two equal sets: Q_{11} consisting of top 2^{n-1} alternatives according to $\succeq$, and Q_{10}, consisting of bottom 2^{n-1} alternatives. Let the second question of language Q be: "does alternative a belong to top 2^{n-2} alternatives in the set Q_{11}, or to top 2^{n-2} alternatives in the set Q_{10}?" By continuing to design questions $1, 2, ..., n$ in a similar fashion, we design an adaptive language Q, where $\lambda_i = 2^{n-i}$ are weights of attributes $i = 1, ..., n$ in an additive utility representation of $\succeq$.

Finally, using an adaptive language Q designed above, we can construct a decision rule $\psi^+ \in \Psi^+$ discussed in Section 4 that solves that choice problem $(Q, \succeq)$. For that, it suffices to choose an appropriate vector of vanishing transition probabilities $\epsilon_i = \epsilon^{\lambda_i}$, $i = 1, .., n$, for some $\epsilon \longrightarrow 0$.

5.2 Theorem 2—sketch of the proof

To prove the first statement of Theorem 2, I use a universal procedure to design a decision rule ψ that solves the choice problem $(Q, \succeq)$ for an arbitrary adequate language Q. The set of flexible memory states of the decision rule ψ has a binary tree structure with root $s = 1$. Each node of this tree (memory state $s \in S^0_\psi$) is associated with a non-singleton set A^s comprised of alternatives $a \in A$ such that the investigation of a could lead to this node.

For an arbitrary node s of the tree, since Q is adequate, there always exists a question $i \in N$ such that both answers $j = 0$ and $j = 1$ are possible for alternatives in A^s. Set up the interrogation rule $\iota(s) = i$. If a subset of alternatives in A^s such that $a_i = j$ contains more than one alternative, create a successor node s' for this subset, and set up a transitional probability $\tau(s, s', j) = 1$. Otherwise, there is a single alternative $a \in A^s$ with $a_i = j$. In this case, create a loop link by setting up $\tau(s, s, 1) = 1 - \epsilon_s$, and a link that leads to the decision state by setting up $\tau(s, \diamond, 1) = \epsilon_s$.

Repeating the procedure described above for each node of the tree until there are no more successor nodes, we construct a decision rule ψ. By choosing appropriate sequences of probabilities ϵ_s, we can achieve a solution of $(Q, \succeq)$ by ψ. The number of links of ψ turns out to be $3|A| - 2$. In addition, it is possible to set up $\epsilon_s = 1$ for one of the states $s \in S^o$ to get $\tau(s, s, 1) = 0$ and decrease the number of links by one, proving the first statement of the Theorem.

To get the intuition behind the proof of statement (ii), suppose $m = |A|$; thus, there is no pair of indifferent alternatives for the agent. Consider language Q consisting of questions "is $a = b$?" for all alternatives $b \in A$ except of the best alternative. All of these questions should be asked in a decision rule ψ that solves the choice problem $(Q, \succeq)$. There are $2|A| - 2$ strong links associated. Lemma 4 implies that ψ should also contain at least $\lceil \log_2(m) \rceil$ weak links, proving the second statement of the Theorem.

5.3 Theorem 3—sketch of the proof

The intuition behind the first statement of Theorem 3 is discussed in Section 4. Let us, therefore, sketch the proof of the statement that all simple languages should be adaptive when $(3/4) \cdot 2^n < m \leq 2^n$ for a natural n. Note that $n = \lceil \log_2(m) \rceil$.

Let us assume that the first statement of Proposition 5 holds; the corresponding proof is relatively more straightforward, and some of its intuition has already been discussed. It suffices, therefore, to prove the only if part of the second statement of Proposition 5. Thus, we want to show the following assertion: for the considered range of m, if the decision rule ψ solves the choice problem $(Q, \succeq)$ and achieves the lowest possible complexity $\kappa(\psi) = 3\lceil \log_2(m) \rceil$, then $\psi \in \Psi_n^+$.

Let ψ be such a rule. Building on the proof of Theorem 1, we conclude that ψ should contain exactly $2n$ strong links—and, therefore, exactly n memory states—and also, exactly n weak links. Moreover, those weak links should not be loops—if a weak link is a loop, it cannot be used in the highest-probability path for any alternative.

Suppose that the allocation of the outgoing weak links among the memory states in ψ is $y = (y_1, \ldots, y_n)$, where y_s is the number of weak links, outgoing from state s. Then we can show that Lemma 4 implies the following Claim:

Claim 1. *If ψ solves the choice problem $(Q, \succeq)$, then* $\prod_{s \in S}(y_s + 1) \geq m$.

Note that if $y = (1, \ldots, 1)$, then the product in the Claim above is equal to 2^n. The next algebraic observation highlights the role of the richness assumption $(3/4) \cdot 2^n < m \leq 2^n$:

Claim 2. *Let $\sum_s y_s = n$. If the allocation of weak links y is such that $y \neq (1, \ldots, 1)$, then there is an allocation y' with $\sum_s y'_s = n$ such that* $\prod_{s \in S}(y_s + 1) \leq (3/4) \cdot \prod_{s \in S}(y'_s + 1)$

Since the number of weak links is n, any allocation of weak links except of $y = (1, \ldots, 1)$ fails to satisfy the necessary condition, given by Claim 1, whenever $(3/4) \cdot 2^n < m \leq 2^n$. Summarizing our observations, we obtain the following Claim:

Claim 3. *In the decision rule ψ described above, there are $|S^o_\psi| = n$ memory states, and at each memory state, there are exactly two outgoing strong links, and exactly one outgoing weak link.*

Thus, to prove our assertion, we need to analyze, for each state, where those two strong and one weak link go. Consider the state $s = 1$. Suppose that question $i = 1$ is asked in state 1, and there are two links, associated with the answer "$a_1 = 0$": a weak link $(1, s, 0)$ and a strong link $(1, s', 0)$. Thus, there is only one strong link $(1, s'', 1)$, associated with the answer "$a_1 = 1$". For this sketch, consider a situation when s and s' are memory states such that $s, s' \neq 1$. Let v be the weak link, outgoing from the state s'.

Claim 4. *The weak links $(1, s, 0)$ and v described above cannot appear together in the highest-probability path for any alternative $a \in A$.*

Using Lemma 4, we are able to conclude that in this situation, the number of indifference classes should be $m \leq (3/4) \cdot 2^n$, contradicting the richness assumption. Continuing this analysis, we obtain:

Claim 5. *The strong link $(1, s', 0)$ described above is a loop; that is, $s' = 1$.*

For the last part of the analysis, note that either the set of alternatives $a \in A$ such that $a_i = 1$, or the set of those with $a_i = 0$ should contain m' indifference classes that satisfy $(3/4) \cdot 2^{n-1} < m' \leq 2^{n-1}$. Suppose it is the set with $a_i = 1$; denote it by A'. Recall that a strong link $(1, s'', 1)$ goes to some state s''. Consider an auxiliary decision rule ψ' that uses the same state space except of the state $s = 1$; instead, the investigation process according to this rule starts at state s'' and uses the same transitions between states $v, v' \neq 1$ as the rule ψ. Next, we replace the transitions $(v, 1, j)$ of ψ going from states $v \neq 1$ to state 1 by loops (v, v, j) in ψ'. Finally, we delete the three transitions, outgoing from the state 1 in ψ. The next Claim is the key observation in the last part of our analysis.

Claim 6. *An auxiliary decision rule described above has exactly* $3n-3$ *transitions and it solves an auxiliary choice problem* $(Q, \succeq')$*, where* $\succeq'$ *is the restriction of* $\succeq$ *on the subset* $A' \subset A$.

We, therefore, can use an induction in n to argue that $\psi' \in \Psi^+_{n-1}$. This allows us to establish the configuration of links, outgoing from all states except of $s = 1$. Using this finding, Lemma 4, and the richness assumption $(3/4) \cdot 2^n < m \leq 2^n$, we can figure out that the strong link $(1, s'', 1)$ and the weak link $(1, s, 0)$ should go to the same state $s = s''$. It follows that $\psi \in \Psi^+_n$, proving our assertion.

6 Conclusion

In this paper, I introduce a model of decision making that captures a highly inattentive consumer who nonetheless achieves near optimal behavior by virtue of making decisions slowly. This type of behavior is reminiscent of browsing or window shopping.

My model also exhibits a close connection to the Luce model from discrete choice theory (Luce (1959)). As in the discrete choice theory, agents in my model behave stochastically and with some (albeit small) error. The relative likelihood of choosing one alternative over the other is independent of the menu of choices, as required in Luce's choice axiom. In discrete choice theory, the source of randomness is typically interpreted as a utility shock or an error, while in my setting random choice is a tool that allows the agent to achieve near optimal outcomes with limited cognitive resources.

My model assumes that the agent has no control over the search process, and also that when a new alternative arrives, the agent does not retain memory from the past examination of alternatives. Relaxing these two assumptions would require the agent to allocate some cognitive resources to record the results of investigation of other items or to classify the current step of the search process. This introduces a trade-off between preserving information regarding the whole search—for instance, a statistic describing the best investigated alternative at the moment, or a subset of already investigated alternatives—and processing information regarding the currently investigated alternative.

While I do not analyze the above-described trade-off in this paper, I conjecture that if these assumptions were relaxed, my main results concerning the complexity of languages would remain unchanged. For instance, the ability to avoid multiple investigations of the obviously inferior alternatives could make the agent's choice faster, but would not matter for the infinitely-patient agent.[9]

[9] Put it differently, the increased frequency of sampling desirable alternatives from the menu resulting

Whether it is always optimal for the agent to focus entirely on the current alternative versus allocating some memory to record past observations is a more nuanced question, however there is an intuitive argument in favor of focusing on the current alternative. Imagine that the total number of different items is 2^n—for instance, any combination of n attributes is possible. If the agent wants to recall the value of the best encountered alternative, she needs to allocate 2^n automaton's states (with more than 2^n transitions attached) only for that. Instead, a patient agent with an adaptive language can solve the choice problem with only n memory states and $3n$ transitions without recalling anything about the previous alternatives.

Another simple extension of the model is to allow the agent to drop the currently investigated alternative and pick a new one from the menu intentionally[10], without waiting for the external attention shock. The complexity results would not change in this case, because all that matters, according to Lemma 4, is a simple path from the initial to the choosing state of the automaton. Whether the agent cycles between the memory states multiple times or simple drops the investigation does not matter for the limiting analysis.

All of the discussed above extensions matter for the analysis of the resulting random choice away from the limiting case of the infinitely patient agent considered in this paper. This is an intriguing avenue for the future research.

To summarise, this paper shows that a very cognitively limited agent operating in an environment with an uncertain choice set and subjected to exogenous attention shocks is able to solve a complex choice problem nearly optimally. For that, the agent only need ample time and a convenient way—that is, an adaptive language—to think about the alternatives. Moreover, an intuitive procedure that consists of looking at each attribute sequentially and advancing in this investigation with higher probability upon learning good news turns out to be the simplest possible algorithm, and often—the only simplest algorithm. From a broader perspective, by characterizing simple languages as adaptive, this paper provides a foundation for the usage of the separable utility functions in models with patient agents.

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from a more directed search does not matter for this paper's optimality criterion, as could be seen from the fact that none of the statements regarding the languages' complexity depend on the sampling distribution.

[10]Formally, we can introduce one more state of the automaton that represents this decision of the agent, and demand that the agent does not distinguish between state 1 and this new special state. Naturally, we also should count the transitions to this new state in the calculation of the decision rule's complexity.

Appendix

This Appendix has the following structure. In Appendix A, I analyze a more general setup that provides tools for the proofs of lemmas, propositions and theorems provided in Sections 2-4, and also allows to talk about some of the extensions discussed in the Conclusion.

In particular, Appendix A introduces a more general setup and formalizes the dynamics of the random choice and investigation process performed by the agent in this setup. Section A.1 introduces and analyzes a class of Markov chains that facilitates the subsequent analysis, then Section A.2 introduces a more general setup and analyses the "global" dynamics that considers multiple investigation processes, performed by the agent. Next, Section A.3 considers a "local" dynamics that considers a single investigation process. Finally, Section A.4 shows how to transform a problem of solvability of a choice problem via a decision rule into a graph theory problem.

Next, Appendix B returns to the setup, introduced in Section 2 of the paper, but uses tools, developed in Appendix A. Section B.1 considers useful Combinatorics statements that are built on the analysis, performed in Section A.4, including the proof of the lower bound on the language complexity. Section B.2 introduces a universal decision rule that solves the choice problem for any adequate language and analysis its properties. Section B.3 discusses the automata that can be used for the solution of problems with languages that admit separable utility representation. Section B.3 also provides other technical results necessary for the proofs of statements in Section 4 of the paper. Section B.4 provides the proof of Proposition 4 in advance, since it is used in the proofs of other theorems.

Section B.5 uses tools developed in the sections before to prove all statements from Sections 2-4 of the paper. An interested reader may start from Section B.5 to see what lemmas in this Appendix are used in the proofs of a particular theorem or proposition from the main part of the paper. Section B.6 contains references for proofs of the statements given in Section 5 of the paper. Finally, section B.7 formalises the dynamics of the system, although a more general version of this formalism is considered in Section A.2.

A Lemmas for a more general setup

A.1 Preliminaries

I first consider a more abstract setup that covers the baseline model together with the extensions.

Let $\mathbf{Z} = (Z_0, Z_1, \ldots)$ be a stationary Markov chain, where Z_t takes finitely many values $z_t \in \mathcal{Z}$. Let the sample space consists of all sequences of $(z_0, z_1, \ldots) \in \mathcal{Z}^{\mathbb{N}}$. Let Ω_0 be the algebra generated by events $\{Z_t = z_t \ \forall t \in \{0, \ldots, T\}\}$ for $T = 0, 1, 2, \ldots$. Consider probability μ on Ω_0, associated with the stochastic matrix P with elements $P^{z}_{z'} = Pr(Z_{t+1} = z \mid Z_t = z')$ and the distribution of the initial states $P^z_o = Pr(Z_0 = z)$ in a natural way:

$$\mu\Big(Z_t = z_t \ \forall t \in \{0, \ldots, T\}\Big) \ = \ P_o^{z_0} \cdot \prod_{t=1}^{T} P^{z_t}_{z_{t-1}}$$

The formula above defines μ for events $\{Z_t = z_t \ \forall t \in \{0, \ldots, T\}\}$; there is a unique extension of μ by additivity for all events in Ω_0. Let Ω be the smallest sigma-algebra that contains Ω_0. The Kolmogorov Extension Theorem allows for the unique extension of the probability μ to a probability measure on Ω. Similarly, for an event $E = \{Z_T = z_T\} \in \Omega$, I define a conditional probability measure $\mu(\cdot|E)$ as follows:

$$\mu\Big(Z_t = z_t \ \forall t \in \{T+1, \ldots, T'\} \ \Big| \ Z_T = z_T\Big) \ = \ \prod_{t=T+1}^{T'} P^{z_t}_{z_{t-1}}$$

and extend it to other events accordingly.

I analyze a specific class of stationary Markov chains $\mathcal{C}$ with finite state space $\mathcal{Z}$, which I call the set of *absorbing Markov chains with renewal property*, defined as follows. First, $\mathcal{Z}$ contains two non-empty disjoint subsets of states $\overline{\mathcal{Z}}$ and $\mathcal{Z}^o$. Set $\overline{\mathcal{Z}}$ admits a partition $\left(\overline{\mathcal{Z}}_a\right)_{a \in A}$, where A is a finite set. Elements of set $\mathcal{Z}^o$ are indexed by $d \in D$; that is, $\mathcal{Z}^o = \{z^o_d\}_{d \in D}$. I call elements of $\mathcal{Z}^o$ *starting states*. Next, for any $\mathbf{Z} \in \mathcal{C}$, there is a probability distribution f on D and number $\eta > 0$ such that the following conditions hold:

$$\begin{array}{lllcl}
(i) & \forall z \in \overline{\mathcal{Z}}: & \mu\Big(Z_{t+1} = z \mid Z_t = z\Big) & = & 1 \\
(ii) & & \mu\Big(Z_0 \in \mathcal{Z}^o\Big) & = & 1 \\
(iii) & \forall d \in D: & \mu\Big(Z_t = z^o_d \mid Z_t \in \mathcal{Z}^o\Big) & = & f(d) \\
(iv) & \forall z \in \mathcal{Z}: & \mu\Big(Z_{t+1} \in \mathcal{Z}^o \mid Z_t = z\Big) & \geq & \Big(1 - \mu\Big(Z_{t+1} \in \overline{\mathcal{Z}} \mid Z_t = z\Big)\Big) \cdot \eta
\end{array} \tag{8}$$

where I omit figure brackets in the notations of events such as $Z_{t+1} = z$ instead of $\{Z_{t+1} = z\}$, etc. for the ease of notations.

Condition (i) tells that set $\overline{\mathcal{Z}}$ is a set of absorbing states[11], and conditions (ii)-(iv) tell that each period, conditional on not hitting any absorbing state from set $\overline{\mathcal{Z}}$, with probability weakly greater than η, the Markov chain $\mathbf{Z}$ starts its evolution from the beginning, with initial distribution supported at $\mathcal{Z}^o = \{z^o_d\}_{d\in D}$ with likelihoods $f(d)$, $d \in D$. Denote by

$$\begin{aligned} q(d,a) &\equiv \mu\Big(\Big\{\exists r > 0: \; Z_{t+r} \in \overline{\mathcal{Z}}_a \text{ and } Z_{t+r'} \notin \mathcal{Z}^o \text{ for all } 1 \le r' < r \mid Z_t = z^o_d\Big\}\Big) \\ q(d,\varnothing) &\equiv \mu\Big(\Big\{\exists r > 0: \; Z_{t+r} \in \mathcal{Z}^o \mid Z_t = z^o_d\Big\}\Big) \end{aligned} \tag{9}$$

Note that $q(d,b)$ and $q(d,\varnothing)$ do not depend on t, since Markov chain $\mathbf{Z}$ is stationary; hence, $q(d,b)$ and $q(d,\varnothing)$ are well-defined. Thus, $q(d,a)$ is the probability that $\mathbf{Z}$ hits an absorbing state $z \in \overline{\mathcal{Z}}_a$ conditional on starting at state z^o_d without going through starting states $z' \in \mathcal{Z}^o$; similarly, $q(d,\varnothing)$ is the probability that the state returns to some $z \in \mathcal{Z}^o$ without hitting any absorbing state from set $\overline{\mathcal{Z}}$.

Lemma 5. *Let* $\mathbf{Z}$ *be an absorbing Markov chain with renewal property. Then*

$$\sum_{a\in A} q(d,a) \; + \; q(d,\varnothing) \; = \; 1 \quad \forall d \in D$$

Proof. Let $E^t_d \equiv \Big\{Z_t = z^o_d\Big\}$. Consider events $E^t_{d\varnothing} \equiv \Big\{\exists r > 0 : Z_{t+r} \in \mathcal{Z}^o\Big\} \cap E^t_d$, $E^t_{da} \equiv \Big\{\exists r > 0 : Z_{t+r} \in \overline{\mathcal{Z}}_a \text{ and } Z_{t+r'} \notin \mathcal{Z}^o \text{ for all } 1 \le r' < r\Big\} \cap E^t_d$, and $\widetilde{E}^t_d \equiv E^t_d \setminus \Big(\bigcup\limits_{a\in A\cup\{\varnothing\}} E^t_{da}\Big)$. Note that for any $r > 0$, we have

$$\mu\Big(\widetilde{E}^t_d \mid E^t_d\Big) \; \le \; \mu\Big(\Big\{Z_{t+r'} \notin \overline{\mathcal{Z}} \cup \mathcal{Z}^0 \; \forall \; 0 < r' \le r\Big\} \;\Big|\; E^t_d\Big) \; \le \; (1-\eta)^r \; \overset{r\to\infty}{\longrightarrow} \; 0$$

Since $q(d,a) = \mu(E^t_{da}|E^t_d)$ and $q(d,\varnothing) = \mu(E^t_{d\varnothing}|E^t_d)$, the lemma follows. ■

Denote by $H_a \;\equiv\; \Big\{\exists t : Z_t \in \overline{Z}_a\Big\}$ and $H_\varnothing \equiv \Omega \setminus \bigcup\limits_{a\in A} H_a$. Then $\mu(H_a \cap H_b) = 0$ for any $a, b \in A$ such that $a \ne b$, and $H_a \cap H_\varnothing = \varnothing$ for any $a \in A$.

Lemma 6. *Let* $\mathbf{Z}$ *be an absorbing Markov chain with renewal property. If* $\exists d \in D$ *such that* $f(d)\cdot(1-q(d,\varnothing)) > 0$, *then* $\mu(H_\varnothing) = 0$, *and*

$$\mu(H_a) \; = \; \frac{\sum_{d\in D} f(d)\cdot q(d,a)}{\sum_{d\in D}\sum_{b\in A} f(d)\cdot q(d,b)};$$

otherwise, $\mu(H_\varnothing) = 1$.

[11] Our definition allows for the case when the set of starting states $\mathcal{Z}^o = \{z^o\}$ is singleton, and z^o is an absorbing state. Thus we don't say that $\overline{\mathcal{Z}}$ is *the* set of all absorbing states.

Proof. Suppose first $\exists d \in D$ such that $f(d)\cdot(1-q(d,\varnothing))>0$. Let

$$T^k = \Big\{(t_1,\dots,t_k)\in(\{0\}\cup\mathbb{N})^k \;\Big|\; 0=t_1<\dots<t_k\Big\},$$

and for $(t_1,\dots,t_k)\in T^k$, denote by

$$H^k(t_1,\dots,t_k)=\Big\{\forall t: 0\le t\le t_k \;\; Z_t\in\mathcal{Z}^o \iff t\in\{t_1,\dots,t_k\}\Big\}$$

Thus, $H^k(t_1,\dots t_k)$ is the event that the state returns to the set $\mathcal{Z}^o$ at periods $t_1,\dots,t_k$, and does not return to $\mathcal{Z}^o$ at any other period before t_k. Next, for any $a\in A$, denote by

$$H_a^k=\Big\{\exists(t_1,\dots,t_k)\in T^k, t'>t_k \;\Big|\; [Z_t\in\mathcal{Z}^o \iff t\in\{t_1,\dots,t_k\}] \text{ and } Z_{t'}\in\overline{\mathcal{Z}}_a\Big\}$$

Thus, H_a^k is the event that the state has been at the set of starting states $\mathcal{Z}^o$ exactly k times at periods $t_1,\dots,t_k$, and then went to the absorbing state $z\in\overline{\mathcal{Z}}_a$. By condition (iii) from eq. (8) for $\mathbf{Z}\in\mathcal{C}$,

$$\mu\Big(Z_{t_{k-1}}=z_d^0 \;\Big|\; H^{k-1}(t_1,\dots,t_{k-1})\Big) \;=\; f(d)$$

Then, using the definition of $q(d,\varnothing)$, we get

$$\sum_{t_k>t_{k-1}}\mu\Big(H^k(t_1,\dots,t_k)\;\Big|\;H^{k-1}(t_1,\dots,t_{k-1})\Big) \;=\; \sum_{d\in D}f(d)\cdot q(d,\varnothing)$$

Recursively,

$$\begin{aligned}
&\sum_{(t_1,\dots,t_k)\in T^k}\mu\Big(H^k(t_1,\dots,t_k)\Big) \;=\\
&=\sum_{t_k>t_{k-1}}\;\sum_{(t_1,\dots,t_{k-1})\in T^{k-1}}\mu\Big(H^k(t_1,\dots,t_k)\;\Big|\;H^{k-1}(t_1,\dots,t_{k-1})\Big)\cdot\mu\Big(H^{k-1}(t_1,\dots,t_{k-1})\Big) \;=\\
&=\Big(\sum_{d\in D}f(d)\cdot q(d,\varnothing)\Big)\cdot\sum_{(t_1,\dots,t_{k-1})\in T^{k-1}}\mu\Big(H^{k-1}(t_1,\dots,t_{k-1})\Big) \;=\\
&=\;\dots\;=\\
&=\Big(\sum_{d\in D}f(d)\cdot q(d,\varnothing)\Big)^{k-1}\cdot\sum_{t_1\in T^1}\mu\Big(H^1(t_1)\Big) \;=\; \Big(\sum_{d\in D}f(d)\cdot q(d,\varnothing)\Big)^{k-1}
\end{aligned}$$

where we used $T^1=\{0\}$ by the definition of T^k and $\mu(H^1(0))=1$ by condition (ii) from eq. (8). Note that by our assumption,

$$\sum_{d\in D}f(d)\cdot q(d,\varnothing) \;=\; 1-\sum_{d\in D}f(d)(1-q(d,\varnothing)) \;<\; 1$$

Next, by the definition of $q(d,a)$ and condition (iii) from eq. (8),

$$\mu\Big(H_a^k \,\Big|\, H^k(t_1,\dots,t_k)\Big) \;=\; \sum_{d\in D} f(d)q(d,a)$$

Then

$$\begin{aligned}\mu\Big(H_a^k\Big) \quad &= \quad \sum_{(t_1,\dots,t_k)\in T^k} \mu\Big(H_a^k \,\Big|\, H^k(t_1,\dots,t_k)\Big)\cdot\mu\Big(H^k(t_1,\dots,t_k)\Big) \;= \\ &= \quad \Big(\sum_{d\in D} f(d)q(d,a)\Big)\cdot\Big(\sum_{d\in D} f(d)\cdot q(d,\varnothing)\Big)^{k-1}\end{aligned}$$

Finally,

$$\begin{aligned}\mu\Big(H_a\Big) \quad &= \quad \sum_{k\in\mathbb{N}}\mu\Big(H_a^k\Big) \;=\; \sum_{k\in\mathbb{N}}\Big(\sum_{d\in D} f(d)q(d,a)\Big)\cdot\Big(\sum_{d\in D} f(d)\cdot q(d,\varnothing)\Big)^{k-1} \;= \\ &= \quad \frac{\sum_{d\in D} f(d)q(d,a)}{1-\sum_{d\in D} f(d)q(d,\varnothing)} \;=\; \frac{\sum_{d\in D} f(d)q(d,a)}{\sum_{d\in D}\sum_{b\in A} f(d)q(d,b)}\end{aligned}$$

where I used Lemma 5 and the fact that $\sum_{d\in D} f(d) = 1$ in the last equality. It follows that

$$\mu\Big(H_\varnothing\Big) \;=\; 1-\sum_{a\in A}\mu\Big(H_a\Big) \;=\; 0$$

Suppose now that $f(d) > 0$ implies $q(d,\varnothing) = 1$; then by Lemma 5, $\sum_{d\in D} f(d)q(d,a) = 0$ for any $a \in A$. Using the same analysis, we get

$$\mu\Big(H_a^k\Big) \;=\; \sum_{d\in D} f(d)q(d,a) \;=\; 0$$

It follows that $\mu(H_a) = \sum_{k\in\mathbb{N}}\mu(H_a^k) = 0$, hence $\mu\Big(H_\varnothing\Big) = 1$. ■

A.2 Global dynamics

I call "local dynamics" the behavior of state variables in a set of subsequent periods such that the decision maker investigates and, maybe, choose some alternative without drawing a new one. I call global dynamics the behavior of state variables in all periods. In this section, I apply Lemmas 5 and 6 to analyze the global dynamics.

To accommodate the extensions of the baseline model, I consider several generalizations. First, I allow no questions being asked at a state. Thus, the interrogation function becomes $\iota:\; S^o \to N\cup\{0\}$, where $\iota(s) = \varnothing$ represents no questions asked. Define a signal space $\mathcal{I}$ with generic element $\theta\in\mathcal{I}$ by $\mathcal{I} = \Big(N\times\{0,1\}\Big)\cup\{\varnothing\}$, where $\theta = (i,j)$ represents question $i \in N$ asked and answer $j \in \{0,1\}$ received, and $\theta = \varnothing$ represents no questions asked.

Second, I consider a stochastic interrogation rule given by $\iota : S^o \to \triangle(N \cup \{\varnothing\})$, where $\iota_s(i)$ is the probability to ask question $i \in N$ or do not ask any question ($i = \varnothing$) in state $s \in S$. A deterministic interrogation rule is a special case of a stochastic interrogation rule with ι being a delta-function: $\iota_s(i) = \delta^i_{\iota(s)}$ for $i \in N \cup \{\varnothing\}$, where $\iota(s)$ is the question asked in state s in the baseline model with the deterministic interrogation rule.

Third, I allow the agent to draw a new alternative intentionally. Thus, I consider an automaton with the state space $S = S^o \cup \{\diamond\} \cup \{new\}$, where state $s = new$ stands for agent's decision to draw a new alternative from the menu. The stochastic transition rule becomes $\tau : S^o \times \mathcal{I} \to S \cup \{\diamond\} \cup \{new\}$ in this case; $\tau(s', s, \theta)$ is the probability to go to state $s \in S^o \cup \{\diamond\} \cup \{new\}$ from state $s' \in S$ after signal $\theta \in \mathcal{I}$. The case when the agent cannot draw a new alternative intentionally is a special case when $\tau(s', new, \theta) = 0$ for all $s' \in S^o$, $\theta \in \mathcal{I}$. Recall that in the baseline model, the transitional probability is given by function $\tau : S^o \times \{0,1\} \to S$. To connect the two versions, define $\tau(s', s, j) = \tau(s', s, \iota(s'), j)$, where the left hand side of the equation denotes the transitional probability in the baseline model, and the right hand side denotes the transitional probability in the general model with the restriction that the signal is $\theta = (\iota(s'), j)$, where $\iota(s')$ is the question, asked in state s'.

In the general model, the automaton strategy (or simply strategy) $\sigma = (S, \iota, \tau)$ consists of the state space S, the stochastic interrogation rule ι and the stochastic transition rule τ described above. As in the baseline model, I denote by Σ the set of all such strategies with finite state space.

The set of the positive-probability transitions in the general case becomes

$$\mathcal{T}_\sigma := \Big\{(s', s, \iota) \;\Big|\; \iota_s(\iota) \cdot \tau(s', s, \iota) > 0\Big\} \tag{10}$$

The set of transitions defined for the baseline model by eq. (2) is a special case of eq. (10), which one can see from

$$\mathcal{T}_\sigma = \Big\{(s', s, i, j) \;\Big|\; \iota_s(i) \cdot \tau(s', s, i, j) > 0\Big\} \cup \Big\{(s', s, \varnothing) \;\Big|\; \iota_s(\varnothing) \cdot \tau(s', s, \varnothing) > 0\Big\}$$

Define the global state space $\mathcal{Z} = (\mathcal{W}, \mathcal{Y}, \mathcal{I})$, where $w \in \mathcal{W} \equiv \{new, old, \diamond\}$ encodes whether some item has already been chosen ($w = \diamond$), or a new item is drawn ($w = new$), or the investigated item remains the same as in the previous period ($w = old$). Next, $y = (y, b) \in \mathcal{Y} \equiv S^o \times A$ describes the flexible state of the automaton and the investigated item (provided that the item has not been chosen previously—otherwise, it describes the last flexible state of the automaton from where it transitioned to $s = \diamond$ and the item that has been chosen). Finally, $\mathcal{I}$ is a signal space, defined above. The dynamics of the model is then described by a Markov chain $Z_0, Z_1, Z_2, \ldots$. I denote by $Z_t = (W_t, Y_t, \Theta_t)$ the corresponding t-th period random state of the chain, and by $z_t = (w_t, y_t, \theta_t)$ its realization.

The strategy $\sigma = (S, \iota, \tau)$ and menu B induce the stochastic $|\mathcal{Z}|\times|\mathcal{Z}|$ matrix $P^z_{z'} \equiv Pr(Z_{t+1} = z | Z_t = z')$ as follows:

$$P^z_{z'} \;=\; Pr\Big(Z_{t+1} = (w, s, a, \theta) \;\Big|\; Z_t = (w', s', a', \theta')\Big) \;=$$

$$= \begin{cases} \delta^a_{a'} \cdot \delta^s_{s'} \cdot \Big(\delta^{\diamond}_{w'} + (1-\delta^{\diamond}_{w'}) \cdot \tau(s', \diamond, \iota')\Big) & \text{if} \quad w = \diamond \\ (1-\delta^{\diamond}_{w'}) \cdot \Big[(1-\tau(s', \diamond, \theta')) \cdot \eta + \tau(s', new, \theta') \cdot (1-\eta)\Big] \cdot \delta^1_s \cdot \rho^B(a) \cdot f^a_1(\theta) & \text{if} \quad w = new \\ (1-\delta^{\diamond}_{w'}) \cdot \Big[1 - \tau(s', \diamond, \theta') - \tau(s', new, \theta')\Big] \cdot (1-\eta) \cdot \delta^{a'}_a \cdot \tau(s', s, \theta') \cdot f^a_s(\theta) & \text{if} \quad w = old \end{cases}$$

where

$$f^a_s(\theta) \;\equiv\; \begin{cases} \iota_s(i) \cdot \delta^{a_i}_j & \text{if} \quad i \in N \\ \iota_s(\varnothing) & \text{if} \quad i = \varnothing \end{cases}$$

The distribution of the initial state is given by a $|\mathcal{Z}|$-dimensional row vector P_o with components

$$P^z_o \;\equiv\; Pr\Big(Z_0 = (w, s, a, \varnothing)\Big) \;=\; \delta^{new}_w \cdot \delta^1_s \cdot \rho^B(a) \cdot f^a_1(\theta)$$

Note that any result, proven for an arbitrary Markov chain induced by menu B and strategy $\sigma = (S, \iota, \tau)$ with generic functions $\iota : S \to \triangle(N \cup \{\varnothing\})$ and $\tau : S \times \mathcal{I} \to S \cup \{\diamond\} \cup \{new\}$ remains valid if we put restrictions on ι (such as ι being a delta-function, and $\iota_s(\varnothing) = 0$), or τ (such as $\tau(s', new, \iota) = 0$ for all $s' \in S$, $\theta \in \mathcal{I}$), or on both these functions. In particular, the results are applicable to the baseline model.

To apply Lemmas 1-2 to the Markov chain $\mathbf{Z}$ induced by menu B and the decision rule σ, I identify set A with the universal set of items A, and consider $D = A \times \mathcal{I}$. Thus, $\overline{\mathcal{Z}}_a = \Big\{(\diamond, s, a, \varnothing)\Big\}_{s \in S^o}$ and $z^o_{a,\theta} = (new, 1, a, \theta)$. Next, for a fixed menu B and strategy σ, consider $f(a, \varnothing) = \rho^B(a) \cdot \iota_1(\varnothing)$, $f(a, i, j) = \rho^B(a) \cdot \iota_1(i) \cdot \delta^{a_i}_j$ for $i \in N$, $j \in \{0, 1\}$.

Lemma 7. *The Markov chain* $\mathbf{Z}$ *induced by menu* B *and strategy* $\sigma \in \Sigma$ *is an absorbing Markov chain with renewal property such that the set* $\overline{\mathcal{Z}} = \Big\{\overline{\mathcal{Z}}_a\Big\}_{a=A}$*, where* $\overline{\mathcal{Z}}_a = \Big\{(\diamond, s, a, \varnothing)\Big\}_{s \in S^o}$*, is the set of absorbing states,* $D = A \times \mathcal{I}$*, the set* $\mathcal{Z}^o = \{z^o_d\}_{d \in D} = \Big\{(new, 1,\ a, \iota)\Big\}_{(a,\iota) \in A \times \mathcal{I}}$ *is the set of starting states, and* $f(a, \theta) = \rho^B(a) \cdot f^a_1(\theta)$.

Proof. First, note that sets $\overline{\mathcal{Z}}$ and $\mathcal{Z}^o$ are non-empty and disjoint. Second, let us verify conditions (i)-(iv) from eq. (8) for Markov chain $\mathbf{Z}$. Property (i) follows from $P^{\diamond,s,a}_{\diamond,s',a',\emptyset} = \delta^a_{a'} \cdot \delta^s_{s'}$. Property (ii) follows from the formula for P_o. For property (iii), recall that $z^o_{a,\theta} = (new, 1, a, \theta)$ and consider separately period $t = 0$ and periods $t > 0$. For $t = 0$, we have:

$$\mu\Big(Z_0 = z^o_d \Bigm| Z_0 \in \mathcal{Z}^o\Big) \;=\; \mu\Big(Z_0 = z^o_{a,\theta}\Big) \;=\; \rho^B(a) \cdot f^a_1(\theta) \;=\; f(a,\theta) \;=\; f(d)$$

For $t > 0$, we have:

$$\mu\Big(Z_t = z^o_d \Bigm| Z_{t-1} = z'\Big) \;=\; (1-\delta^\diamond_{w'}) \cdot \Big[\tau(s', new, \theta') \cdot (1-\eta) + (1-\tau(s', \diamond, \theta')) \cdot \eta\Big] \cdot f(d)$$

$$\mu\Big(Z_t \in \mathcal{Z}^o \Bigm| Z_{t-1} = z'\Big) \;=\; \sum_{e \in D} \mu\Big(Z_t = z^o_e \Bigm| Z_{t-1} = z'\Big) \;=\; \sum_{e \in D} x(z') \cdot f(e) = x(z')$$

where I denote by

$$x(z') \;\equiv\; (1-\delta^\diamond_{w'}) \cdot \Big[\tau(s', new, \theta') \cdot (1-\eta) + (1-\tau(s', \diamond, \theta')) \cdot \eta\Big]$$

Then

$$\mu\Big(Z_t \in \mathcal{Z}^o\Big) \;=\; \sum_{z' \in \mathcal{Z}} \mu\Big(Z_t \in \mathcal{Z}^o \Bigm| Z_{t-1} = z'\Big) \cdot \mu\Big(Z_{t-1} = z'\Big) \;=\; \sum_{z' \in \mathcal{Z}} x(z')$$

$$\mu\Big(Z_t = z^o_d\Big) \;=\; \sum_{z' \in \mathcal{Z}} \mu\Big(Z_t = z^o_d \Bigm| Z_{t-1} = z'\Big) \cdot \mu\Big(Z_{t-1} = z'\Big) \;=\; \sum_{z' \in \mathcal{Z}} x(z') \cdot f(d)$$

It follows

$$\mu\Big(Z_t = z^o_d \Bigm| Z_t \in \mathcal{Z}^o\Big) \;=\; \frac{\mu\Big(Z_t = z^o_d, Z_t \in \mathcal{Z}^o\Big)}{\mu\Big(Z_t \in \mathcal{Z}^o\Big)} \;=\; \frac{\mu\Big(Z_t = z^o_d\Big)}{\mu\Big(Z_t \in \mathcal{Z}^o\Big)} \;=\; \frac{\sum_{z' \in \mathcal{Z}} x(z') \cdot f(d)}{\sum_{z' \in \mathcal{Z}} x(z')} \;=\; f(d)$$

To prove property (iv), we use the previous calculations:

$$\mu\Big(Z_t \in \mathcal{Z}^o \Bigm| Z_{t-1} = z'\Big) \;=\; x(z') \geq (1-\delta^\diamond_{w'}) \cdot \Big[(1-\tau(s', \diamond, \theta')) \cdot \eta\Big] = \Big(1 - \mu\Big(Z_t \in \overline{\mathcal{Z}} \Bigm| Z_{t-1} = z'\Big)\Big) \cdot \eta$$

■

Lemma 8. *Let* $\mathbf{Z}$ *be the Markov chain induced by menu* B *and strategy* σ*. Then* $q((b,\theta), a) = 0$ *for all* $a, b \in A$ *such that* $a \neq b$*.*

Proof. The lemma follows from

$$\mu\Big(\exists r > 0 : a_{t+r} \neq b, w_{t+r'} \neq new \;\forall 1 \leq r' \leq r \Bigm| w_t \neq \diamond,\; s_t = 1,\; a_t = b\Big) \;=\; 0$$

since $a' \neq a$ implies $Pr(Z_{t+1} = (old, s, a, \theta) | Z_t = (w', s', a', \theta')) = 0$ and $Pr(Z_{t+1} = (\diamond, s, a, \theta) | Z_t = (w', s', a', \theta')) = 0$. ■

Lemma 9. *Let* $\mathbf{Z}$ *be the Markov chain induced by menu* B *and strategy* σ. *For* $a \in A$, *denote by*

$$q_\sigma(a) := \mu\Big(\exists r > 0 : w_{t+r} = \diamond, a_{t+r} = a, w_{t+r'} \neq new \ \forall 1 \leq r' \leq r \ \Big|\ w_t = new, s_t = 1, a_t = a\Big) \quad (11)$$

then: (i) $q_\sigma(\cdot)$ *does not depend on menu* B, *and (ii) the following holds:*

$$\sum_{d \in D} f(d) \cdot q(d,a) \ = \ \rho^B(a) \cdot q_\sigma(a) \qquad \forall a \in A$$

Proof. By Lemma 8, we get

$$\begin{aligned}
\sum_{d\in D} f(d)\cdot q(d,a) \ &= \ \sum_{b\in A}\sum_{\theta\in I} \rho^B(b)\cdot q((b,\theta),a) \ = \ \sum_{\theta\in I}\rho^B(a)\cdot q((a,\theta),a) \ = \\
&= \ \sum_{\theta\in I}\rho^B(a)\cdot \mu\Big(\exists r>0:\ w_{t+r}=\diamond, a_{t+r}=a, w_{t+r'}\neq new\ \forall 1\leq r'<r\ \Big|\ Z_t=(new,1,a,\theta)\Big) \ = \\
&= \ \rho^B(a)\cdot\mu\Big(\exists r>0:\ w_{t+r}=\diamond, a_{t+r}=a, w_{t+r'}\neq new\ \forall 1\leq r'\leq r\ \Big|\ w_t\neq\diamond,\ s_t=1,\ a_t=a\Big) \ = \\
&= \ \rho^B(a)\cdot q_\sigma(a)
\end{aligned}$$

proving statement (ii). Statement (i) follows from the fact that $Pr(Z_{t+1} = (old, s, a, \theta)|Z_t = z')$ and $Pr(Z_{t+1} = (\diamond, s, a, \varnothing)|Z_t = z')$ do not depend on B for any $z' \in \mathcal{Z}$. ■

Lemma 10. *Let* $\mathbf{Z}$ *be the Markov chain induced by menu* B *and strategy* σ. *If* $\exists a \in B$ *such that* $\rho^B(a) \cdot q_\sigma(a) > 0$, *then* $\mu(H_\varnothing) = 0$, *and*

$$\mu(H_a) \ = \ \frac{\rho^B(a)\cdot q_\sigma(a)}{\sum_{b\in B}\rho^B(b)\cdot q_\sigma(b)} \quad (12)$$

for any $a \in B$; *otherwise,* $\mu(H_\varnothing) = 1$.

Proof. The Lemma follows straightforwardly from Lemma 6, Lemma 7, Lemma 9 and $\text{supp}(\rho^B) = B$. ■

Lemma 11. *A decision rule* $\{\sigma_r\}_{r=1,2,...} \in \Psi$ *solves choice problem* $(Q, \succeq)$ *if and only if the following two conditions hold: (i)* $a \succ b$ *implies* $q_{\sigma_r}(b)/q_{\sigma_r}(a) \longrightarrow 0$ *for all* $a, b \in A$, *and (ii)* $\exists \bar{r}$: $q_{\sigma_r}(a) > 0$ *for all* $r > \bar{r}$ *for all* $a \in A$.

Proof. Suppose first that σ_r solves $(Q, \succeq)$. Consider arbitrary $a \in A$. Towards a contradiction, assume there is a subsequence σ_{r^l} such that $q_{\sigma_{r^l}}(a) = 0$ and consider menu $B = \{a\}$. Then by Lemma 7 and Lemma 5, $q((a,\theta),\varnothing) = 0$ for all $\theta \in \mathcal{I}$. Since $B = \{a\}$, then $f((b,\theta)) = 0$ for $b \neq a$. Thus by Lemma 10, $\mu(H_\varnothing) = 1$, contradicting $\mu(H_a) \xrightarrow{l\to\infty} 1$. Thus, property (ii) holds; moreover, eq. (12) is well-defined.

Suppose $a \succ b$ and consider menu $B = \{a, b\}$. Since σ_r solves $(Q, \succeq)$, then

$$1 \;=\; \lim_{r\to\infty} \mu(H_a) \;=\; 1 - \frac{\rho^B(a)}{\rho^B(b)} \cdot \lim_{r\to\infty} \frac{q_\sigma(b)}{q_\sigma(a)}$$

and statement (ii) follows, since $\rho^B(a), \rho^B(b) > 0$.

Suppose now that conditions (i) and (ii) hold. For an arbitrary menu B, let $\overline{B} = \{a \in B | a \succeq b \;\forall b \in B\}$ be the set of maximizers of $\succeq$ over B, and $B_- = B \backslash \overline{B}$. Condition (ii) guarantees that $\rho^B(a) \cdot q_{\sigma_r}(a) > 0$ for any $a \in B$ for large enough r. Then by Lemma 10,

$$\lim_{r\to\infty} \mu(\overline{B}) \;=\; 1 - \lim_{r\to\infty} \frac{\sum_{b\in B_-} \rho^B(b) \cdot q_{\sigma_r}(b)}{\sum_{b\in B_-} \rho^B(b) \cdot q_{\sigma_r}(b) + \sum_{a\in \overline{B}} \rho^B(a) \cdot q_{\sigma_r}(a)} \;=\; 1$$

where we used the fact that $\rho^B(a) > 0$ for all $a \in B$ and $q_{\sigma_r}(b)/q_{\sigma_r}(a) \longrightarrow 0$ for any $b \in B_-$, $a \in \overline{B}$ by condition (i). Thus σ_r solves $(Q, \succeq)$. ∎

A.3 Local dynamics

Lemma 11 shows that for our analysis, it suffices to learn the limiting properties of function $q_{\sigma_r}(a)$ given by eq. (11). In this section, I assume that the current alternative $a \in A$ that the agent investigates fixed, and omit it if it does not cause confusion. By stationarity of the Markov chain $\mathbf{Z}$,

$$q_\sigma(a) \;=\; \mu\Big(\exists t > 0 : w_t = \diamond, a_t = a, w_{t'} \neq new \;\forall 1 \leq t' \leq t \;\Big|\; w_0 = new, s_0 = 1, a_0 = a\Big)$$

It is useful to note that, when analysing $q_\sigma(a)$, we are interested only in the probability of realizations of Markov chain $\mathbf{Z}$ up to the point where either the current alternative $a \in A$ is chosen, or a new alternative is drawn. Thus, we may omit w_t and a_t in the description of the state $z \in \mathcal{Z}$, and instead focus on s_t and ι_t. To account for the cases when $w_r = \diamond$ or $w_r = new$—these cases correspond to the last period of our analysis relevant for the calculation of $q(a)$—I consider a local state space $\mathcal{X} = \left(S^o \times \mathcal{I}\right) \cup \{\diamond\} \cup new$.

To analyse $q_\sigma(a)$, I, therefore, consider Markov chain $\mathbf{X} = (X_0, X_1, \ldots)$ with realizations $(x_0, x_1, \ldots) \in \mathcal{X}^{\mathbb{N}}$ induced by strategy σ and alternative $a \in A$ via a mapping $\chi : \mathcal{Z}^{\mathbb{N}} \to \mathcal{X}^{\mathbb{N}}$ from the set of realizations of the Markov chain $\mathbf{Z}$ induced by menu $B = \{a\}$ to the set of realizations of the Markov chain $\mathbf{X}$ given by the following formula.

$$\chi((z_0, z_1, \ldots)) = (x_0, x_1, \ldots) : \begin{cases} x_t = (s_t, \theta_t) & \text{if} \quad w_{t'} = old \;\forall 0 < t' \leq t \\ x_t = \diamond & \text{if} \quad \exists t'' \leq t : \; w_{t'} = old \;\forall 0 < t' < t'', \; w_{t''} = \diamond \\ x_t = new & \text{if} \quad \exists t'' \leq t : \; w_{t'} = old \;\forall 0 < t' < t'', \; w_{t''} = new \end{cases}$$

where $z_t = (w_t, s_t, a_t, \theta_t)$. Abusing notations denote by P be the $\mathcal{X} \times \mathcal{X}$ stochastic matrix, associated with the Markov chain $\mathbf{X}$, then:

$$P_{x'}^{x} \ = \ Pr\Big(X_{t+1} = (x, \theta) \ \Big| \ X_t = (x', \theta')\Big) \ =$$

$$= \begin{cases} \delta_{x'}^{\diamond} + (1 - \delta_{x'}^{\diamond} - \delta_{x'}^{new}) \cdot \tau(s', \diamond, \theta') & \text{if } x = \diamond \\ \\ \delta_{x'}^{new} + (1 - \delta_{x'}^{\diamond} - \delta_{x'}^{new}) \cdot [\eta + (1-\eta) \cdot \tau(s', new, \theta')] & \text{if } x = new \\ \\ (1 - \delta_{x'}^{\diamond} - \delta_{x'}^{new}) \cdot (1-\eta) \cdot \tau(s', s, \theta') \cdot f_s(\theta) & \text{otherwise} \end{cases}$$

where

$$f_s(\theta) \ \equiv \ \begin{cases} \iota_s(i) \cdot \delta_j^{a_i} & \text{if} \quad \theta = (i, j) \\ \iota_s(\varnothing) & \text{if} \quad \theta = \varnothing \end{cases}$$

The distribution of the initial state X_0 is given by

$$P_o^{s,\theta} \ \equiv \ Pr\Big(X_0 = (s, \theta)\Big) \ = \ \delta_s^1 \cdot f_s(\theta)$$

The stochastic matrix P and initial distribution P_o induce a probability distribution μ on the sigma-algebra, generated by events $\{X_t = x_t \ \forall \{0, ..., T\}\}$ according to the formula below and its extension by countable additivity on other events:

$$\mu\Big(X_t = x_t \ \forall \{0, ..., T\}\Big) \ = \ P_o^{x_0} \cdot \prod_{t=1}^{T} P_{x_{t-1}}^{x_t} \tag{13}$$

This probability distribution is a pushforward measure given by $\mu(\chi(C)) = \mu(C)$ for event C of the original sigma-algebra of events Ω, associated with the global dynamics. In the new terms, quantity $q_\sigma(a)$ looks as

$$q_\sigma(a) \ = \ \tilde{\mu}\Big(\exists t > 0 : x_t = \diamond\Big)$$

Lemma 12. *The probability that the state $X_t \in \mathcal{X}$ is not absorbing vanishes with each period at rate at least $(1-\eta)$; that is,*

$$\tilde{\mu}(X_t \neq \diamond, new) \ \leq \ (1-\eta)^t$$

Proof. The lemma is straightforward. ■

A.4 A Graph Theory Approach

In this section, I transform an analysis of the solution of the choice problem via a decision rule into the analysis of properties of specific directed graphs that represent transitions between the automaton's states, and in particular, into the analysis of simple paths going from the starting state $s = 1$ to the decision state $s = \diamond$.

Given a strategy σ, define a mapping $\omega : \mathcal{T} \to (0,1]$ by

$$\omega(s', s, \theta) \;=\; \begin{cases} \iota_{s'}(i) \cdot \tau(s', s, i, j) & \text{if} \quad \theta = (i, j) \in N \times \{0, 1\} \\ \iota_{s'}(\varnothing) \cdot \tau(s', s, \varnothing) & \text{if} \quad \theta = \varnothing \end{cases} \tag{14}$$

I call $\omega(s', s, \iota)$ the *weight* of the link (s', s, ι). When needed, I use a subscript σ to show the dependence of ω on the strategy σ.

For an arbitrary alternative $a \in A$ and strategy $\sigma \in \Sigma$, define the set of links

$$\mathcal{T}^a_\sigma \;=\; \left\{(s', s, \theta) \in \mathcal{T}_\sigma \;\middle|\; f^a_{s'}(\iota) \cdot \tau(s', s, \theta) > 0\right\}$$

where

$$f^a_{s'}(\theta) \;=\; \begin{cases} \iota_{s'}(i) \cdot \delta^{a_i}_j & \text{if} \quad i \in N \\ \iota_{s'}(\varnothing) & \text{if} \quad i = \varnothing \end{cases}$$

Thus, $\mathcal{T}^a_\sigma$ is the set of links that can be potentially used for transitions while investigating alternative a using strategy σ. Note that

$$\mathcal{T}^a_\sigma \;=\; \left\{(s', s, i, j) \in \mathcal{T}_\sigma \;\middle|\; \iota_s(i) \cdot \delta^{a_i}_j \cdot \tau(s', s, i, j) > 0\right\} \cup \left\{(s', s, \varnothing) \in \mathcal{T}_\sigma \;\middle|\; \iota_s(\varnothing) \cdot \tau(s', s, \varnothing)\right\}$$

Consider a directed weighted graph $\mathcal{G} = \mathcal{G}(a, \sigma) = (S_\sigma, \mathcal{T}^a_\sigma, \omega_\sigma)$, defined as follows. Set up $S = S \cup \{\diamond\}$ to be the set of vertexes, set up $\mathcal{T}^a = \mathcal{T}^a_\sigma$ to be the set of transitions, given by eq. (A.4), where we interpret $l = (s', s, \theta) \in \mathcal{T}^a_\sigma$ as a directed link from state $s' \in S^o$ to state $s \in S^o \cup \{\diamond\}$, labeled by a signal $\theta \in \mathcal{I}$. Finally, $\omega(s', s, \theta)$ is the weight of the link $(s', s, \theta) \in \mathcal{T}^a_\sigma$. Note that

$$(s', s, \theta) \in \mathcal{T}^a_\sigma \implies \omega(s', s, \theta) \;=\; \begin{cases} \iota_{s'}(i) \cdot \delta^{a_i}_j \cdot \tau(s', s, i, j) & \text{if} \quad \iota = (i, j) \\ \iota_{s'}(\varnothing) \cdot \tau(s', s, \varnothing) & \text{if} \quad \iota = \varnothing \end{cases} \;=\; f^a_s(\iota) \cdot \tau(s', s, \theta)$$

Define a valid (directed) path l on the graph $\mathcal{G}$ to be a finite ordered set of labeled directed links $(l_1, ..., l_T) \in \mathcal{T}^T$ such that the following properties hold:

(i) The first link, l_1, starts at state $s = 1$;

(ii) If link $l_t = (s'_t, s_t, \iota_t)$, $t < T$, ends at state s_t, then link $l_{t+1} = (s'_{t+1}, s_{t+1}, \iota_{t+1})$ starts at state $s'_{t+1} = s_t$;

(iii) The last link, l_T, ends at $s = \diamond$;

Denote by $\mathcal{L}$ the set of all valid paths.

For an arbitrary valid path $l = (l_1, ..., l_T) \in \mathcal{L}$, define its weight as a product of the weights of its links times $(1-\eta)^{T-1}$; that is, $\omega : \mathcal{L} \to \mathbb{R}$ given by

$$\omega(l) \ = \ \omega^a_\sigma(l) = (1-\eta)^{T-1} \cdot \prod_{t=1}^{T} \omega(l_t)$$

Lemma 13. *Let $\mathcal{G} = (S, \mathcal{T}, \omega)$ be a directed graph, $\mathcal{L}$ be the set of valid paths, and ω be the weighting function defined above, then*

$$q_\sigma(a) \ = \ \sum_{l \in \mathcal{L}} \omega(l)$$

Proof. Consider event

$$(x_0, ..., x_T) \ = \ \{X_t = x_t \ \forall 0 \le t \le T\}$$

such that $x_T = \diamond$, and $x_t \neq \diamond, new$ for $t < T$. Let $\mathcal{D}$ be the set of such events, that is,

$$\mathcal{D}_T \ = \ \left\{(x_0, ..., x_T) \,\middle|\, x_t \in \mathcal{X} \backslash \Big(\{\diamond\} \cup \{new\}\Big) \ \forall 0 \le t < T, \ x_T = \diamond \right\}, \qquad \mathcal{D} \ = \ \bigcup_{t=0}^{\infty} \mathcal{D}_T$$

Then $\mathcal{D}$ is a partition of the event $\{\exists t > 0 : x_t = \diamond\}$; thus,

$$q_\sigma(a) \ = \ \sum_{D \in \mathcal{D}} \mu(D)$$

Let

$$\overline{\mathcal{D}} = \{D \in \mathcal{D} \mid \mu(D) > 0\}$$

Since the set $\mathcal{D}$ is countable, it follows that

$$q_\sigma(a) \ = \ \sum_{D \in \overline{\mathcal{D}}} \mu(D)$$

Define a mapping $\zeta : \overline{\mathcal{D}} \to \mathcal{L}$ as follows. For $D = \Big((s_0, \theta_0), (s_1, \theta_1), ..., (s_{T-1}, \theta_{T-1}), \diamond\Big) \in \overline{\mathcal{D}}$, $\zeta(D) = (l_1, ..., l_T) \in \mathcal{L}$ such that $l_t = (s_{t-1}, s_t, \theta_{t-1})$ for $t = 1, ..., T$. The mapping ζ is well-defined, since for $(x_0,\ldots,x_T)\in\overline{\mathcal{D}}$ we have $P^{x_t}_{x_{t-1}}>0$, and hence, $(s_{t-1},s_t,\theta_{t-1})\in\mathcal{T}$ for all $t=1,\ldots,T$; moreover, $\mu(D)>0$ implies $\left(P_o\right)_{s_0,\theta_0}>0$, hence $s_0=1$. Next,

$$\mu(D) \;=\; \left(P_o\right)_{s_0,\theta_0}\cdot P^{s_1,\theta_1}_{s_0,\theta_0}\cdot\ldots\cdot P^{s_{T-1},\theta_{T-1}}_{s_{T-2},\theta_{T-2}}\cdot P^{\diamond}_{s_{T-1},\theta_{T-1}} \;=$$

$$f_s(\theta_0)\cdot(1-\eta)\cdot\tau(s_0,s_1,\theta_0)\cdot f_{s_1}(\theta_1)\cdot\ldots\cdot(1-\eta)\cdot\tau(s_{T-2},s_{T-1},\theta_{T-2})\cdot f_{s_{T-1}}(\theta_{T-1})\cdot\tau(s_{T-1},s_T,\theta_{T-1})=$$

$$(1-\eta)^{T-1}\cdot\prod_{t=1}^{T} f_{s_{t-1}}(\theta_{t-1})\cdot\tau(s_{t-1},s_t,\theta_{t-1}) \;=\; (1-\eta)^{T-1}\cdot\prod_{t=1}^{T}\omega(\zeta(D)_t) \;=\; \omega(\zeta(D))$$

The lemma follows. ■

I say that a valid path is simple if $t\neq r$ implies $s'_t\neq s'_r$, where $l_t=(s'_t,s_t,\theta_t)$, $l_r=(s'_r,s_r,\theta_r)$. Thus, a simple valid directed path goes via a particular state s at most one time. I denote by $\mathcal{L}_0\subset\mathcal{L}$ the set of all simple valid directed paths.

For an arbitrary valid path $l\in\mathcal{L}$, denote by $\pi(l)=\left(\pi(l)_1,\ldots,\pi(l)_{\overline{r}}\right)$ the result of the application of the following recursive procedure π with steps $r=1,2,\ldots,\overline{r}$:

1. At the first step, $r=1$, the procedure starts at state $s^1=1$;
2. At step r, assume the state $s^r\in S$ is given. For $l=(l_1,\ldots,l_T)$, define by t_r be maximum t such that the link l_t starts from the state s^r, that is,

$$t_r \;=\; \max\{t\in\{1,\ldots,T\}\mid s'_t=s^r \text{ for } l_t=(s'_t,s_t,\theta_t)\};$$

3. Define $\pi(l)_r=l_{t_r}$;
4. If the ending state of l_{t_r} is $s_{t_r}=\diamond$, then $r=\overline{r}$, and the procedure finishes;
5. If the ending state of l_{t_r} is $s_{t_r}\neq\diamond$, define $s^{r+1}=s_{t_r}$ and continue the procedure for step $r+1$.

Lemma 14. *Consider procedure π, defined above, then: (i) the procedure π always finishes at step $\overline{r}\leq T$, (ii) the result $\pi(l)$ is a simple valid path, i.e., $\pi(l)\in\mathcal{L}_0$, (iii) if $l\in\mathcal{L}_0$, then $\pi(l)=l$.*

Proof. Suppose r and $v=r+1$ are two consecutive steps of the procedure. Since v is one of the steps, then $s^v=s^{r+1}\in S$. Moreover, $l_{t_r}=(s'_{t_r},s_{t_r},\theta_{t_r})=(s'_{t_r},s^{t_{r+1}},\theta_{t_r})$, hence $l_{t_r+1}=(s^{t_{r+1}},s_{t_r+1},\theta_{t_r+1})$. It follows that $t_{r+1}\geq t_r+1>t_r$. Inductively, $v>r$ implies $t_v>t_r$. Since $t_r\in\{1,\ldots,T\}$, the procedure finishes after a finite number of steps $\overline{r}\leq T$, proving (i).

Next, since $s^1 = 1$, then $\pi(l)_1 = (1, s_{t_1}, \theta_{t_1})$, i.e., the first link of $\pi(l)$ begins at state $s = 1$. Since the procedure finishes at step $\overline{r}$ such that $s_{t_r} = \diamond$, then $\pi(l)_{\overline{r}} = (s'_{t_{\overline{r}}}, \diamond, \theta_{t_{\overline{r}}})$, i.e. the last link of $\pi(l)$ ends at $s = \diamond$. Consider $\pi(l)_r = (s'_{t_r}, s_{t_r}, \iota_{t_r})$. If $r < \overline{r}$, then $s^{r+1} = s_{t_r}$, hence $\pi(l)_{r+1} = (s_{t_r}, s_{t_{r+1}}, \theta_{t_{r+1}})$; thus, the next link of path $\pi(l)$ begins at the state at which the previous link ends. Thus, $\pi(l) \in \mathcal{L}$.

To show that $\pi(l) \in \mathcal{L}_0$, consider link $\pi(l)_r$ that begins at state s^r. Towards a contradiction, assume that $\pi(l)_{r'}$ for $r' > r$ begins at state $s^{r'} = s^r$ as well. Then $s'_{t_{r'}} = s^{r'} = s^r$, hence $t_{r'} \leq t_r$, in contradiction to the proven assertion $t_{r'} > t_r$. Statement (ii) is proven.

Suppose $l \in \mathcal{L}_0$. Then $l_t = (s'_t, s_t, \theta_t)$ implies $s'_v \neq s'_t$ for $v \neq t$. Then, inductively, $\pi(l)_r = l_r$ for all $r = 1, \ldots, T$, hence $\pi(l) = l$. ∎

Lemma 14 implies that π is a surjective mapping from $\mathcal{L}$ to $\mathcal{L}_0$. For $l \in \mathcal{L}_0$, denote by $\pi^{-1}(l)$ the pre-image of l with respect to π, that is,

$$\pi^{-1}(l) \; = \; \{l' \in \mathcal{L} \mid \pi(l') = l\}$$

Lemma 15. *Let $k = |S|$ be the number of flexible memory states. Then for any $l \in \mathcal{L}_0$,*

$$\omega(l) \; \geq \; \eta^k \cdot \sum_{l' \in \pi^{-1}(l)} \omega(l')$$

Proof. We prove the lemma by induction in k.

First, suppose $k = 1$. The set of simple valid paths is then $\mathcal{L}_0 = \Big\{\{(1, \diamond, \theta)\} \,\Big|\, (1, \diamond, \theta) \in \mathcal{T}\Big\}$. Let $l = (l_1) \in \mathcal{L}_0$. For any path $l' = (l'_1, \ldots, l'_T) \in \pi^{-1}(l)$ we have $l'_T = l_1$ and $l_t = (1, 1, \theta_t)$ for all $t < T$. Then

$$\begin{aligned}
\sum_{l' \in \pi^{-1}(l)} \omega(l') \; &= \\
= \; \omega(l_1) \cdot \sum_{T=1}^{\infty} (1-\eta)^{T-1} \cdot \sum_{(\theta_1, \ldots, \iota_{T-1})} \prod_{t=1}^{T-1} f_1(\theta_t) \cdot \tau(1, 1, \theta_t) \\
= \; \omega(l_1) \cdot \sum_{T=1}^{\infty} (1-\eta)^{T-1} \cdot \sum_{\theta_{T-1}} f_1(\theta_{T-1}) \cdot \tau(1, 1, \theta_{T-1}) \cdot \Big(\sum_{(\theta_1, \ldots, \theta_{T-2})} \prod_{t=1}^{T-2} f_1(\theta_t) \cdot \tau(1, 1, \theta_t) \Big) \; \leq \\
\leq \; \omega(l_1) \cdot \sum_{T=1}^{\infty} (1-\eta)^{T-1} \cdot \Big(\sum_{(\theta_1, \ldots, \theta_{T-2})} \prod_{t=1}^{T-2} f_1(\theta_t) \cdot \tau(1, 1, \theta_t) \Big) \; \leq \; \ldots \; \leq \\
\leq \; \omega(l_1) \cdot \sum_{T=1}^{\infty} (1-\eta)^{T-1} \; = \; \eta^{-1} \cdot \omega(l_1) \; = \; \eta^{-1} \cdot \omega(l)
\end{aligned}$$

where we used $\sum_{\theta \in \mathcal{I}} f_{s'}(\theta) = 1$ and $\tau(s', s, \theta) \leq 1$.

Second, suppose the lemma holds for $k = \overline{k}$, and consider $k = \overline{k} + 1$. Consider arbitrary $l \in \mathcal{L}_0$. Assume $|l| > 1$, then $l_1 = (1, v, \overline{\theta})$ for some $v \in S \backslash \{1\}$ and $\overline{\theta} \in \mathcal{I}$. Denote also $l = (l_1, ..., l_{\overline{T}})$. For $L \in \pi^{-1}(l)$, let

$$t_1(L) \;=\; \max\{t \in \{1, ..., T\} \mid s'_t = 1 \text{ for } L_t = (s'_t, s_t, \theta_t)\}$$

Thus, $t_1(L)$ is the last period when valid path $L \in \pi^{-1}(l)$ goes via state $s = 1$, and $t_1(L) = t_1$ when procedure π is applied to the path L.

For $L = (L_1, ..., L_T) \in \pi^{-1}(l)$, denote by $L_+ = (l_1, ..., L_{t_1(L)})$, and $L_- = (L_{t_1(L)+1}, ... L_T)$; thus, $L = (L_+, L_-)$. Let

$$\mathcal{L}_+(l) \;=\; \bigcup_{t=1}^{\infty} \left\{ (\tilde{L}_1, ..., \tilde{L}_t) \,\middle|\, \exists L \in \pi^{-1}(l) : (\tilde{L}_1, ..., \tilde{L}_t) = L_+ \right\}$$

and

$$\mathcal{L}_-(l) \;=\; \bigcup_{t=1}^{\infty} \left\{ (\tilde{L}_1, ..., \tilde{L}_t) \,\middle|\, \exists L \in \pi^{-1}(l) : (\tilde{L}_1, ..., \tilde{L}_t) = L_- \right\}$$

Claim 7. *The following decomposition holds:*

$$\pi^{-1}(l) \;=\; \left\{ L \in \mathcal{L} \,\middle|\, \exists L^1 \in \mathcal{L}_+(l), L^2 \in \mathcal{L}_-(l) : L = (L^1, L^2) \right\}$$

Proof of the Claim. For all paths $L \in \pi^{-1}(l)$, the last link of the sequence L_+ is $l_1 = (1, v, \theta)$, and the first link of sequence L_- starts from state v. It follows that if $L, \tilde{L} \in \pi^{-1}(l)$, then $(L_+, \tilde{L}_-) \in \pi^{-1}(l)$. Thus,

$$\pi^{-1}(l) \;\supseteq\; \left\{ L \in \mathcal{L} \,\middle|\, \exists L^1 \in \mathcal{L}_+(l), L^2 \in \mathcal{L}_-(l) : L = (L^1, L^2) \right\}$$

To see that

$$\pi^{-1}(l) \;\subseteq\; \left\{ l \in \mathcal{L} \,\middle|\, \exists L^1 \in \mathcal{L}_+(l), L^2 \in \mathcal{L}_-(l) : L = (L^1, L^2) \right\},$$

take $L^1 = L_+$ and $L^2 = L_-$ for $L \in \pi^{-1}(l)$. □

Claim 8. *For $l \in \mathcal{L}_0$ and $\mathcal{L}_+(l)$ defined above,*

$$\sum_{L^1 \in \mathcal{L}_+(l)} (1 - \eta)^{|L^1| - 1} \cdot \prod_{t=1}^{|L^1|} \omega(L^1_t) \;\leq\; \eta^{-1} \cdot \omega(l_1)$$

Proof of the Claim. For $y = 1, 2, \ldots$, denote by

$$\mathcal{L}_+^y(l) \;=\; \left\{ L^1 \in \mathcal{L}_+(l) \,\middle|\, |L^1| = y \right\}$$

Then $\left\{\mathcal{L}_+^y(l)\right\}_{y=1,2,..}$ is a partition of $\mathcal{L}_+(l)$. Thus,

$$\sum_{L^1 \in \mathcal{L}_+(l)} (1-\eta)^{|L^1|} \cdot \prod_{t=1}^{|L^1|} \omega(L_t^1) \;=\; \sum_{y=1}^{\infty} (1-\eta)^y \cdot \sum_{L^1 \in \mathcal{L}_+^y(l)} \prod_{t=1}^{y} \omega(L_t^1)$$

Recall that $l_1 = (1, v, \theta) = L_y^1$ for all $L^1 \in \mathcal{L}_+^y(l)$. For any $L^1 = \left((s_{t-1}^1, s_t^1, \theta_{t-1}^1)\right)_{t=1}^{y} \in \mathcal{L}_+^y(l)$, consider the event $\zeta^+(L^1)$ given by

$$\zeta^+(L^1) \;=\; \left\{ X_t = (s_t^1, \theta_t^1) \;\forall 0 \le t \le y-2,\; S_{y-1} = s_{y-1}^1 \right\}$$

Clearly, if $L^1, L^3 \in \mathcal{L}_+^y(l)$, $L^1 \neq L^3$, then $\zeta^+(L^1) \neq \zeta^+(L^3)$; moreover, $\zeta^+(L^1) \cap \zeta^+(L^3) = \varnothing$. Next, for $L^1 = \left((s_{t-1}^1, s_t^1, \theta_{t-1}^1)\right)_{t=1}^{y} \in \mathcal{L}_+^y(l)$, if $y = 1$, then $\mu(\zeta^+(L^1)) = \mu(S_0 = 1) = 1$, and if $y > 1$, using the fact that $s_t^1 \in S$ for all $t = 1, \ldots, y-1$ and the formula for the stochastic matrix P, we get

$$\begin{aligned}
\mu(\zeta^+(L^1)) \;&=\; f_1(\theta_0^1) \cdot \left[\prod_{t=1}^{y-2} \left((1-\eta) \cdot \tau(s_{t-1}^1, s_t^1, \theta_{t-1}^1) \cdot f_{s_t^1}(\theta_t^1) \right) \right] \cdot (1-\eta) \cdot \tau(s_{y-1}^1, s_{y-1}^1, \theta_{y-2}^1) \;= \\
&=\; (1-\eta)^{y-1} \cdot \prod_{t=1}^{y-1} f_{s_{t-1}^1}(\theta_{t-1}^1) \cdot \tau(s_{t-1}^1, s_t^1, \theta_{t-1}^1) \;= \\
&=\; (1-\eta)^{y-1} \cdot \prod_{t=1}^{y-1} \omega(s_{t-1}^1, s_t^1, \theta_{t-1}^1) = (1-\eta)^{y-1} \cdot \prod_{t=1}^{y-1} \omega(l^1)
\end{aligned}$$

Thus, for all $y = 1, 2, \ldots$,

$$\mu(\zeta^+(L^1)) \;=\; (1-\eta)^{y-1} \cdot \prod_{t=1}^{y-1} \omega(L^1)$$

Then, using $\zeta^+(L^1) \subset \{S_{y-1} = 1\}$ for all $L^1 \in \mathcal{L}_+^y(l)$, $\zeta^+(L^1) \cap \zeta^+(L^3) = \varnothing$ for $L^1 \neq L^3$ and Lemma 12, we get

$$(1-\eta)^{y-1} \cdot \sum_{L^1 \in \mathcal{L}_+^y(l)} \prod_{t=1}^{y-1} \omega(L_t^1) \;=\; \sum_{L^1 \in \mathcal{L}_+^y(l)} \tilde{\mu}(\zeta^+(L^1)) \;\le\; \mu(S_{y-1} = 1) \;\le\; (1-\eta)^{y-1}$$

It follows

$$\sum_{L^1 \in \mathcal{L}_+(l)} (1-\eta)^{|L^1|-1} \cdot \prod_{t=1}^{|L^1|} \omega(L^1_t) \;\leq\; \sum_{y=1}^{\infty} (1-\eta)^{y-1} \cdot \omega(L^1_y) \;=\; \eta^{-1} \cdot \omega(L_1)$$

proving the claim. $\square$

Note that state v is the end of the link L_1, and it is the end of the link $L_{t_1(L)} = l_1$ for all $L \in \pi^{-1}(l)$. Let $\widehat{S} = \{1, \ldots, \overline{k}\} \cup \{\diamond\}$, and $\widehat{S}^o = \{1, \ldots, \overline{k}\}$. Define a mapping $\chi : \{2, \ldots, \overline{k}+1\} \cup \{\diamond\} \to \{1, \ldots, \overline{k}\} \cup \{\diamond\}\}$ as follows: $\chi(v) = 1$, $\chi(s) = s$ for all $s < v$, $\chi(s) = s - 1$ for all $s > v$, $\chi(\diamond) = \diamond$. Clearly, χ is a bijection. Let χ^{-1} be the inverse mapping, and

$$\widehat{\mathcal{T}}^a = \left\{ (\widehat{r}, \widehat{s}, \theta) \in \widehat{S}^o \times \widehat{S} \times \mathcal{I} \;\middle|\; (\chi^{-1}(\widehat{r}), \chi^{-1}(\widehat{s}), \theta) \in \mathcal{T}^a \right\}$$

Let $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{L}}_0$ be the set of valid paths and simple valid paths on $\widehat{S}$, $\widehat{\pi}$ be the corresponding mapping between $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{L}}_0$, and $\widehat{\omega}$ be the corresponding weighting function, where

$$\widehat{\omega}(\widehat{r}, \widehat{s}, \theta) \;=\; f_{\chi^{-1}(\widehat{r})}(\theta) \cdot \tau(\chi^{-1}(\widehat{r}), \chi^{-1}(\widehat{s}), \theta) \;=\; \omega(\chi^{-1}(\widehat{r}), \chi^{-1}(\widehat{s}), \theta)$$

For a link $(s', s, \theta) \in \mathcal{T}^a$ such that $s', s \neq \{1\}$, define $\chi((s', s, \theta)) = (\chi(s'), \chi(s), \theta) \in \widehat{\mathcal{T}}^a$.

Claim 9. *Mapping χ, applied to the set of links as defined above, is a bijection between the set $\{(s', s, \theta) \in \mathcal{T}^a \mid s', s \neq 1\}$ and the set $\widehat{\mathcal{T}}^a$.*

Proof of the Claim. The Claim follows from the definition of $\widehat{\mathcal{T}^a}$ and the fact that $\chi : \{2, \ldots, \overline{k}+1\} \to \{1, \ldots, \overline{k}\}$ is a bijection. $\square$

For an ordered set of links $(L_{t'+1}, L_{t'+2}, \ldots, L_{T'})$ such that $(L_1, \ldots, L_{T'}) \in \mathcal{L}$, $L_{t'+1}$ begins at v, and L_t does not go via state $s = 1$ for $t > t'$, define $\chi((L_{t'+1}, L_{t'+2}, \ldots, L_{T'})) = (\chi(L_{t'+1}), \chi(L_{t'+2}), \ldots, \chi(L_{T'})) \in \widehat{\mathcal{L}}$.

Claim 10. *The mapping χ, applied to the sequences of links as defined above, is a bijection between the set $\mathcal{L}_-(l)$ and the set $\widehat{\pi}^{-1}(\chi(L_2), \ldots, \chi(L_{\overline{T}}))$.*

Proof of the Claim. First, note that links $L_2, \ldots, L_{\overline{T}}$ do not go via $s = 1$, thus links $\chi(L_2), \ldots, \chi(L_{\overline{T}})$ are well-defined. The beginning of the link L_{t+1} is the end of the link L_t, thus the beginning of the link $\chi(L_{t+1})$ is the end of the link $\chi(L_t)$. Also, link L_2 starts at $s = v$, thus $\chi(L_2)$ starts at $\chi(v) = 1$, and $L_{\overline{T}}$ ends at $s = \diamond$, hence $\chi(L_{\overline{T}})$ ends at $\chi(\diamond) = \diamond$. Therefore, $(\chi(L_2), \ldots, \chi(L_{\overline{T}})) \in \widehat{\mathcal{L}}$. Next, since every $s \in S^o$ is a beginning of at most one of the links

$L_2, ..., L_{\overline{T}}$, then every $\widehat{s} \in \widehat{S}^o$ is the beginning of at most one of the links $\chi(L_2), ..., \chi(L_{\overline{T}})$, thus $(\chi(L_2), ..., \chi(L_{\overline{T}})) \in \widehat{\mathcal{L}}_0$, and the set $\widehat{\pi}^{-1}(\chi(L_2), ..., \chi(L_{\overline{T}}))$ is well-defined.

Next, consider $L^2, L^3 \in \mathcal{L}_-(l)$ such that $\chi(L^2) = \chi(L^3)$. Then $|L^2| = |L^3| = K$ for some $K > 0$, and $\chi(L^2)_t = \chi(L^3)_t$ for all $t \in \{1, ..., K\}$, hence $\chi(L_t^2) = \chi(L_t^3)$ for all $t \in \{1, ..., K\}$, hence, by Claim 9, $L_t^2 = L_t^3$ for all $t \in \{1, ..., K\}$. Thus, $L^2 = L^3$, and χ is an injection from $\mathcal{L}_-(l)$ to $\widehat{\mathcal{L}}$.

Consider arbitrary $L^2 \in \mathcal{L}_-(l)$. For $r = 2, ..., \overline{T}$, let t_r be the maximum t such that L_t^2 goes via state $(s')^r$ such that $L_r = ((s')^r, s^r, \theta^r)$. Then t_r is also the maximum t such that $\chi(L^2)_t$ goes via state $\chi((s')^r)$ such that $\chi(l)_r = (\chi((s')^r), \chi(s^r), \theta^r)$. It follows that $\chi(L^2) \in \widehat{\pi}^{-1}(\chi(l_2), ..., \chi(l_{\overline{T}}))$. Thus, χ is an injection from $\mathcal{L}_-(l)$ to $\widehat{\pi}^{-1}(\chi(l_2), ..., \chi(l_{\overline{T}}))$.

Finally, take arbitrary $L = (L_1, ..., L_T) \in \widehat{\pi}^{-1}(\chi(l_2), ..., \chi(l_{\overline{T}}))$ and consider $L^2 = (\chi^{-1}(L_1), ..., \chi^{-1}(L_T))$. For $r = 2, ..., \overline{T}$, let t_r be the maximum t such that $\chi(L^2)_t$ goes via state $\chi((s')^r)$ such that $\chi(l)_r = (\chi((s')^r), \chi(s^r), \theta^r)$, then t_r is also the maximum t such that L_t^2 goes via state $(s')^r$ such that $l_r = ((s')^r, s^r, \theta^r)$. It follows that $L^2 \in \mathcal{L}_-(l)$. Since $\chi(L^2) = L$, then χ is also a surjection from $\mathcal{L}_-(l)$ to $\widehat{\pi}^{-1}(\chi(l_2), ..., \chi(l_{\overline{T}}))$, proving the claim. $\square$

Using Claims 7, 8, 10 and the induction assumption, we get:

$$\sum_{L\in\pi^{-1}(l)}\omega(L)=\sum_{L^1\in\mathcal{L}_+(l)}\sum_{L^2\in\mathcal{L}_-(l)}\omega(L^1,L^2)=\sum_{L^1\in\mathcal{L}_+(l)}\sum_{L^2\in\mathcal{L}_-(l)}(1-\eta)^{|L^1|+|L^2|-1}\cdot\prod_{t=1}^{|L^1|}\omega(L^1_t)\cdot\prod_{t''=1}^{|L^2|}\omega(L^2_{t''})=$$

$$=\sum_{L^1\in\mathcal{L}_+(l)}(1-\eta)^{|L^1|}\cdot\prod_{t=1}^{|L^1|}\omega(L^1_t)\cdot\left[\sum_{L^2\in\mathcal{L}_-(l)}(1-\eta)^{|L^2|-1}\cdot\prod_{t''=1}^{|L^2|}\omega(L^2_{t''})\right]=$$

$$=\sum_{L^1\in\mathcal{L}_+(l)}(1-\eta)^{|L^1|}\cdot\prod_{t=1}^{|L^1|}\omega(L^1_t)\cdot\left[\sum_{\widehat{L}\in\widehat{\mathcal{L}}:\ \widehat{\pi}(\widehat{L})=(\chi(l_2),\dots,\chi(l_{\overline{T}}))}(1-\eta)^{|\widehat{L}|-1}\cdot\prod_{t''=1}^{|\widehat{L}|}\widehat{\omega}(\widehat{L}_{t''})\right]=$$

$$=\sum_{L^1\in\mathcal{L}_+(l)}(1-\eta)^{|L^1|}\cdot\prod_{t=1}^{|L^1|}\omega(L^1_t)\cdot\left[\sum_{\widehat{L}\in\widehat{\pi}^{-1}(\chi(l_2),\dots,\chi(l_{\overline{T}}))}\widehat{\omega}(\widehat{L})\right]\leq$$

$$\leq\sum_{L^1\in\mathcal{L}_+(l)}(1-\eta)^{|L^1|}\cdot\prod_{t=1}^{|L^1|}\omega(L^1_t)\cdot\left[\eta^{-\overline{k}}\cdot\omega'(\chi(l_2),\dots,\chi(l_{\overline{T}}))\right]\leq$$

$$\leq(1-\eta)\cdot\eta^{-1}\cdot\omega(l_1)\cdot\left[\eta^{-\overline{k}}\cdot\widehat{\omega}(\chi(l_2),\dots,\chi(l_{\overline{T}}))\right]=$$

$$=(1-\eta)\cdot\eta^{-1}\cdot\omega(l_1)\cdot\left[\eta^{-\overline{k}}\cdot(1-\eta)^{\overline{T}-2}\prod_{t=2}^{\overline{T}}\widehat{\omega}(\chi(l_t))\right]=$$

$$=(1-\eta)\cdot\eta^{-1}\cdot\omega(l_1)\cdot\left[\eta^{-\overline{k}}\cdot(1-\eta)^{\overline{T}-2}\prod_{t=2}^{\overline{T}}\omega(l_t)\right]=$$

$$=\eta^{-\overline{k}-1}\cdot(1-\eta)^{\overline{T}-1}\prod_{t=1}^{\overline{T}}\omega(l_t)\ =\ \eta^{-\overline{k}-1}\cdot\omega(l)$$

which is the statement of the Lemma for $k=\overline{k}+1$.

When $l=\{l_1\}$, the same analysis applies, but this time $\mathcal{L}_-(l)=\emptyset$, and we can skip the usage of induction assumption and Claims 9 and 10. The Lemma is proven. ■

I now provide conditions on $\mathcal{L}_0$ and $\omega(\cdot)$ that are necessary and sufficient for a sequence of decision rules to solve the choice problem.

For an arbitrary alternative $a \in A$ and strategy $\sigma \in \Sigma$, define $l^*_\sigma(a) \in \mathcal{L}_0$ and $\omega^*_\sigma(a) \in [0,1]$ as follows:

$$\omega^*_\sigma(a) \;=\; \max_{l \in \mathcal{L}_0(a,\sigma)} \omega^a_\sigma(l), \qquad l^*_\sigma(a) = \arg\max_{l \in \mathcal{L}_0(a,\sigma)} \omega^a_\sigma(l)$$

where, if the set of maximizers of $\omega^a_\sigma(l)$ is not a singleton, then $l^*_\sigma(a)$ is chosen from this set according to a fixed total order on $\mathcal{L}_0$, that I don't introduce explicitly to ease the notations.

Lemma 16. *A sequence decision rule $\{\sigma_r\}_{r=1,2,\dots} \in \Psi$ solves choice problem $(Q,\succeq)$ if and only if the following conditions hold:*
(i) for all $a,b \in A$: $a \succ b$ implies $\omega^*_{\sigma_r}(b)/\omega^*_{\sigma_r}(a) \longrightarrow 0$;
(ii) for all $a \in A$, $\exists \bar{r}$: $\omega^*_{\sigma_r}(a) > 0 \;\forall r > \bar{r}$.

Proof. Since two links of a simple path cannot begin from the same state, the number of simple valid paths $|\mathcal{L}_0|$ is bounded from above by

$$K \;=\; \prod_{s \in S} \Big(\big| \{(s,\theta) \in S \times \mathcal{I} \mid (s',s,\theta) \in \mathcal{T}\} \big| + 1 \Big)$$

By Lemmas 13, 14, 15 and the definition of $\omega^*_\sigma(a)$,

$$q_\sigma(a) \;=\; \sum_{l \in \mathcal{L}} \omega(l) \;=\; \sum_{l \in \mathcal{L}_0} \sum_{l' \in \pi^{-1}(l)} \omega(l') \;\leq\; \eta^{-k} \cdot \sum_{l \in \mathcal{L}_0} \omega(l) \;\leq\; \eta^{-k} \cdot K^{-1} \cdot \omega^*_\sigma(a)$$

From the other hand,

$$\omega^*_\sigma(a) \;=\; \omega(l^*_\sigma(a)) \;\leq\; \sum_{l \in \mathcal{L}} \omega(l) \;=\; q_\sigma(a)$$

Hence,

$$\omega^*_\sigma(a) \;\leq\; q_\sigma(a) \;\leq\; \eta^{-k} \cdot K^{-1} \cdot \omega^*_\sigma(a)$$

Since η^{-k} and K^{-1} are constant along the sequence of strategies σ_r that comprises the decision rule—recall that the set $\mathcal{T}$ remains constant for all σ_r—then $q_{\sigma_r}(b)/q_{\sigma_r}(a) \longrightarrow 0$ if and only if $\omega^*_{\sigma_r}(b)/\omega^*_{\sigma_r}(a) \longrightarrow 0$ and $\exists \bar{r}$: $q_{\sigma_r}(a) > 0 \;\forall r > \bar{r}$ if and only if $\exists \bar{r}$: $\omega^*_{\sigma_r}(a) > 0 \;\forall r > \bar{r}$. The statement of the Lemma then follows from Lemma 11. ■

For a state $s \in S^o$, denote by $\mathcal{T}(s)$ the subset of links that begin in state s and end in some other state $v \neq s$; I omit index σ from $\mathcal{T}_\sigma$ and $\mathcal{T}_\sigma(s)$ when it does not cause confusion. Thus,

$$\mathcal{T}(s) \;=\; \Big\{ (s',v,\theta) \in \mathcal{T} \mid s' = s, v \neq s \Big\} \tag{15}$$

and define similarly $\mathcal{T}^a(s)$:

$$\mathcal{T}^a(s) = \left\{(s', v, \theta) \in \mathcal{T}^a \mid s' = s, v \neq s\right\}$$

where I omit index σ for brevity as well. Note that

$$\mathcal{T}(s) = \bigcup_{a \in A} \mathcal{T}^a(s), \qquad \mathcal{T} = \bigcup_{a \in A} \mathcal{T}^a$$

Define also sets

$$\overline{\mathcal{T}}(s) = \mathcal{T}(s) \cup \{\varnothing\}, \qquad \mathbb{T} = \times_{s \in S} \overline{\mathcal{T}}(s) \tag{16}$$

and, similarly,

$$\overline{\mathcal{T}}^a(s) = \mathcal{T}^a(s) \cup \{\varnothing\}, \qquad \mathbb{T}^a = \times_{s \in S} \overline{\mathcal{T}}^a(s)$$

I interpret $\overline{\mathcal{T}}(s)$ as a set of links outgoing from state s that could potentially occur in a simple valid path $l \in \mathcal{L}(a, \sigma)$, where $\varnothing \in \overline{\mathcal{T}}(s)$ stands for path l not going via state s. For an element $e = (e_1, ..., e_{|S|}) \in \mathbb{T}$, define its weight by

$$\omega_\sigma(e) = \prod_{s \in S} \omega_\sigma(e_s), \quad \text{where} \begin{cases} \omega_\sigma(e_s) = \omega_\sigma(s', s, \theta) & \text{if} \quad e_s = (s', s, \theta) \\ \omega_\sigma(e_s) = 1 & \text{if} \quad e_s = \varnothing \end{cases}$$

I say that a decision rule $\{\sigma_r\}_{r=1,2,...} \in \Psi$ is *regular*, if the following two conditions hold:

(i) For all $a \in A$ for all r, r', $l^*_{\sigma_r}(a) = l^*_{\sigma_{r'}}(a)$;

(ii) For any $e, e' \in \mathbb{T}$, there exists $\lim_{k \to \infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e') \in [0, \infty) \cup \{\infty\}$

I denote by $\Psi_0 \subset \Psi$ the set of regular decision rules.

Lemma 17. *Every decision rule $\{\sigma_r\}_{r=1,2,...} \in \Psi$ contains a regular decision rule as a subsequence.*

Proof. The Lemma follows from the finiteness of sets A, $\mathcal{L}_0(a, \sigma)$ and $\mathbb{T}$. ∎

Given a regular decision rule $\psi = \{\sigma_r\}_{r=1,2,...} \in \Psi_0$, define the following auxilliary binary relation $\hat{\succsim} \subseteq \mathbb{T} \times \mathbb{T}$ (where $\mathbb{T} = \mathbb{T}_\psi$ is given by eq. (16) with the state space $S = S_\psi$ and the set of transitions $\mathcal{T} = \mathcal{T}_\psi$):

$$e \mathrel{\hat{\succsim}_\psi} e' \iff \lim_{r \to \infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e') > 0 \tag{17}$$

where I use the convention $\{\infty\} > 0$.

Lemma 18. *Let $\psi = \{\sigma_r\}_{r=1,2,\ldots} \in \Psi_0$ be a regular decision rule, then the binary relation $\hat{\succsim}_\psi$ defined above, is complete and transitive.*

Proof. Since $\psi \in \Psi_0$, then for any $e, e' \in \mathbb{T}$, there exist the limits of the ratios $\omega_{\sigma_r}(e)/\omega_{\sigma_r}(e')$, $\omega_{\sigma_r}(e')/\omega_{\sigma_r}(e)$, and at least one of them is not zero. If $\lim_{r\to\infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e') > 0$ and $\lim_{r\to\infty} \omega_{\sigma_r}(e')/\omega_{\sigma_r}(e'') > 0$, then

$$\lim_{r\to\infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e'') \;=\; \Big(\lim_{r\to\infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e')\Big)\cdot\Big(\lim_{r\to\infty} \omega_{\sigma_r}(e')/\omega_{\sigma_r}(e'')\Big)$$

implying $\lim_{r\to\infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e'') > 0$, thus, $\hat{\succsim}_\psi$ is transitive. ■

Denote by symbols $\hat{\sim}_\psi$ and $\hat{\succ}_\psi$ the symmetric and asymmetric parts of $\hat{\succsim}_\psi$. Note that

$$e \mathrel{\hat{\sim}_\psi} e' \quad\Longleftrightarrow\quad \lim_{r\to\infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e') \in (0,\infty),$$

$$e \mathrel{\hat{\succ}_\psi} e' \quad\Longleftrightarrow\quad \lim_{r\to\infty} \omega_{\sigma_r}(e)/\omega_{\sigma_r}(e') = \infty \quad\Longleftrightarrow\quad \lim_{r\to\infty} \omega_{\sigma_r}(e')/\omega_{\sigma_r}(e) = 0$$

Define a mapping $\phi : A \to \mathbb{T}$ as follows:

$$\phi_s(a) \;=\; \begin{cases} (s', s, \theta) & \text{if} \quad (s', s, \theta) \text{ is a link of } l^*_\psi(a) \\ \varnothing & \text{if} \quad \nexists (s,\theta):\ (s', s, \theta) \text{ is a link of } l^*_\psi(a) \end{cases}$$

Note that $\phi(\cdot)$ is well-defined, since there is at most one link (s', s, θ) outgoing from a state s' in a simple path $l^*_\psi(a)$.

Lemma 19. *If a regular decision rule $\psi = \{\sigma_r\}_{r=1,2,\ldots} \in \Psi_0$ solves choice problem $(Q, \succeq)$, then $a \succ b$ implies $\phi(a) \mathrel{\hat{\succ}_\psi} \phi(b)]$ for all $a, b \in A$.*

Proof. Consider arbitrary $a, b \in A$ such that $a \succeq b$. By Lemma 16, $\omega^*_{\sigma_r}(a)/\omega^*_{\sigma_r}(b) \longrightarrow 0$. Thus,

$$\lim_{r\to\infty} \frac{\omega_{\sigma_r}(\phi(b))}{\omega_{\sigma_r}(\phi(a))} = \lim_{r\to\infty} \frac{\prod_{s\in S} \omega(\phi(b))}{\prod_{s\in S} \omega(\phi(a))} = \lim_{r\to\infty} \frac{\prod_{t=1}^{|l^*(b)|} \omega(l^*(b)_t)}{\prod_{t'=1}^{|l^*(a)|} \omega(l^*(a)_{t'})} = \lim_{r\to\infty} \cdot \frac{(1-\eta)^{|l^*(a)|} \cdot \omega(l^*(b))}{(1-\eta)^{|l^*(b)|} \cdot \omega(l^*(a))} = 0$$

It follows that $\phi(a) \mathrel{\hat{\succ}_\psi} \phi(b)$. ■

Note that if a decision rule $\psi = \{\sigma_r\}_{r=1,2,\ldots}$ is regular, then for any link (s', s, θ), there exists $\lim_{r\to\infty} \omega(s', s, \theta) \in [0,1]$—to see this, notice that $\omega(\varnothing, \ldots, \varnothing) = 1$ for $(\varnothing, \ldots, \varnothing) \in \mathbb{T}$, and $e \in \mathbb{T}$, where $e_{s''} = \varnothing$ if $s'' \neq s$, and $e_s = (s', s, \theta)$.

Thus, for a regular decision rule $\psi = \{\sigma_r\}_{r=1,2,\dots}$, I call a link (s', s, θ) *weak* if $\lim_{r\to\infty} \omega(s', s, \theta) = 0$; otherwise, I call a link *strong*. Thus, $\lim_{r\to\infty} \omega(s', s, \theta) \in (0, 1]$ for a strong link (s', s, θ). Define also $\varnothing \in \overline{\mathcal{T}}(s)$ for any s to be a strong link as well. Denote by

$$\mathcal{T}^{weak}(s) = \Big\{(s, v, \theta) \in \mathcal{T}(s) \;\Big|\; \lim_{r\to\infty} \omega(s, v, \theta) = 0\Big\}, \qquad \mathcal{T}^{weak} = \bigcup_{s\in S} \mathcal{T}^{weak}(s)$$

$$\mathcal{T}^{strong}(s) = \Big\{(s, v, \theta) \in \mathcal{T}(s) \;\Big|\; \lim_{r\to\infty} \omega(s, v, \theta) \in (0, 1]\Big\}, \qquad \mathcal{T}^{strong} = \bigcup_{s\in S} \mathcal{T}^{weak}(s)$$

Note that $\Big\{\mathcal{T}^{weak}(s), \mathcal{T}^{strong}(s)\Big\}$ is a partition of $\mathcal{T}(s)$.

For an arbitrary alternative $a \in A$, define

$$\mathcal{F}(a) = \bigcup_{s\in S:\ \phi_s(a)\in\mathcal{T}^{weak}} \phi_s(a)$$

Thus, the set $\mathcal{F}(a)$ consists of all weak links that are used in a simple path $l^*(a)$ that maximizes the probability to go from state 1 to state $\diamond$ among all simple paths for a. Denote also by

$$S^{weak}(a) = \{s \in S \mid (l^*(a))_s \in \mathcal{T}^{weak}\}$$

Thus, $S^{weak}(a)$ is the set of states $s \in S$ such that there is a link in the path $l^*(a)$, outgoing from state s, and this link is weak.

Lemma 20. *Let a regular decision rule $\psi = \{\sigma_r\}_{r\in\mathbb{N}} \in \Psi_0$ be given. Suppose $e, e' \in \mathbb{T}_\psi$ be such that for all $s \in S$ if $e_s \in \mathcal{T}^{weak}(s)$ or $e'_s \in \mathcal{T}^{weak}(s)$, then $e'_s = e_s$. Then $e \hat{\sim}_\psi e'$.*

Proof. Note that $e_s \in \mathcal{T}^{weak}(s)$ if and only if $e'_s \in \mathcal{T}^{weak}(s)$. Let $\tilde{S} = \{s \in S | e_s \notin \mathcal{T}^{weak}(s)\}$. Then

$$\lim_{r\to\infty} \frac{\omega(e)}{\omega(e')} = \lim_{r\to\infty} \prod_{s\in S\setminus\tilde{S}} \frac{\omega(e_s)}{\omega(e'_s)} \cdot \prod_{s\in\tilde{S}} \frac{\omega(e_s)}{\omega(e'_s)} = \lim_{r\to\infty} \prod_{s\in\tilde{S}} \frac{\omega(e_s)}{\omega(e'_s)} = \prod_{s\in\tilde{S}} \frac{\lim_{r\to\infty} \omega(e_s)}{\lim_{r\to\infty} \omega(e'_s)} \in (0, \infty)$$

since $\lim_{r\to\infty} \omega(e_s), \lim_{r\to\infty} \omega(e'_s) \in (0, 1]$ for all $s \in \tilde{S}$. The statement of the Lemma follows from the definition of $\hat{\succsim}_\psi$. ■

Lemma 21. *If a regular decision rule $\psi = \{\sigma_r\}_{r=1,2,\dots} \in \Psi_0$ solves the choice problem $(Q, \succeq)$, then for any $a, b \in A$, $a \succ b$ implies $\mathcal{F}(a) \neq \mathcal{F}(b)$.*

Proof. Towards a contradiction, assume that $a \succ b$, but $\mathcal{F}(a) = \mathcal{F}(b)$; hence, $S^{weak}(a) = S^{weak}(b) = \overline{S}$ for some $\overline{S} \subseteq S$. Then

$$\begin{aligned}\lim_{r\to\infty} \frac{\omega^*_{\sigma_r}(b)}{\omega^*_{\sigma_r}(a)} &= \lim_{r\to\infty} \prod_{s\in\overline{S}} \frac{\omega(\phi_s(b))}{\omega(\phi_s(a))} \cdot \prod_{s\in S\setminus\overline{S}} \frac{\omega(\phi_s(b))}{\omega(\phi_s(a))} = \lim_{r\to\infty} \prod_{s\in S\setminus S^o} \frac{\omega(\phi_s(b))}{\omega(\phi_s(a))} = \\ &= \prod_{s\in S\setminus\overline{S}} \frac{\lim_{r\to\infty}\omega(\phi_s(b))}{\lim_{r\to\infty}\omega(\phi_s(a))} > 0\end{aligned}$$

where we used $\phi_s(b) = \phi_s(a)$ for $s \in \overline{S}$, and $\lim_{r\to\infty}\omega(\phi_s(b)) > 0$, $\lim_{r\to\infty}\omega(\phi_s(a)) > 0$ for $s \in S\setminus\overline{S}$. Therefore, $\phi(b) \,\hat{\succsim}_\psi\, \phi(a)$, in contradiction to Lemma 19. ■

Lemma 22. *For a regular decision rule $\psi \in \Psi_0$, the number of indifference classes of the preference relation $\hat{\succsim}_\psi$ defined by eq. (17) is less than or equal to $\prod_{s\in S}\left(|\mathcal{T}^{weak}_\psi(s)|+1\right)$.*

Proof. Consider a subset $\hat{\mathbb{T}} \subseteq \mathbb{T}$ given by $\hat{\mathbb{T}} = \times_{s\in S}\left(\mathcal{T}^{weak}(s) \cup \{\varnothing\}\right)$. By Lemma 20, every element $e \in \mathbb{T}$ is indifferent to some element $e' \in \hat{\mathbb{T}}$. Since $|\hat{\mathbb{T}}| = \prod_{s\in S}\left(|\mathcal{T}^{weak}_\psi(s)|+1\right)$, the Lemma follows. ■

B Proofs for the baseline model

In this section, as well as in the rest of the Appendix, I consider a setup of the baseline model; that is, the interrogation rule is deterministic; and also $\iota : S \to N$. Thus, the agent always asks a question in a state. Next, the agent cannot draw a new alternative intentionally. Thus, the stochastic transition rule is $\tau : S^o \times \{0,1\} \to \triangle(S \cup \{\diamond\})$.

B.1 Other Combinatorics statements

Lemma 23. *For $k \in \mathbb{N}$, let functions $f^k, g^k : (\mathbb{N}\cup\{0\})^k \to \mathbb{N}$ be given by*

$$f^k(x_1,...,x_k) = \prod_{i=1}^{k}(x_i+1), \qquad g^k(x_1,...,x_k) = \sum_{i=1}^{k} x_i$$

If $x_i = 0$, $x_j > 1$ for some $i, j \in \{1,...,k\}$ then $f^k(x') \geq (4/3)\cdot f^k(x)$ and $g^k(x') = g^k(x')$, where $x'_l = x_l$ for $l \neq i, j$, $x'_i = 1$, $x'_j = x_j - 1$.

Proof. Let x, x' are as above, then $f^k(x') = f^k(x) \cdot 2 \cdot (x_j/(x_j+1)) \geq (4/3) \cdot f^k(x)$, since $(x_j/(x_j+1)) \geq 2/3$ for $x_j \geq 2$. ∎

Lemma 24. *Let the preference relation $\succeq$ have m indifference classes, and a decision rule $\psi = \{\sigma_r\}_{r=1,2,\ldots} \in \Psi$ solves choice problem $(Q, \succeq)$. Let $Q' \subseteq Q$ be a collection of questions that are asked with positive probability in some state $s \in S_\psi$. Then $|Q'| \geq \lceil \log_2(m) \rceil$.*

Proof. Pick one representative a^k from each indifference class of the preference relation $\succeq$. Towards a contradiction, assume that $|Q'| < n \equiv \lceil \log_2(m) \rceil$. Then the maximum number of different vectors of binary answers is bounded above by 2^{n-1}. Since $m > 2^{n-1}$, then by the pigeonhole principle, there is a couple of alternatives a^k, $a^{k'}$ such that $a^k_i = a^{k'}_i$ for all i in the index set of Q'. It follows that all probabilities of transitions between the states of the automaton ψ are equal for a and b. Therefore, $q_\sigma(a) = q_\sigma(b)$ and hence, by Lemma 11, ψ does not solve $(Q, \succeq)$, contradiction. ∎

Lemma 25. *Let $\succeq$ have m indifference classes and $n = \lceil \log_2(m) \rceil$. Suppose that the decision rule $\psi = \{\sigma_r\}_{r=1,2,\ldots} \in \Psi$ solves the choice problem $(Q, \succeq)$, then $\kappa(\psi) \geq 3n$.*

Proof. Assume first that a decision rule $\psi = \{\sigma_r\}_{r=1,2,\ldots} \in \Psi$ solves the choice problem $(Q, \succeq)$. By Lemma 17, there is a regular subsequence $\psi_0 \subseteq \psi$ that also solves $(Q, \succeq)$. Without loss of generality, assume $\psi_0 = \psi$. Pick one representative a^k from each indifference class of the preference relation $\succeq$. By Lemma 21, the sets of weak links corresponding to the alternatives from different indifference classes should be distinct, that is, $\mathcal{F}(a^k) \neq \mathcal{F}(a^{k'})$ for $k \neq k'$. Since $\mathcal{F}(a) \subseteq \mathcal{T}^{weak}$ for all $a \in A$, then by the pigeonhole principle, $2^{|\mathcal{T}^{weak}|} \geq m$, hence $|\mathcal{T}^{weak}| \geq \log_2(m)$, implying $|\mathcal{T}^{weak}| \geq \lceil \log_2(m) \rceil$, since $|\mathcal{T}^{weak}|$ is an integer.

Next, by Lemma 24, there should be at least $n = \lceil \log_2(m) \rceil$ questions being asked in different states of S_ψ. Each question can have one of the two answers: 1 or 0, and, conditional on receiving one of the answers, there should be at least one transition with positive in the limit probability to some other (or the same) state—otherwise transitional probabilities from the corresponding state conditional on answer received would not sum up to one. Recall that, given a regular sequence of decision rules $\psi\{\sigma_r\}_{r=1,2,\ldots} \in \Psi_0$, a link $(s, s', \iota) \in \mathcal{T}$ is strong if the corresponding transitional probability converges to non-zero number. It follows that the number of strong links is bounded above by $|\mathcal{T}^{strong}| \geq 2 \cdot n = 2 \cdot \lceil \log_2(m) \rceil$. Then

$$\kappa(\psi) = |\mathcal{T}_\psi| = |\mathcal{T}_\psi^{weak}| + |\mathcal{T}_\psi^{strong}| \geq \lceil \log_2(m) \rceil + 2 \cdot \lceil \log_2(m) \rceil = 3 \cdot \lceil \log_2(m) \rceil$$

proving that the lower bound on complexity of an arbitrary language Q is $3 \cdot \lceil \log_2(m) \rceil$. ■

B.2 Universal decision rule

In this section, I introduce a universal decision rule that serves for the proof of Lemma 2 and provides an upper bound for Theorem 2.

For an arbitrary preference relation $\succeq$ on A and language Q, adequate for $\succeq$, consider the automaton $\overline{\psi} \in \Psi$, defined via the recursive procedure $\mathcal{R}$ described below; I write $\overline{\psi} = \mathcal{R}(Q, \succeq)$ for the resulting automaton.

Recall that $C(a) = \{a' \in A | a' \sim a\}$ is the indifference class of alternative a with respect to $\succeq$. Define $m(a)$ to be the number of indifference classes C of $\succeq$ such that $a' \succeq a$ for any $a' \in C$; thus, $m(a) = 1$ for any first-best alternative, $m(b) = 2$ for any second-best alternative, etc. Take some parameter $\epsilon \in (0,1)$—we will consider $\epsilon \longrightarrow 0$ later on.

Procedure $\mathcal{R}$: define recursively by steps $r = 1, 2, \ldots$ the following sets, interrogation and stochastic transition rules, and auxiliary functions

1. For $r = 1$, define $s(r) = 1$, $\mathcal{K}^1 = \{1\}$, $S^1 = \{1\}$, $\mathcal{J}^1 = \varnothing$, $V^1 = \varnothing$, $A^1 = A$.
2. For $r \geq 1$, if $\mathcal{K}^r = \varnothing$, finish the procedure. Otherwise, define $s(r) = \min \mathcal{K}^r$, denote by $s = s(r)$ and assume that sets $\mathcal{J}^s$, V^s and $A^s \subseteq A$, such that $|A^s| > 1$ are given.

Case α: $A^s = \{a, b\}$ for some $a, b \in A$.

Let $i \in \{1, \ldots, N\}$ be the smallest index such that $a_i \neq b_i$. Define

$$
\begin{array}{lcl}
type(s) & = & \alpha \\
\iota(s) & = & i \\
\tau(s, \diamond, a_i) & = & \epsilon^{m(a)-1} \\
\tau(s, s, a_i) & = & 1 - \epsilon^{m(a)-1} \\
\tau(s, \diamond, b_i) & = & \epsilon^{m(b)-1} \\
\tau(s, s, b_i) & = & 1 - \epsilon^{m(b)-1} \\
\mathcal{K}^{r+1} & = & \mathcal{K}^r \backslash \{s\} \\
S^{r+1} & = & S^r
\end{array}
$$

and repeat the procedure for step $r + 1$.

Case β: $|A^s| > 2$, and $\exists i \in N$, $a \in A^s$ such that $a_i \neq b_i$ for all $b \in A^s \backslash \{a\}$.

Let $s' = \max S^r$, and let $i \in N$ be the smallest index such that there exists $a \in A^s$

such that $a_i \neq b_i$ for all $b \in A^s \backslash \{a\}$. Define

$$\begin{array}{lcl} type(s) & = & \beta \\ \iota(s) & = & i \\ \tau(s, \diamond, a_i) & = & \epsilon^{m(a)-1} \\ \tau(s, s, a_i) & = & 1 - \epsilon^{m(a)-1} \\ \tau(s, s'+1, 1-a_i) & = & 1 \\ \mathcal{K}^{r+1} & = & \left(\mathcal{K}^r \backslash \{s\}\right) \cup \{s'+1\} \\ S^{r+1} & = & S^r \cup \{s'+1\} \\ \mathcal{J}^{s'+1} & = & \mathcal{J}^s \cup \{(i, 1-a_i)\} \\ V^{s'+1} & = & V^s \cup \{s\} \\ A^{s'+1} & = & A^s \backslash \{a\} \end{array}$$

and repeat the procedure for step $r+1$.

Case γ: $|A^s| > 2$, and $\nexists i \in N$, $a \in A^s$ such that $a_i \neq b_i$ for all $b \in A^s \backslash \{a\}$.
Let $s' = \max S^r$, and let $i \in N$ be the smallest index such that there exists $a, b \in A^s$ such that $a_i \neq b_i$. Define

$$\begin{array}{lcl} type(s) & = & \gamma \\ \iota(s) & = & i \\ \tau(s, s'+1, 1) & = & 1 \\ \tau(s, s'+2, 0) & = & 1 \\ \mathcal{K}^{r+1} & = & \left(\mathcal{K}^r \backslash \{s\}\right) \cup \{s'+1\} \cup \{s'+2\} \\ S^{r+1} & = & S^r \cup \{s'+1\} \cup \{s'+2\} \\ \mathcal{J}^{s'+1} & = & \mathcal{J}^s \cup \{(i, 1)\} \\ \mathcal{J}^{s'+2} & = & \mathcal{J}^s \cup \{(i, 0)\} \\ V^{s'+1} & = & V^s \cup \{s\} \\ V^{s'+2} & = & V^s \cup \{s\} \\ A^{s'+1} & = & \{a \in A^s | a_i = 1\} \\ A^{s'+2} & = & \{a \in A^s | a_i = 0\} \end{array}$$

and repeat the procedure for step $r+1$.

3. If the procedure finishes at step $\overline{r}$, define $S = S^{\overline{r}}$.

Thus, the procedure $\mathcal{R}$ defines an automaton with a binary tree structure, where the vertexes are the states $s \in S$; for each s, the set V^s encodes all predecessors of s up to the root $s = 1$, the set A^s is the set of alternatives for which the automaton can reach state s, the set $\mathcal{J}$ is the set of signals that partitions A into A^s and $A \backslash A^s$. The sets $\mathcal{K}^r$ and S^r with $r < \overline{r}$ are auxiliary for the procedure: the set S^r is a set of states that have been introduced by the procedure, and the set $\mathcal{K}^r \subseteq S^r$ is the set of states such that the procedure hasn't applied one of the cases α, β or γ yet and, in particular, hasn't defined the interrogation rule g and transition rule τ. Finally, $type \in \{\alpha, \beta, \gamma\}$ stands for the type of a vertex that is defined by the corresponding condition for cases α, β, γ.

Lemma 26. *The decision rule $\psi = \{(S, g, \tau)_\epsilon\}_{\epsilon \to 0}$, where $(S, g, \tau)_\epsilon = \mathcal{R}(Q, \succeq)$ is well-defined:*
(i) If $s = s(r)$ for some step r, then $\mathcal{J}^s$, V^s, $A^s \subseteq A$ have been already defined, and $|A^s| > 1$;
(ii) The cases α, β, γ are mutually exclusive and exhaust all possibilities;
(iii) For each of the cases α, β, γ, the index i is well-defined;
(iv) The procedure finishes after a finite number of steps $\overline{r}$;
(v) The interrogation rule ι and the stochastic transition rule τ are defined for all states $s \in S$;
(vi) The state space S, the interrogation rule ι and the set of positive-probability transitions $\mathcal{T}$ do not depend on $\epsilon \in (0, 1)$.

Proof. The proofs of statements (i),(ii),(v),(vi) are straightforward, but bulky, and omitted. The statement (iii) is true since the language Q is adequate for the preference relation $\succeq$. To prove statement (iv), consider

$$h(r) \;=\; \sum_{s \in \mathcal{K}^r} 10^{|A^s|}$$

Let us show that $h(r+1) < h(r)$ if the procedure was not finished at steps r and $r+1$. Suppose Case α realized at step r, then $\mathcal{K}^{r+1} = \mathcal{K}^r \backslash \{s\}$, thus $h(r) - h(r+1) = 10^{|A^s|} > 0$. Suppose Case α realized at step r, then $\mathcal{K}^{r+1} = \left(\mathcal{K}^r \backslash \{s\}\right) \cup \{s' + 1\}$ and $A^{s'+1} = A^s \backslash \{a\}$, hence $h(r) - h(r+1) = 10^{|A^s|} - 10^{|A^s \backslash \{a\}|} > 0$. Suppose Case γ realized at step r, then $\mathcal{K}^{r+1} = \left(\mathcal{K}^r \backslash \{s\}\right) \cup \{s' + 1\} \cup \{s' + 2\}$, $A^{s'+1} = \{a \in A^s | a_i = 1\}$, $A^{s'+2} = \{a \in A^s | a_i = 0\}$. Since function h' given by $h'(x) = 10^x$ is strictly convex, then $h(r) - h(r+1) = 10^{|A^{s'+1}| + |A^{s'+2}|} - 10^{|A^{s'+1}|} - 10^{|A^{s'+2}|} > 0$.

Therefore, h is a decreasing integer-valued function of the step r. It follows that $r \leq h(1) = 10^{|A|}$ for any step r of procedure $\mathcal{R}$, proving statement (iv). ■

Lemma 27. *Let $\psi = \{\sigma_\epsilon\}_{\epsilon \to 0} = \mathcal{R}(Q, \succeq)$ be the decision rule, constructed via procedure $\mathcal{R}$. Then:*

(i) ψ solves the choice problem $(Q, \succeq)$.

(ii) $\kappa(\psi) = 3|A| - 2 - x$, *where* $x = |\{a \in A | a \succeq a' \; \forall a' \in A\}|$;

(iii) $|S_\psi| = |A| - 1$.

Proof. First, let us prove statement (i). Note that for an arbitrary alternative $b \in A$, there is a unique valid simple path $l = l^*(b)$; indeed, for each memory state of the automaton that is reached with positive probability while investigating $a \in A$, there is only one link to the other state. Next, this path goes via states for which the procedure $\mathcal{R}$ applies either case γ or case α for some $a \neq b$; in this cases, the probability of a transition to the next state in the path is 1. Finally, finally $l^*(b)$ reaches a state s' such that either case β applies, or case α applies with $a = b$. In both cases, the last link in the path $l^*(b)$ has probability $\tau_{s'}(\diamond, g(s'), b_{g(s')}) = \epsilon^{m(b)-1}$. Thus, for all $b \in A$,

$$\omega^*(b) \;=\; \epsilon^{m(b)-1}$$

By Lemma 16, the decision rule $\psi = \mathcal{R}(Q, \succeq)$ solves the choice problem $(Q, \succeq)$, proving statement (i).

Second, let us prove statement (ii). Note that $type(s) = \alpha$ if and only if there are no $s' \in S$ such that $s \in V^{s'}$; thus, states where case α applies are leafs of the binary tree of transitions between the flexible memory states, and only such states are leafs. For a state $s \in S$, denote by $\xi(s)$ the maximum length of the path to the state $\diamond$; in other words, define $\xi(s)$ recursively as follows:

(1) $\xi(s) = 1$ for all s such that $type(s) = \alpha$;

(2) if $type(s) = \beta$ or $type(s) = \gamma$, then $\xi(s) = \max_{v:\, s \in V^v} \xi(s) + 1$.

Next, for $s \in S_\psi$, define

$$D(s) \;=\; \bigcup_{s':\, s \in V(s)} |\mathcal{T}_\psi(s)|, \qquad \overline{k}(B) \;=\; \{a \in B \mid m(a) = 1\}$$

Thus, $D(s)$ is the number of transitions outgoing from the states of the branch of the tree that starts at state s, and $\overline{k}(A^s)$ is the number of first-best alternatives that reach this branch of the tree.

Claim 11. *If* $D(s)$ *and* $\overline{k}(A^s)$ *are as defined above, then:*

$$D(s) = 3|A^s| - 2 - \overline{k}(A^s)$$

Proof of the Claim. First, note that the Claim holds for all s such that $\xi(s) = 1$. Indeed, if $\theta(s) = \alpha$, then $D(s) = 4$ if $m(a), m(b) > 1$, $D(s) = 3$ if $m(a) = 1$, $m(b) > 1$ or if $m(a) > 1$,

$m(b) = 1$, and $D(s) = 2$ is $m(a) = m(b) = 1$. Next, assume that the Claim holds for all s such that $\xi(s) \le \overline{\xi}$, and consider arbitrary state with $\xi(s) = \overline{\xi} + 1$.

Consider the case when $type(s) = \beta$, then $|\mathcal{T}(s)| = 3$ if $m(a) > 1$ and $|\mathcal{T}(s)| = 2$ if $m(a) = 1$. Therefore,

$$D(s) \;=\; 3 - \overline{k}(\{a\}) + D(v) \;=\; 3 - \overline{k}(\{a\}) + 3|A^v| - 2 - \overline{k}(A^v) \;=\; 3|A^s| - 2 - \overline{k}(A^s)$$

where $v = s' + 1$, and $s' = \max S^r$ is the state defined according to the case β by procedure $\mathcal{R}$; i.e. $s'+1$ is the successor of s such that $\tau_s(s'+1, g(s), 1-a_i) = 1$. In the above equation, we used $A^v = A^{s'+1} = A^s \backslash \{a\}$ and the fact that $\xi(s) = \xi(v)+1$, hence $\xi(v) = \overline{\xi}$ and the induction assumption applies.

Consider the case when $type(s) = \gamma$, then $|\mathcal{T}(s)| = 2$, and

$$D(s) \;=\; 2+D(s'+1)+D(s'+2) = 2+3|A^{s'+1}|-2+\overline{k}(S^{s'+1})+3|A^{s'+2}|-2+\overline{k}(S^{s'+2}) \;=\; 3|A^s|-2-\overline{k}(A^s)$$

where $s' = \max S^r$ is the state defined according to the case γ by procedure R, and states $s' + 1$ and $s' + 2$ are the successors of the state s. In the above equation, we used the fact that $\{S^{s'+1}, S^{s'+2}\}$ is a partition of S^s, and the fact that $\xi(s) = \max\{\xi(s' + 1), \xi(s' + 2)\} + 1$, hence $\xi(s' + 1), \xi(s' + 2) \le \overline{\xi}$ and the induction assumption applies. □

Statement (ii) of the Lemma follows from Claim 11 applied to the root of the tree: $\kappa(\psi) = D(1) = 3|A| - 2 - \overline{k}(A)$.

Finally, let us prove statement (iii). Define

$$E(s) \;=\; |\{s' : \; s \in V^{s'}\}| + 1$$

Thus, $E(s)$ is the number of successors of the state s in a tree plus 1.

Claim 12. *If $E(s)$ is as defined above, then $E(s) = |A^s| - 1$*

Assume $\xi(s) = 1$, then $E(s) = 1$. Suppose now that the Claim is true for all s such that $\xi(s) \le \overline{\xi}$. If $type(s) = \beta$, then

$$E(s) \;=\; 1 + E(s' + 1) \;=\; 1 + |A^{s'+1}| - 1 \;=\; |A^s| - 1$$

where state $s'+1$ is a unique successor of the state s defined by case β of the procedure $\mathcal{R}$, and we used the induction assumption for $s' + 1$. If $type(s) = \gamma$, then

$$E(s) \;=\; 1 + E(s' + 1) + E(s' + 2) \;=\; 1 + |S^{s'+1}| - 1 + |S^{s'+2}| - 1 \;=\; |A^s| - 1$$

where states $s'+1$ and $s'+2$ are successors of the state s defined by case γ of the procedure $\mathcal{R}$, and we used the induction assumption for $E(s'+1)$ and $E(s'+2)$. $\square$

Statement (iii) of the Lemma follows from Claim 11 applied to the root of the tree: $|S_\psi| = E(1) = |A| - 1$. $\blacksquare$

B.3 Additive languages and automata

I say that language Q with the index set of questions N is *additive* with respect to preference relation $\succeq$ if there exists a vector $\lambda \in R^N$ such that $\lambda_i \neq 0$ for all $i \in N$, and

$$a \succeq b \iff v(a) \geq v(b), \qquad \text{where } v(a) = \sum_{i \in N} \lambda_i a_i \text{ for } a \in A$$

For an additive language Q, let $\psi^+(Q, \succeq) = \{\sigma_\epsilon\}_{\epsilon \to 0} \in \Psi$ be the decision rule, defined as follows: $\sigma_\epsilon = (S, \iota, \tau_\epsilon)$, where $|S| = |N|$, $\iota(i) = i$, and the stochastic transition rule τ is

$$\begin{cases} \tau(s, s+1, 1) = 1, & \tau(s, s+1, 0) = \epsilon^{\lambda_s}, & \tau(s, s, 0) = 1 - \epsilon^{\lambda_s} & \text{if } \lambda_s > 0 \\ \tau(s, s+1, 0) = 1, & \tau(s, s+1, 1) = \epsilon^{-\lambda_s}, & \tau(s, s, 1) = 1 - \epsilon^{-\lambda_s} & \text{if } \lambda_s < 0 \\ \text{where} & \tau(|S|, |S|+1, \cdot) \equiv (|S|, \diamond, \cdot) & & \end{cases}$$

The set of positive-probability transitions of $\psi^+(Q, \succeq)$ is

$$\mathcal{T} = \Big\{(s, s+1, 1)\Big\} \cup \Big\{(s, s+1, 0)\Big\} \cup \Big\{(s, s, 1) | \lambda_s < 0\Big\} \cup \Big\{(s, s, 0) | \lambda_s > 0\Big\}$$

where $(|S|, |S|+1, \cdot) \equiv (|S|, \diamond, \cdot)$. Clearly, $\mathcal{T}$ does not depend on ϵ, so the decision rule $\psi^+(Q, \succeq) \in \Psi$ is well-defined.

Lemma 28. *Let Q be an additive language for preference relation Q, and $\psi^+(Q, \succeq)$ be the decision rule, described above, then:*
(i) $\psi^+(Q, \succeq)$ solves the choice problem $(Q, \succeq)$;
(ii) The complexity of the decision rule $\psi^+(Q, \succeq)$ is $3|N|$, i.e.

$$\kappa(\psi^+(Q, \succeq)) = 3|N|$$

Proof. For the decision rule ψ^+, a simple path l for any alternative $a \in A$ should go via states $1, 2, \ldots, |S|, \diamond$ consequently, and also, for every state s and any alternative $a \in A$, there is only one link outgoing from state s that can be a part of the simple path, namely, link

$(s, s+1, a_{\iota(s)})$. It follows that for any alternative $a \in A$, the set of simple paths is a singleton: $\mathcal{L}_0(a, \sigma_\epsilon) = \{l^*(a)\}$, where $l^*(a) = (l_1^*(a), ..., l_{|S|}^*(a))$, $l_s^*(a) = (s, s+1, a_{\iota(s)})$ with the convention that $(|S|, |S|+1, a_{\iota(s)}) = (|S|, \diamond, a_{\iota(s)})$. Therefore,

$$(1-\eta)^{1-|S|}\omega^*(a) = \prod_{s \in S:} \omega((s, s+1, a_{\iota(s)})) =$$

$$= \left(\prod_{s \in S:\ \lambda_{g(s)} > 0,\ a_i = 0} \epsilon^{\lambda_{\iota(s)}} \right) \cdot \left(\prod_{s \in S:\ \lambda_{g(s)} < 0,\ a_i = 1} \epsilon^{-\lambda_{\iota(s)}} \right) = \left(\prod_{s \in S} \epsilon^{\lambda_{\iota(s)} \cdot (1 - a_{\iota(s)})} \right) \cdot \left(\prod_{s \in S:\ \lambda_{\iota(s)} < 0} \epsilon^{-\lambda_{\iota(s)}} \right)$$

Thus, $\omega^*(a) > 0$ for all $a \in A$. Moreover, if $a \succ b$, then $v(a) > v(b)$, and

$$\omega^*(b)/\omega^*(a) = \prod_{s \in S} \epsilon^{\lambda_{\iota(s)} \cdot (a_{\iota(s)} - b_{\iota(s)})} = \epsilon^{v(a) - v(b)} \longrightarrow 0$$

By Lemma 16, the decision rule $\psi^+(Q, \succeq)$ solves the choice problem $(Q, \succeq)$, proving statement (i).

Statement (ii) follows from

$$\mathcal{T}_{\psi^+(Q,\succeq)} = \Big\{(s, s+1, 1)\Big\} \cup \Big\{(s, s+1, 0)\Big\} \cup \Big\{(s, s, 1)\Big| \lambda_{\iota(s)} < 0\Big\} \cup \Big\{(s, s, 0)\Big| \lambda_{\iota(s)} > 0\Big\}$$

where $(|S|, |S|+1, \cdot) \equiv (|S|, \diamond, \cdot)$. ∎

For a given preference relation $\succeq$ and language Q with $|N| \geq n = \lceil \log_2(m) \rceil$ questions, I denote by Ψ_n^{++} the set of decision rules $\psi = \sigma_\epsilon$, where $\epsilon = (\epsilon_1, ..., \epsilon_n) \in \mathbb{R}_{++}^n$ with $\epsilon_i \longrightarrow 0$ for all $i \in \{1, ..., n\}$, that have the following properties:

(i) The state space is $S = \{1, ..., n\}$;

(ii) The interrogation rule $\iota : S \to N$ is an injection;

(iii) There is $x = (x_1, ..., x_n) \in \{0, 1\}^n$ and permutation $h : \{1, ..., n\} \to \{1, ..., n\}$ such that $h(1) = 1$, and the transitional probabilities satisfy the following properties for all $s \in S$:

  (a) $\tau(s, h^{-1}(h(s)+1), x_s) = 1$, where $h^{-1}(n+1)$ denotes $\diamond$;

  (b) $\tau(s, h^{-1}(h(s)+1), 1 - x_s) = \epsilon_s$, where $h^{-1}(n+1)$ denotes $\diamond$;

  (c) There exists a unique s' such that $h(s') \leq h(s)$ such that $\tau(s, s', 1 - x_s) = 1 - \epsilon_s$.

Note that ψ is well-defined, since transitional probabilities conditional on any event (s, ι) that may occur sum up to one. Note also that there are no other positive-probability transitions of ψ except those described in (a),(b),(c).

Note also that $\Psi_n^{++} = \Psi_n^+$, where Ψ_n^+ the set of decision rules, defined in Section 4 of the main part of the paper.

Lemma 29. *Let the preference relation $\succeq$ have m indifference classes, and $n = \lceil \log_2(m) \rceil$. Then for any $\psi \in \Psi_n^{++}$, $\kappa(\psi) = 3n$*

Proof. In each of the n states, there are exactly 3 transitions outgoing: $(s, h^{-1}(h(s)+1), x_s)$, $(s, h^{-1}(h(s)+1), 1-x_s)$ and $(s, s', 1-x_s)$. Hence, $\kappa(\psi) = 3n$. ∎

Lemma 30. *Let $\psi \in \Psi_n^{++}$. Then:*
(i) For an arbitrary alternative $a \in A$ for arbitrary valid simple path $l \in \mathcal{L}_0(a)$, l goes consequently via states $h(1) = 1$, $h(2)$,...,$h(n)$, $\diamond$;
(ii) For $a \in A$, any simple path that consists of positive-probability links starting from a state s such that $s = h(k)$ and ending in state $\diamond$, consequently goes via states $h(k), h(k+1), \ldots, h(n), \diamond$; moreover, the total number of different simple paths with the described properties is 2^{n-k+1}.

Proof. A valid simple path should start at $s = 1$. The only two positive-probability transitions outgoing from state 1 and ending not in s, end at state $s' = h(2)$, hence one of these to links should be l_1. Recursively, there are exactly 2 positive-probability transitions outgoing from state $s = h(k)$ that end at state s' such that $h^{-1}(s') > k$, and these transitions end at state $s' = h(k+1)$, where $h(n+1)$ denotes $\diamond$. The third transition outgoing from state $s = h(s)$ goes via one of the already visited states $\{s \in S | h^{-1}(s) \le k\}$ and, hence, cannot be a part of a simple path. Therefore, one of these two transitions should be a link l_k of $l \in \mathcal{L}_0$.

The same argument works for a simple path that starts from an arbitrary state $s \in S$. Note that in state s, there are exactly two possible links that can be used in such simple path, and a path goes consequently via $h(k), h(k+1), \ldots, h(n), \diamond$, hence the total number of such paths is 2^{n-k+1}. ∎

The next lemma facilitates the proof of Theorem 3. Let us first introduce the following notation:

$$\tilde{\mathbb{N}} \;=\; \left\{ m \in \mathbb{N} \backslash \{1\} \;\middle|\; \exists n \in \mathbb{N} : \; (3/4) \cdot 2^n < m \le 2^n \right\}$$

Thus, $\tilde{\mathbb{N}} = \{2, 4, 7, 8, 13, 14, 15, 16, \ldots\}$ is the set of natural numbers that represents the condition on m given in statement (ii) of Theorem 3.

Lemma 31. *Let the number of indifference classes of preference relation $\succeq$ be $m \in \tilde{\mathbb{N}}$, and $n = \lceil \log_2(m) \rceil$. If a regular decision rule $\psi = \{\sigma_r\}_{r=1,2,\dots} \in \Psi_0$ solves the choice problem $(Q, \succeq)$, and $\kappa(\psi) = 3n$, then $\psi \in \Psi^{++}$.*

Proof. First, let's prove the following claims.

Claim 13. *If a regular decision rule $\psi = \{\sigma_r\}_{r=1,2,\dots} \in \Psi_0$ solves the choice problem $(Q, \succeq)$ above, and $\kappa(\psi) = 3n$, then:*
(1) The state space of ψ is $S_\psi = \{1, \dots, n\}$;
(2) The interrogation rule $\iota : S \to N$ is an injection;
(3) There are exactly n weak links, and from each state there is exactly one outgoing weak link.

Proof of the Claim. First, note that by Lemma 24, language Q should contain at least n questions. Each question is associated with a state at which it is asked and with at least 2 strong links. Hence, the number of strong links is at least $2n$. By lemma 21 and the pigeonhole principle, the number of weak links should satisfy $2^{|\mathcal{T}^{weak}|} \geq m$, hence $|\mathcal{T}^{weak}| \geq n$. Since $\kappa(\psi) = 3n$, the number of weak links should be exactly n, and the number of strong links should be exactly $2n$. Since n state are associated with $3n$ links, and $\kappa(\psi) = 3n$, then $S_\psi = \{1, \dots, n\}$; since there are n difference questions asked, then $g : S \to N$ is an injection. This proves statements (1) and (2).

Next, Let x_s be the number of weak links outgoing from state s. By Lemmas 20 and 19, for ψ to solve $(Q, \succeq)$, it should be that $f^n(x) \geq m$, where $f^n : (\mathbb{N} \cup \{0\})^n \to \mathbb{N}$ is given by $f^n(x) = \prod_{s=1}^n (x_s + 1)$.

Towards a contradiction, assume that $x_s \neq 1$ for some s. Since $\sum_{s=1}^n x_i = n$, then there are s', s'' such that $x_{s'} = 0$ and $x_{s''} > 1$. Consider $z \in (\mathbb{N} \cup \{0\})^n$ given by $z_l = x_l$ for $l \neq s', s''$, $z_{s'} = 1$, and $z_{s''} = x_{s''} - 1$. By Lemma 23, $f^n(z) \geq (3/4) \cdot f^n(x)$. Note that the number of states such that $z_s = 0$ is the number of states such that $x_s = 0$ minus one. Keep replacing z by $\hat{z} \in (\mathbb{N} \cup \{0\})^n$ in the manner described above, until $z_s > 0$ for all s—this process stops in no more than n steps. Note that the result of this procedure is $z = (1, \dots, 1)$. Let k be the number of steps of the described above procedure that is needed to transform x into $(1, \dots, 1)$. Then $f^n(1, \dots, 1) \geq (4/3)^k \cdot f^n(x) \geq (4/3) \cdot f^n(x)$, since $k > 0$ for $x \neq (1, \dots, 1)$. Then,

$$f^n(x) \;\leq\; (3/4) \cdot f^n(1, .., 1) \;=\; (3/4) \cdot 2^n \;<\; m$$

in contradiction to $f^n(x) \geq m$, proving statement (3) of the Claim is proven. □

Claim 14. *Suppose $\psi = \{\sigma_r\}_{r \in \mathbb{N}} \in \Psi_0$ solves the choice problem $(Q, \succeq)$. Define the decision rule $\psi' = \{\sigma'_r\}_{r \in \mathbb{N}}$ as follows:*

1. *$S_{\psi'} = S_\psi$ and $\iota_{\psi'} = \iota_\psi$;*
2. *If $s = 1$ or $s' \neq 1, s$, then $\tau_{\psi'}(s,s',j) = \tau_\psi(s,s',j)$;*
3. *$\tau_{\psi'}(s,1,j) = 0$ for $s \neq 1$;*
4. *If $s \neq 1$, then $\tau_{\psi'}(s,s,j) = \tau_\psi(s,1,j) + \tau_\psi(s,s,j)$ for all $s \neq 1$.*

Then the decision rule ψ' solves the choice problem $(Q,\succeq)$, and $\kappa(\psi') \leq \kappa(\psi)$.

Proof of the Claim. Since links (s,s,j) and $(s,1,j)$ cannot appear in the simple path $l^*(a)$ for any alternative a, then $\omega^*_{\psi'}(a) = \omega^*_\psi(a)$ for all $a \in A$. Since ψ solves $(Q,\succeq)$, then by Lemma 16, ψ' solves $(Q,\succeq)$ as well. Next, if $(s,s,j) \in \mathcal{T}_{\psi'} \backslash \mathcal{T}_\psi$, then $(s,1,j) \in \mathcal{T}_\psi \backslash \mathcal{T}_{\psi}$. It follows that $|\mathcal{T}_{\psi'}| \leq |\mathcal{T}_\psi|$, proving the Claim. □

Consider $m = 2$, then $n = \lceil \log_2(m) \rceil = 1$. Suppose ψ solves $(Q,\succeq)$, and $\kappa(\psi) = 3n$. By Lemma 21, ψ should have at least one weak link, and by lemma 24, at least one question i should be asked, thus there are at least two strong links, associated with this question at a state s. It follows that ψ should have one state $s = 1$ with $g(1) = i$ for some $i \in N$, one transition—strong link—corresponding to some answer $Q_i(a) = x_1$, i.e. link $(1,\diamond,x_1)$, and two transitions—strong and weak links—corresponding to the opposite answer $Q_i(a) = 1 - x_1$. If link $(1,\diamond,1-x_1)$ is strong, then $q(a) > 0$ in the limit for all alternatives, and by Lemma 11, ψ does not solve $(Q,\succeq)$, in contradiction. Hence, link $(1,\diamond,1-x_1)$ is strong, and $(1,1,1-x_1)$ is weak. Denote by $\epsilon_1 = \tau(1,\diamond,1-x_1)$, then $\psi \in \Psi^{++}$.

Suppose that the statement of the Lemma holds for all $m_0 \in \mathbb{N}$ such that $\lceil \log_2(m_0) \rceil \leq n_0$, where $n_0 \geq 1$ and consider arbitrary $m \in \mathbb{N}$ such that $n_0 < \lceil \log_2(m) \rceil \leq n_0 + 1$. Consider state $s = 1$. By Claim 13, there is exactly one weak link, outgoing from this state. Without loss of generality, this link is $v_1 = (1,\hat{s},1-x_1)$ for some $\hat{s} \in S\backslash\{1\}$ and $x_1 \in \{0,1\}$—recall that a weak link cannot end at the same state as it begins. Since there are two strong links associated with each state $s \in S$, then the other two outgoing links from $s = 1$ are $v_2 = (1,s',1-x_1)$ and $v_3 = (1,s'',x_1)$ for some $s', s'' \in S$.

If $s'' = 1$, then an alternative with $a_{g(1)} = x_1$ is not chosen from a singleton menu, consisting of this alternative—note that by the definition of a language, there should be at least one such alternative, since the partition $\{Q_{g(1)x_1}, Q_{g(1)1-x_1}\}$ of A is non-trivial. Hence, $s'' \neq 1$.

Towards a contradiction, assume that $s'' = \diamond$, then for all alternatives $a \in A$ such that $a_{g(1)} = x_1$ we have $q(a) = 1$ and $\omega(\phi(a)) = 1$. Consider $\hat{\mathbb{T}} \subseteq \mathbb{T}$ given by $\hat{\mathbb{T}} = \Big\{(1,\hat{s},1-x_1)\Big\} \times \Big[\times_{s>1}\left(\mathcal{T}^{weak} \cup \{\varnothing\}\right)\Big]$. Since $a_{\iota(1)} = 1 - x_1$ implies $(\phi(a))_1 = (1,\hat{s},1-x_1)$, then by Lemma

20, for all $a \in A$ such that $a_{g(1)} = 1 - x_1$ there exists $e \in \hat{\mathbb{T}}$ such that $\phi(a) \stackrel{\sim}{\succ}_\psi e$. Thus, $\phi(A) = \left\{e \in \mathbb{T} \middle| \exists a \in A: \; e = \phi(a)\right\}$ contains at most $1+\prod_{s>1}\left(|\mathcal{T}^{weak}|+1\right) = 1+2^{n-1}$ indifference classes, then by Lemma 19, $\succeq$ cannot contain more than $1 + 2^{n-1}$ indifference classes. Thus, $m \leq 1 + 2^{n-1} \leq (3/4) \cdot 2^n < m$ in contradiction, where we used $1 + 2^{n-1} \leq (3/4) \cdot 2^n$ for $n = n_0 + 1 > 1$. Hence, $s'' \neq \varnothing$.

Claim 15. *If $(1, s', 1 - x_1)$ is a strong link described about, then $s' = 1$.*

Proof of the Claim. Towards a contradiction, assume that $s' \neq 1$. If $s' = \diamond$, then for all $a \in A$ such that $a_{\iota(1)} = 1 - x_1$, we have $\omega(\phi(a)) = 1$. Then, we can repeat the argument above with $\hat{\mathbb{T}} = \left\{(1, s'', x_1)\right\} \times \left[\times_{s>1}\left(\mathcal{T}^{weak} \cup \{\varnothing\}\right)\right]$ and get a contradiction $m \leq 1 + 2^{n-1} \leq (3/4) \cdot 2^n < m$. Thus, $s' \in \overline{S} \backslash \{1, \diamond\}$.

Let v_4 be a weak link, outgoing from state s'; that is, $\mathcal{T}^{weak} = \{v_4\}$. Towards a contradiction, assume that there is $a \in A$ such that the path $l^*(a)$ contains both a weak link $v_1 = (1, \hat{s}, 1 - x_1)$ and a weak link v_4. Thus, $l^*(a) = v_1 l^1 v_4 l^2$ for some (possibly, empty) paths l^1, l^2. Note that an alternative path $\tilde{l} \in \mathcal{L}(a)$ is $v_2 v_4 l^2$. Then

$$\frac{\omega(l^*(a))}{\omega(\tilde{l})} \leq \frac{\omega(v_1)}{\omega(v_2)} \longrightarrow 0$$

since v_2 is a strong link, and v_1 is a weak link. Hence, $\omega(\tilde{l}) > \omega(l^*(a))$, contradicting the definition of $l^*(a)$. Thus, for all alternatives $a \in A$, both weak links v_1 and v_4 are never in use in a path $l^*(a)$. Let $\hat{\mathbb{T}} \subset \mathbb{T}$ be the following set:

$$\hat{\mathbb{T}} = \left\{e \in \mathbb{T}' \;\middle|\; (e_1, e_{s'}) \neq (v_1, v_4)\right\}, \quad \text{where } \; \mathbb{T}' = \times_{s \in S}\left(\mathcal{T}^{weak}(s) \cup \{\varnothing\}\right)$$

By Lemma 20 and the proven fact that v_1, v_4 are never used together in $l^*(a)$, for any $a \in A$, there is $e \in \hat{\mathbb{T}}$ such that $\phi(a) \stackrel{\sim}{\succ}_\psi e$. Note that $|\hat{\mathbb{T}}| = 3 \cdot 2^{n-2}$, since only 3 out of four combinations $\{v_1, \varnothing\} \times \{v_4, \varnothing\}$ are possible, and $|\mathcal{T}^{weak}(s)| = 1$ for all s. By Lemma 19, $(3/4) \cdot 2^n = 3 \cdot 2^{n-2} \geq m$, contradicting $(3/4) \cdot 2^n < m$. The Claim is proven. □

Thus, the 3 links outgoing from state 1 are: a weak link $v_1 = (1, \hat{s}, 1 - x_1)$ with transitional probability $\tau(s, \hat{s}, 1 - x_1) \equiv \epsilon_1 \longrightarrow 0$, a strong link $v_2 = (1, 1, 1 - x_1)$ with transitional probability $\tau(1, 1, 1 - x_1) = 1 - \epsilon_1 \longrightarrow 1$, and a strong link $v_3 = (1, s'', x_1)$ with transitional probability $\tau(1, s'', x_1) = 1$.

Let $A_1 = \{a \in A | a_{\iota(1)} = x_1\}$, and $A_2 = \{a \in A | a_{\iota(1)} = 1 - x_1\}$, then (A_1, A_2) is a partition of A. Let m_1 be the number of indifference classes of $\succeq$ on A_1, and m_2 be the number of indifference classes of $\succeq$ on A_2, then $m_1 + m_2 \geq m$. It follows that there is $i \in \{1,2\}$ such that $m_i > (3/4) \cdot 2^{n-1}$.

Suppose $m_2 > (3/4) \cdot 2^{n-1}$. Let ψ' be the decision rule constructed according to Claim 14, then it has the same set of links, outgoing from state 1 such that ψ', it solves $(Q,\succeq)$, and $\kappa(\psi') \leq \kappa(\psi) = 3n$. By Lemma 25, $\kappa(\psi') \geq 3n$, hence $\kappa(\psi') = 3n$. Moreover, by Lemma 17, there is a regular decision rule ψ'' that consists of a subsequence of automatons of ψ' and solves the choice problem $(Q,\succeq)$. Without loss of generality, $\psi'' = \psi'$.

Consider the following auxiliary choice problem: the set of alternatives is A_2, the preference relation is $\succeq$, restricted on A_2, and the language is Q. Construct the decision rule $\hat{\psi}$ as follows. Let $\chi : \{2,...,n\} \to \{1,...,n-1\}$ be the following bijection (recall that state $\hat{s}$ is the state at which the weak link beginning from state 1 in the decision rules ψ and ψ' ends): $\chi(\hat{s}) = 1$, $\chi(s) = s$ for all $s < \hat{s}$, and $\chi(s) = s-1$ for all $s > \hat{s}$. Then, the flexible state space of $\hat{\psi}$ is $\hat{S} = \{1,...,n-1\}$, the interrogation function is $\hat{\iota}(s) = \iota(h^{-1}(s))$ for $s \in \{1,...,n-1\}$, and the transitional probabilities are $\hat{\tau}(s,s',j) = \tau'(\chi^{-1}(s),\chi^{-1}(s'),j)$, where τ' are the transitional probabilities of the decision rule ψ', described above.

Claim 16. *The decision rule $\hat{\psi}$ solves the auxiliary choice problem with the set of alternatives A_2, preference relation $\succeq$ and language Q.*

Proof of the Claim. For an arbitrary alternative $a \in A_2$, let $\hat{\mathcal{L}}_0(a)$ be the set of simple paths defined with respect to the decision rule $\hat{\psi}$, and $\mathcal{L}_0(a)$ be the set of simple paths defined with respect to the decision rule ψ'. Note that for any $a \in A_2$ for any $l \in \mathcal{L}_0$, $l_1 = v_1 = (1,\hat{s},1-x_1)$. Then $\chi : \{2,...,n\} \to \{1,...,n-1\}$ defined above induces a natural a bijection between $\mathcal{L}_0(a)$ and $\hat{\mathcal{L}}_{0(a)}$: for $l = v_1 v_{j_1} ... v_{j_k}$, $\chi(l) = \chi(v_{j_1})...\chi(v_{j_k})$, where $\chi(s,s',j) = (\chi(s),\chi(s'),j)$. Note that

$$(1-\eta)\cdot\tau(1,\hat{s},1-x_1)\cdot\omega(\chi(l)) \; = \; \omega(l)$$

Denote by $\hat{\omega}^*(a) = \max_{\hat{l}\in\hat{\mathcal{L}}_0}(\omega(\hat{l}))$. Then for all $a \in A_2$ we have

$$\hat{\omega}^*(a) \; = \; (\omega^*(a))/((1-\eta)\cdot\tau(1,\hat{s},1-x_1))$$

Since ψ' solves the original choice problem, then by Lemma 16, $\omega^*(a) > 0$ for all $a \in A_2$, and for all $a,b \in A_2$ such that $a \succ b$, $(\omega^*(b))/(\omega^*(a)) \to 0$. It follows that $\hat{\omega}^*$ has the same properties, thus by Lemma 16, $\hat{\psi}$ solves the auxiliary choice problem. $\square$

Note that $\kappa(\hat{\psi}) = 3n-3$, since it has the same (after re-numeration of states via mapping χ) set of links as ψ', except of the 3 links, outgoing from the state $s = 1$ in ψ'. Since $\hat{\psi}$

solves the auxiliary choice problem, by Lemma 25, it should be that $m_2 \le 2^{n-1}$. Thus, $(3/4) \cdot 2^{n-1} < m_2 \le 2^{n-1}$, and we can apply the induction assumption to show that $\hat{\psi} \in \Psi^{++}_{n-1}$. In particular, there is $\hat{x} = (\hat{x}_1, ..., \hat{x}_{n-1}) \in \{0,1\}$, $(\hat{\epsilon}_1, ...\hat{\epsilon}_{n-1}) \in (0,1)^{n-1}$ with $\hat{\epsilon}_i \to 0$, and a permutation $\hat{h} : \{1, ..., n-1\} \to \{1, ..., n-1\}$ such that $\hat{h}(1) = 1$, and

(a) For all $s \in \{1, ..., n-1\}$, $\hat{\tau}(s, \hat{h}^{-1}(\hat{h}(s)+1), \hat{x}_s) = 1$, where $\hat{h}^{-1}(n+1)$ denotes $\diamond$;

(b) For all $s \in \{1, ..., n-1\}$, $\hat{\tau}(s, \hat{h}^{-1}(\hat{h}(s)+1), 1-\hat{x}_s) = \hat{\epsilon}_s$, where $\hat{h}^{-1}(n+1)$ denotes $\diamond$;

(c) For all $s \in \{1, ..., n-1\}$, there exists a unique $s''' \in \{1, ..., n-1\}$ such that $\hat{h}(s''') \le \hat{h}(s)$ such that $\tau(s, s''', 1-\hat{x}_s) = 1 - \hat{\epsilon}_s$.

where $\hat{\tau}$ are the transitional probabilities of $\hat{\psi}$.

Claim 17. *Let $v_1 = (1, s', 1-x_1)$, $v_3 = (1, s'', x_1)$ be links of ψ defined above. Then $s' = s''$.*

Proof of the Claim. Let $\hat{x}$, $\hat{\epsilon}$, $\hat{h}$ are defined above. Towards a contradiction, assume that $s'' \ne s'$, then $k = \hat{h}(\chi(s'')) \ne \hat{h}(\chi(s')) = 1$, hence $k > 1$. Lemma 30 applied for the decision rule $\hat{\psi} \in \Psi^{++}_{n-1}$ implies that, when we consider a decision rule ψ', a valid simple path $l \in \mathcal{L}_0(a)$ for arbitrary alternative $a \in A_1$ should go consequently via states 1, $\chi(\hat{h}(k))$, $\chi(\hat{h}(k+1)), ..., \chi(\hat{h}(n)), \diamond$; moreover, the total number of different simple paths that can be used by all alternatives $a \in A_1$ is $2^{n-1+k+1}$. It follows that

$$\left|\left\{z \in [0,1] \;\middle|\; \exists a \in A_1 : \omega^*(a) = z\right\}\right| \le 2^{n-1+k+1} \le 2^{n-2}$$

where we used $k > 1$, because $s'' \ne s'$. Since ψ' solves the original choice problem, By Lemma 16, A_1 should contain no more than 2^{n-2} indifference classes, that is, $m_1 \le 2^{n-2}$. Since $m_2 \le 2^{n-1}$, it follows that $m \le m_1 + m_2 \le 2^{n-2} + 2^n \le (3/4) \cdot 2^n$, in contradiction. □

Thus, $s' = s''$. Define a permutation h as follows: $h(1) = 1$, $h(s) = \hat{h}(\chi(s)) + 1$ for $s \in \{2, ..., n-1\}$. Note that $h^{-1}(1) = 1$, and $h^{-1}(k) = \chi^{-1}(\hat{h}(k-1))$ for $k \in \{2, ..., n\}$. Next, x_1 and ϵ_1 has already been defined. Define $x_2, ..., x_n$ by $x_s = \hat{x}_{\chi(s)}$, and $\epsilon_2, ..., \epsilon_n$ by $\epsilon_s = \hat{\epsilon}_{\chi(s)}$ for all $s \in \{2, ..., n\}$.

Claim 18. *The decision rule ψ' with h, x, ϵ defined as above, satisfy conditions (i),(ii),(iii)(a,b,c) in the definition of $\psi \in \Psi^{++}_n$.*

Proof of the Claim. We already know that $S_{\psi'} = \{1, ..., n\}$ and $g_{\psi'}$ is an injection. Next, let us check (iii). For $s = 1$, we get $h^{-1}(h(1)+1) = h^{-1}(2) = \chi^{-1}(\hat{h}^{-1}(1)) = \chi^{-1}(1) = s' = s''$. The set of links, outgoing from $s = 1$ is $v_1 = (1, 2, 1-x_1)$ with probability $\tau(1, 2, 1-x_1) = \epsilon_1$,

$v_2 = (1,1,1-x_1)$ with probability $\tau(1,1,1-x_1) = 1-\epsilon_1$ and $v_3 = (1,2,x_1)$ with probability $\tau(1,2,x_1) = 1$. Hence, properties (iii)(a,b,c) hold for state $s = 1$.

Third, let $s > 1$, then, by the definition of the decision rule $\hat{\psi}$, the transitional probabilities of the decision rule ψ', outgoing from state s, are:

(a) $1 = \tau\Big(s, \chi^{-1}[\hat{h}^{-1}(\hat{h}(\chi(s))+1)], \hat{x}_{\chi(s)}\Big) = \tau\Big(s, h^{-1}(h(s)+1), x_s\Big)$;

(b) $\epsilon_s = \hat{\epsilon}_{\chi(s)} = \tau\Big(s, \chi^{-1}[\hat{h}^{-1}(\hat{h}(\chi(s))+1)], 1-\hat{x}_{\chi(s)}\Big) = \tau\Big(s, h^{-1}(h(s)+1), 1-x_s\Big)$;

(c) $1-\epsilon_s = 1-\hat{\epsilon}_{\chi(s)} = \tau\Big(s, \chi^{-1}[\hat{h}^{-1}(\hat{h}(\chi(s)))], 1-\hat{x}_{\chi(s)}\Big) = \tau\Big(s, s, 1-x_s\Big)$.

Thus, property (iii) in the definition of Ψ_n^{++} holds for ψ' as well. □

Claim 19. *If $\psi' \in \Psi_n^{++}$, where ψ' is constructed via $\psi \in \Psi_0$ according to Claim 14, and ψ satisfies the properties stated in Claim 13, then $\psi \in \Psi_n^{++}$.*

Proof of the Claim. Clearly, properties (i) and (ii) in the definition of Ψ_n^{++} hold for ψ. We show that if h, x and ϵ are such that property (iii) of the definition of Ψ_n^{++} holds for ψ' than this property also holds for ψ with the same h, x and ϵ.

Since $\kappa(\psi') = 3n = \kappa(\psi)$, then there is a bijection between the set of links $\mathcal{T}^1 \equiv \mathcal{T}_\psi \backslash \mathcal{T}_{\psi'}$ and $\mathcal{T}^2 \equiv \mathcal{T}_{\psi'} \backslash \mathcal{T}_\psi$. The set $\mathcal{T}^1$ is comprised from the links that go from states $s \in S'$ to state 1 for some $S' \subset S\backslash\{1\}$, and the set of links $\mathcal{T}^2$ is comprised of links that goes from states $s \in S'$ to themselves. Thus, for any $s \in S'$, the link in $\mathcal{T}^2$ outgoing from s is a strong that correspond to case (c) in the definition of Ψ^{++}, but since $h(1) = 1$, the link in $\mathcal{T}^1$, outgoing from s corresponds to case (c) in the definition of Ψ^{++} as well. Since the other links of ψ and ψ' coincide, we conclude that $\psi \in \Psi_n^{++}$ □

The Claim 19 finishes the proof for the case when $m_2 > (3/4)\cdot 2^{n-1}$. The case when $m_1 > (3/4)\cdot 2^{n-1}$ is analyzed by the analogous way. ■

Lemma 32. *Let the preference relation $\succeq$ have m equivalence classes, and $n = \lceil \log_2(m) \rceil$. Then language Q is adaptive if and only if there exists a decision rule $\psi \in \Psi_n^{++}$ that solves the choice problem $(Q, \succeq)$.*

Proof. First, suppose that language Q is adaptive for a preference relation $\succeq$. Then there is a subset $Q' = \{Q_i\}_{i \in N'}$ of questions such that $|N'| = n \equiv \lceil \log_2(m) \rceil$ and $\lambda \in \mathbb{R}^{N'}$ such that

$$a \succ b \quad \Longrightarrow \quad v(a) > v(b), \qquad \text{where} \quad v(a) \;=\; \sum_{i \in N'} \lambda_i a_i$$

Claim 20. *Let preference relations $\succeq'$ and $\succeq$ on A be such that $a \succ b$ implies $a \succ' b$ for all $a, b \in A$, and let Q be an arbitrary language, adequate for $\succeq'$. If a sequence of decision rules $\psi = \{\sigma_r\}_{r=1,2,\ldots} \in \Psi$ solves the choice problem $(Q, \succeq')$, then ψ solves the choice problem $(Q, \succeq)$.*

Proof of the claim. Consider arbitrary $a, b \in A$ such that $a \succ b$, then $a \succ' b$. Since ψ solves the choice problem $(Q, \succeq')$, by Lemma 11, $q_{\sigma_r}(b)/q_{\sigma_r}(b) \longrightarrow 0$. Therefore, by Lemma 11, ψ solves the choice problem $(Q, \succeq)$. □

Let $\succeq'$ be the preference relation on A, induced by the utility function $v(\cdot)$; that is,

$$v(a) \geq v(b) \iff a \succeq' b \qquad \forall a, b \in A$$

It follows that the language Q' is additive for a preference relation $\succeq'$. Then, by Lemma 28, the sequence of decision rules $\psi^+(Q', \succeq')$ solves the choice problem $(Q', \succeq')$, and the complexity of this sequence of decision rules is $\kappa(\psi^+(Q', \succeq')) = 3n$. By Claim 20, $\psi^+(Q', \succeq')$ solves the choice problem $(Q', \succeq)$ as well, and hence it solves the choice problem $(Q, \succeq)$, since $Q' \subseteq Q$.

Claim 21. *Let $\psi = \psi^+(Q', \succeq')$ be the decision rule, defined for an additive (with respect to preference relation $\succeq'$) language Q' as described above, where language Q' contains exactly $|N'| = n = \lceil \log_2(m) \rceil$ questions. Then $\psi^+(Q', \succeq') \in \Psi_n^{++}$.*

Proof of the Claim. First, note that $S_\psi = \{1, \ldots, n\}$, and the interrogation rule $g(i) = i$ is an injection; thus, properties (i) and (ii) in the definition of Ψ_n^{++} hold. Second, consider a unit permutation h given by $h(i) = i$, let $x_i = \mathbb{1}\{\lambda_i > 1\}$, and $\epsilon_i = \epsilon^{|\lambda_i|}$ for all $i \in \{1, \ldots, n\}$. Then it is straightforward to verify that ψ satisfies conditions (iii)(a,b,c) in the definition of Ψ_n^{++}. □

Thus, $\psi^+(Q', \succeq') \in \Psi_n^{++}$ solves the choice problem $(Q, \succeq)$, proving the only if direction.

Suppose now that $\psi \in \Psi_n^{++}$ solves the choice problem $(Q, \succeq)$ with m indifference classes, and $n = \lceil \log_2(m) \rceil$. Let x, h, ϵ be as in the definition of Ψ_n^{++}. Note that for any alternative $a \in A$, its set of simple paths is a singleton: $\mathcal{L}_0 = \{l^*(a)\}$, where $l_k = (h(k), h(k+1), Q_{\iota(h(k))(a)})$ for $k \in \{1, \ldots, n\}$ with the convention that $h(k+1) = \diamond$. Hence,

$$\begin{aligned}
\omega^*(a) &= (1-\eta)^{n-1} \cdot \prod_{k=1}^{n} (h(k), h(k+1), Q_{\iota(h(k))}(a)) = \\
&= (1-\eta)^{n-1} \cdot \prod_{k=1}^{n} \Big[\mathbb{1}\{Q_{\iota(h(k))}(a) = x_{h(k)}\} + \mathbb{1}\{Q_{\iota(h(k))}(a) = 1 - x_{h(k)}\} \cdot \epsilon_{h(k)} \Big] \\
&= (1-\eta)^{n-1} \cdot \prod_{k:\ Q_{\iota(h(k))}(a) = 1 - x_{h(k)}} \epsilon_{h(k)} = (1-\eta)^{n-1} \cdot \prod_{s:\ Q_{\iota(s)}(a) = 1 - x_s} \epsilon_s
\end{aligned}$$

It follows that

$$\log(\omega^*(a)) \;=\; (n-1)\cdot\log(1-\eta) + \sum_{s:\ Q_{\iota(s)}(a)=1-x_s} \log(\epsilon_s)$$

Since ψ solves $(Q,\succeq)$, then by Lemma 16, for all $a,b \in A$, $a \succ b$ implies

$$\omega^*(b)/\omega^*(a) \;\longrightarrow\; 0$$

Thus,

$$\log(\omega^*(b))/\log(\omega^*(a)) \;=\; \frac{\sum_{s:\ Q_{g(s)}(b)=1-x_s} \log(\epsilon_s)}{\sum_{s:\ Q_{g(s)}(a)=1-x_s} \log(\epsilon_s)} \;\longrightarrow\; 0$$

Recall that $\epsilon = (\epsilon_1,\dots,\epsilon_n)$ is, in fact, a sequence converging to $(0,\dots,0)$. Since the number of pairs (a,b) with $a,b \in A$ is finite, there is an element of this sequence such that

$$\sum_{s:\ Q_{\iota(s)}(b)=1-x_s} \log(\epsilon_s) < \sum_{s:\ Q_{\iota(s)}(a)=1-x_s} \log(\epsilon_s) \quad \forall a,b \in A:\ a \succ b \tag{18}$$

For $i \in N' = \iota(S)$, for this element of the sequence ϵ, define

$$\lambda_i \;=\; \lambda_{\iota(s)} \;=\; \begin{cases} -\log(\epsilon_s) & \text{if} \quad x_s = 1 \\ \log(\epsilon_s) & \text{if} \quad x_s = 0 \end{cases}$$

Note that λ_i are well-defined, since $\iota : S \to N$ is an injection. Then for any $a \in A$

$$\sum_{s:\ Q_{\iota(s)}(a)=1-x_s} \log(\epsilon_s) \;=\; \sum_{s:\ x_s=1,\ Q_{\iota(s)}(a)=0} \log(\epsilon_s) + \sum_{s:\ x_s=0,\ Q_{\iota(s)}(a)=1} \log(\epsilon_s) \;=$$

$$= \sum_{s:\ x_s=1} \log(\epsilon_s) \sum_{s:\ x_s=1,\ Q_{\iota(s)}(a)=1} \Big(-\log(\epsilon_s)\Big) + \sum_{s:\ x_s=0,\ Q_{\iota(s)}(a)=1} \log(\epsilon_s) \;=\; -\sum_{s:\ x_s=1} \lambda_s + \sum_{s\in S} \lambda_s a_{\iota(s)}$$

Thus, $a \succ b$ implies

$$\sum_{s\in S} \lambda_s a_{\iota(s)} \;>\; \sum_{s\in S} \lambda_s b_{\iota(s)} \quad\Longrightarrow\quad \sum_{i\in N'} \lambda_s a_i \;>\; \sum_{i\in N'} \lambda_s b_i$$

Since $|S| = n = \lceil \log_2(m) \rceil$, $g(S) = N'$ and ι is an injection, then $|N'| = \lceil \log_2(m) \rceil$. If $\lambda_i = 0$ for some $i \in N'$, then

$$\Big|\Big\{ \sum_{s\in S''} \log \epsilon_s \;\Big|\; \varnothing \subseteq S'' \subseteq S \Big\}\Big| \;\le\; 2^{|S|-1} = 2^{|N'|-1} = 2^{n-1} < m$$

in contradiction to eq. (18). Thus, Q is an adaptive language, proving the if direction of the Lemma. ■

B.4 Other Lemmas

The next lemma proves a generalized version of Proposition 4.

Lemma 33. *For any preference relation $\succeq$ with $m \geq 2$ indifference classes, there is an additive language Q with $n = \lceil \log_2(m) \rceil$ questions.*

Proof. Enumerate indifference classes of preference relation $\succeq$ by $C^0, C^1, ..., C^m$ such that $a \in C^j$, $b \in C^k$ implies $a \succ b$ iff $j > k$. Denote by $k(a)$ the index of the indifference class of a, i.e. $k(a) = k'$ iff $a \in C^{k'}$. Denote by $n = \lceil \log_2(m) \rceil$. Let $x_1 x_2 ... x_n$ be the expression of $k \in \{0, 1, ..., n-1\}$ via a base-2 number—it is well defined because of our choice of n—and denote by $k_i = x_i$ the corresponding digit of the base-2 expression of k. Consider the language Q with $Q_{ij} = \{a \in A \mid k_i(a) = j\}$ for $j \in \{1, 2\}$, and let $\lambda_i = 2^{n-i}$. Consider an additive utility function $v : A \to \mathbb{R}$ with weights λ, then

$$v(a) = \sum_{i=1}^{n} \lambda_i a_i = \sum_{i=1}^{n} 2^{n-i} k_i(a) = k(a)$$

It follows that $v(\cdot)$ represents $\succeq$. ■

B.5 Proofs of statements in Sections 2-4

In this section, I use previous calculations to prove statements from the main part of the paper.

B.5.1 Proof of Lemma 1

The Lemma follows from Lemma 10 by substituting $\rho^B(a) = \rho(a)/\sum_{b \in B} \rho(b)$. ■

B.5.2 Proof of Lemma 2

By Lemma 27, for any preference relation $\succeq$ and adequate language Q, the decision rule $\mathcal{R}(Q, \succeq)$ solves the choice problem $(Q, \succeq)$. ■

B.5.3 Proof of Theorem 1

The lower bound is proven in Lemma 25.

Consider an arbitrary preference relation $\succeq$ on A with $m \geq 2$ indifference classes. By Lemma 33, there is an additive with respect to preference relation $\succeq$ language Q with $n = \lceil \log_2(m) \rceil$ questions. By Lemma 28, the sequence of decision rules $\psi^+(Q, \succeq)$ solves $(Q, \succeq)$; moreover, $\kappa(\psi^+(Q, \succeq)) = 3|N| = 3 \cdot \lceil \log_2(m) \rceil$ proving the tightness of the lower bound. ■

B.5.4 Proof of Proposition 1

First, by Lemma 24, if $\psi \in \Psi$ solves $(Q, \succeq)$, then at least $n = \lceil \log_2(m) \rceil$ should be asked in various states, thus $|S| \geq \lceil \log_2(m) \rceil$.

Second, assume that $\psi \in \Psi$ solves $(Q, \succeq)$ solves $(Q, \succeq)$, and $\kappa(\psi) = 3n = 3 \cdot \lceil \log_2(m) \rceil$. Towards a contradiction, assume $|S| > n = \lceil \log_2(m) \rceil$. Similar to the proof of Theorem 1 given above, by Lemma 21 and pigeonhole principle, $2^{|\mathcal{T}^{weak}|} \geq m$, thus $|\mathcal{T}^{weak}| \geq \lceil \log_2(m) \rceil = n$. Next, by Lemma 24, there should be at least n states with question asked in each state, and there are at least two strong links outgoing from each such state. Since from every state there is an outgoing strong link, then $|\mathcal{T}^{strong}| > 2n$, and hence,

$$\kappa(\psi) \;=\; |\mathcal{T}^{weak}| + |\mathcal{T}^{strong}| > n + 2n = 3n$$

in contradiction to our assumption that $\kappa(\psi) = 3n = 3 \cdot \lceil \log_2(m) \rceil$ ■

B.5.5 Proof of Proposition 2

Consider set $A = \{a\} \cup \{b^1, \ldots, b^k\}$ with $k + 1$ alternatives, and preference relation $\succeq$ on A such that $a \succ b^i$ and $b^i \sim b^{i'}$ for all i, i'. Then $\succeq$ has $m = 2$ indifference classes. Consider language $Q = \{Q_i\}_{i=1}^k$ with $Q_{i1} = \{b^i\}$ and $Q_{i0} = A \backslash \{b^i\}$ for all $i = 1, \ldots, k$. Note that the language Q is adequate, since for any $i \in \{1, \ldots, k\}$, $b^i_i \neq c'_i$ for any $c' \neq b^i$.

Suppose $\psi \in \Psi$ solves $(Q, \succeq)$. Towards a contradiction, assume that question Q_i is not asked in ψ, i.e. $\nexists s \in S : \iota(s) = i$. Then the set of links that can be used for transitions between the states for alternatives a and b^i coincide, that is, $\mathcal{T}^a(s) = \mathcal{T}^{b^i}(s)$ for all $s \in S$. It follows that $q(a) = q(b^i)$, which is a contradiction by Lemma 11, since $a \succ b^i$. Thus, questions $Q_1, \ldots, Q_k$ should be asked in ψ. Note that each question i asked in state s is associated with at least two links $(s, s', 0), (s, s'', 1) \in \mathcal{T}$, and different questions are associated with different links, thus

$$\kappa(\psi) \ = \ |\mathcal{T}| \geq 2k \quad \forall \psi \in \Psi : \ \psi \text{ solves } (Q, \succeq)$$

Thus, $\kappa(Q) > k$, proving the Proposition. ∎

B.5.6 Proof of Theorem 2

Statement (i) of the Theorem follows from the Lemma 27, since the universal decision rule $\overline{\psi} = \mathcal{R}(Q, \succeq)$ solves the choice problem $(Q, \succeq)$ and uses $\kappa(\overline{\psi}) = 3|A| - 3$ transitions.

Let $m \in \{2, ..., |A|\}$. Let $\succeq$ be a preference relation such that $\tilde{a} \succ b$ for all $b \in A$; that is, $\tilde{a}$ is a unique $\succeq$-best element in A, and let $\succeq$ have m indifference classes; clearly, such preference relation exists. Enumerate alternatives in $A \backslash \{\tilde{a}\}$ by $a^1, ..., a^{|A|-1}$, and consider language $Q = \{Q_i\}_{i=1}^{|A|-1}$ with $Q_{i1} = \{a^i\}$, $Q_{i0} = A \backslash \{a^i\}$.

Assume a decision rule ψ solves the choice problem $(Q, \succeq)$; by Lemma 17, it is without loss of generality to assume that ψ is regular. Note that all questions of the language Q should be asked in ψ; otherwise, if Q_i is not asked, then $l^*(\tilde{a}) \in \mathcal{L}(a^i)$, $\omega^*(\tilde{a}) \leq \omega^*(a^i)$, and by Lemma 16, ψ does not solve $(Q, \succeq)$, in contradiction. Thus, $\mathcal{T}_\psi$ contains at least $2 \cdot (|A| - 1)$ strong links. Next, since $\succeq$ has m indifference classes, then by Lemma 21, the sets of weak links used by alternatives from distinct indifference classes should be different. It follows that $2^{|\mathcal{T}_\psi^{weak}|} \geq m$, hence $|\mathcal{T}_\psi^{weak}| \geq \lceil \log_2(m) \rceil$. Hence,

$$\kappa(\psi) \ = \ |\mathcal{T}_\psi| \ = \ |\mathcal{T}_\psi^{strong}| + |\mathcal{T}_\psi^{weak}| \ \geq \ 2|A| - 2 + \lceil \log_2(m) \rceil$$

Since the above inequality holds for arbitrary decision rule ψ that solves the choice problem $(Q, \succeq)$, then $\kappa(Q) \geq 2|A| - 2 + \lceil \log_2(m) \rceil$, proving statement (ii) of the Theorem. ∎

B.5.7 Proof of Proposition 3

The proof of the Proposition mirrors the proof of Theorem 2 above. Statement (i) of the Proposition follows from the Lemma 27, since the universal decision rule $\overline{\psi} = \mathcal{R}(Q, \succeq)$ solves the choice problem $(Q, \succeq)$ and uses $|S_{\overline{\psi}}| = |A| - 1$ memory states.

Let $m \in \{2, ..., |A|\}$. Let $\succeq$ be a preference relation such that $\tilde{a} \succ b$ for all $b \in A$; that is, $\tilde{a}$ is a unique $\succeq$-best element in A, and let $\succeq$ have m indifference classes; clearly, such preference relation exists. Enumerate alternatives in $A \backslash \{\tilde{a}\}$ by $a^1, ..., a^{|A|-1}$, and consider language $Q = \{Q_i\}_{i=1}^{|A|-1}$ with $Q_{i1} = \{a^i\}$, $Q_{i0} = A \backslash \{a^i\}$.

Assume a decision rule ψ solves the choice problem $(Q, \succeq)$; by Lemma 17, it is without loss of generality to assume that ψ is regular. Note that all questions of the language Q should be asked in ψ; otherwise, if Q_i is not asked, then $l^*(\tilde{a}) \in \mathcal{L}(a^i)$, $\omega^*(\tilde{a}) \leq \omega^*(a^i)$, and by Lemma 16, ψ does not solve $(Q, \succeq)$, in contradiction. Since all questions of the language Q must be asked in Ψ, then $|S_\psi| \geq |Q| = |A| - 1$. It follows that $\mathcal{M}(Q) \geq |A| - 1$. By the proven statement (i), $\mathcal{M}(Q) \leq |A| - 1$, hence $\mathcal{M}(Q) = |A| - 1$, proving statement (ii) of the Proposition. ■

B.5.8 Proof of Proposition 4

The Proposition follows from Lemma 33. ■

B.5.9 Proof of Theorem 3

Suppose that language Q is adaptive for a preference relation $\succeq$ with m indifference classes, and $n = \lceil \log_2(m) \rceil$. By Lemma 32, there is a decision rule $\psi \in \Psi_n^{++}$ that solves the choice problem $(Q, \succeq)$. By Lemma 29, $\kappa(\psi) = 3n$, thus the language Q is simple.

Suppose now that $(3/4) \cdot 2^n < m \leq 2^n$ for some natural n, $\psi \in \Psi$ solves the choice problem $(Q, \succeq)$, and $\kappa(\psi) = 3n$. By Lemma 31, $\psi \in \Psi_n^{++}$, and thus by Lemma 32, the language Q is adaptive. ■

B.5.10 Proof of Proposition 5

Statement (i) of the Proposition is proved in Lemma 32. Statement (ii) of the Proposition proved in Lemma 31 ■

B.6 Proofs of lemmas and claims from Section 5

Lemma 3 is proven in Lemma 11 in this Appendix, Lemma 4 is proven in Lemma 16 in this Appendix, Claim 1 follows from Lemma 19 and Lemma 22 in the Appendix, Claim 2 is proven in Lemma 23 in the Appendix, Claim 3 follows from Claim 13 in the Appendix, Claim 4 is proven in Claim 15 in the Appendix, Claim 6 follows from Claim 16 in the Appendix.

B.7 Formalizing the dynamics

The realized menu $B \subseteq A$ and the sampling probability ρ^B from B determine the economic environment. The language Q and the automaton $\sigma = (S, \iota, \tau) \in \Sigma$ determines the agent's decision procedure. Together, the economic environment and the decision procedure, induce a dynamic random choice in a straightforward way.

Let $\mathcal{Y}^B = B \times S$ be the global state space; denote by $y = (a, s)$ its generic element. Thus, $a \in B$ represents the currently drawn alternative, and $s \in S$ represents the state of the automaton. In the following, I will omit a superscript B in the notations whenever the menu B is fixed. Let $\mathbf{Y} = (Y_1, Y_2, \ldots)$ be a Markov chain with realizations $\mathbf{y} = (y_1, y_2, \ldots) \in \mathcal{Y}^{\mathbb{N}}$ that describes the dynamics of the model in periods $t = 1, 2, \ldots$ as follows.

In the first period, a random alternative is drawn and the automaton's state is $s = 1$:

$$Pr\Big(Y_1 = (a,s)\Big) \quad = \quad \rho^B(a) \cdot \delta_1^s$$

where I denote by $\delta_i^k = 1$ if $i = k$ and $\delta_i^k = 0$ if $i \neq k$. The dynamics of the Markov chain $\mathbf{Y}$ in the subsequent periods ($t > 1$) is described by the following stochastic $\mathcal{Y} \times \mathcal{Y}$ matrix:

$$Pr\Big(Y_t = (a,s) \,\Big|\, Y_{t-1} = (b,v)\Big) \; = $$
$$= \begin{cases} \Big(1 - \tau\big(v, \diamond, Q_{\iota(v)}(b)\big)\Big) \cdot \eta \cdot \rho^B(a) \cdot \delta_1^s + (1-\eta) \cdot \delta_b^a \cdot \tau\big(v, s, Q_{\iota(v)}(b)\big) & \text{if} \quad v, s \neq \diamond \\ \tau\big(v, \diamond, Q_{\iota(v)}(b)\big) \cdot \delta_b^a & \text{if} \quad v \neq \diamond, s = \diamond \\ \delta_a^b \cdot \delta_\diamond^s & \text{if} \quad v = \diamond \end{cases} \tag{19}$$

Thus, the first case in eq. (19) tells that a transition between the two memory states occurs in one of the two ways. First, if the alternative has not been chosen in the previous period, then with probability η a new alternative is drawn according to the probability distribution $\rho^B(a)$, and the state moves to $s = 1$, hence the term δ_1^s. Second, it can be that the decision procedure prescribes to go to state s with probability $\tau\big(v, s, Q_{\iota(v)}(b)\big)$ as a result of the information "$Q_{\iota(v)}(b) = j$" with $j \in \{0, 1\}$ acquired in the previous period. If this transition occurs, it must be that a new alternative is not drawn, hence the multiple $(1 - \eta)$, and the current alternative remains the same, hence the multiple δ_b^a. The other two cases in eq. (19) are interpreted similarly.

For each period $t \in \{1, 2, \ldots\}$, the probability that alternative $a \in B$ is chosen in some period $t' < t$ is given by

$$Pr\Big(a \text{ is chosen from } B \text{ in period } t' < t\Big) \; = \; Pr\Big(Y_t^B = (a, \diamond)\Big)$$

In this paper, I am concerned with the total probability of an alternative to be chosen in some period from the menu. Thus, I analyze the stochastic choice rule given by

$$p^B(a) := \lim_{t\to\infty} Pr\Big(Y_t^B = (a, \diamond)\Big) \tag{20}$$

Note that $p^B(a)$ depends on the language Q and the automaton σ via the stochastic matrix given by eq. (19) that governs the evolution of the Markov chain $\mathbf{Y}$. I will use a subscript σ to show this dependence when needed.

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