Foundations of self-progressive choice models*

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Abstract

Consider a population of agents whose choice behaviors are partially *comparable* according to given *primitive orderings*. The set of choice functions admissible in the population specifies a *choice model*. A choice model is *self-progressive* if any aggregate choice behavior consistent with the model is uniquely representable as a probability distribution over admissible choice functions that are comparable. We establish an equivalence between self-progressive choice models and (i) well-known algebraic structures called *lattices*; (ii) the maximizers of supermodular functions over a specific domain of choice functions. Our results provide for a precise recipe to restrict or extend any choice model for unique orderly representation. We characterize the minimal self-progressive extension of rational choice functions, which offers an explanation for why agents might exhibit *choice overload*. Finally, we extend our analysis to choice models which render unique orderly representations independent of the primitive orderings.

Keywords: Random choice, heterogeneity, identification, unique orderly representation, lattice, supermodular optimization, multiple behavioral characteristics.

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1 Introduction

Random choice models are used successfully to identify heterogeneity in the aggregate choice behavior of a population. The success is achieved despite prominent choice models, such as the random utility model, being underidentified in the sense that the explained choice behavior renders different representations within the model. The typical remedy to this challenging matter has been structuring the model in order to obtain a unique representation and achieve point-identification.[1] Here, instead of focusing on a specific choice model, we follow a different route, in that we take *choice models* as our primitive objects, and assume an "orderliness" in the population that allows for partial comparison of agents' choice behaviors.[2] In our analysis, we present testable and optimization based foundations of choice models that guarantee a unique orderly representation for each aggregate choice behavior consistent with the model.

The findings of two recent studies that use the "orderliness" in the population are precursory for our approach. Apesteguia, Ballester & Lu (2017) observe that if a random utility model is represented as a probability distribution over a set of comparable rational choice functions, then the representation must be unique. Filiz-Ozbay & Masatlioglu (2022) observe that each random choice function can be uniquely represented as a probability distribution over a set of choice functions that are comparable to each other. Both studies present intriguing examples in which agents' choices are ordered according to a single characteristic. However, parametrizing agents' choices according to various behavioral characteristics is critical in explaining economically relevant phenomena. A classical example is the equity premium puzzle (Mehra & Prescott 1985) that cannot be

[1] See for example Gul & Pesendorfer (2006) and Dardanoni et al. (2022).

[2] See the discussions by Apesteguia, Ballester & Lu (2017) and Filiz-Ozbay & Masatlioglu (2022).

explained by maximization of CRRA or CARA utilities parameterized by the risk aversion coefficient. As for an explanation, Epstein & Zin (1989) proposed utility functions in which the coefficient of risk aversion and the elasticity of substitution are separated. Another explanation based on agents' choices is Benartzi & Thaler (1995)'s *myopic loss aversion* that combines *loss aversion*–a greater sensitivity to losses than to gains–and a tendency to evaluate outcomes more frequently. Since two parameters should be specified separately, population heterogeneity explained by these models may not be consistent with a fixed set of choice functions ordered according to a single characteristic.

We formulate and analyze *self-progressive* choice models that contain the choice functions used in the unique progressive representation of any aggregate choice behavior consistent with the model. Our findings show that self-progressive models allow for specification of multiple behavioral characteristics. Additionally, we obtain a precise recipe to restrict or extend any choice model as to be self-progressive. We extend our analysis to choice models rendering unique orderly representations independent of the primitive orderings. In what follows, we introduce our approach and findings.

Consider a population of agents who rank alternatives according to a *primitive ordering* that depends on the available alternatives, called a *choice set*. In addition to risk attitudes, social preferences, or prices that may be choice set independent, primitive orderings can accommodate, for example, the temptation or information processing costs that depend on the availability of more tempting or memorable alternatives. In this population, a pair of choice functions are *comparable* if the alternative chosen by one of the choice functions is ranked higher than the alternative chosen by the other for every choice set according to the associated primitive ordering.

The main object of our analysis is a *choice model*, which is simply the set of choice functions that may be adopted by an agent in the population. We call these choice functions *admissible*. Next, we describe our notion of *self-progressiveness*. Suppose that an analyst represents the aggregate choice behavior of a population as a probability distribution over a set of admissible choice functions. We know that the same aggregate choice behavior renders a unique representation as a probability distribution over–possibly different–choice functions that are *comparable* to each other. Self-progressiveness requires these comparable choice functions to be admissible as well. Put differently, a self-progressive choice model provides a language to the analyst that allows for orderly representing any aggregate choice behaviour that is consistent with the model.

In our analysis, we first establish an equivalence between self-progressive choice models and well-known algebraic structures called *lattices*. For each pair of choice functions, their *join* (*meet*) is the choice function, choosing from each choice set the higher(lower)-ranked alternative among the ones chosen by the given pair. A choice model forms a *lattice* if for each pair of admissible choice functions, their join and meet are admissible as well. In Theorem 1, by using a simple probabilistic decomposition procedure, we show that self-progressive choice models are the ones that possess a lattice structure. It follows that self-progressive choice models are not limited to models consisting of comparable choice functions. To demonstrate the relevance of this generality, we present examples of choice models in which multiple behavioral characteristics are parametrized. Additionally, Theorem 1 provides for a precise recipe to restrict or extend any choice model for unique orderly representation. To prove out, we characterize the minimal self-progressive extension of rational choice functions via two choice axioms, which offer an explanation for why agents might exhibit *choice overload*.[3]

[3]Choice overload refers to the phenomena that agents tend to deviate from their accurate preferences

In the second part, we provide an optimization-based foundation for self-progressive choice models. Our departure point is that a choice function can be interpreted as a complete contingent plan to be implemented upon observing available alternatives. Then, we imagine a population of agents evaluating a set of choice functions via a common *value function*, which can be thought of as an indirect utility function associated with the problem of optimally adopting a choice function. Each agent adopts a choice function by maximizing the value function over the specific domain.[4] The population is homogeneous in the sense that each agent evaluates same choice functions via the same value function. The unique source of heterogeneity is the maximizers' multiplicity. This raises the question: What sort of choice heterogeneity allows for self-progressiveness?

To see the answer, suppose that the set of considered choice functions is a lattice, and the value function is *supermodular*. Then, since the maximizers form a (sub)lattice, by Theorem 1, the associated choice model is self-progressive. In Theorem 2, we conversely show that every self-progressive choice model can be obtained as the maximizers of a supermodular value function over an intuitive domain of choice functions.

We have assumed so far that an underlying partial ordering obtained from the primitive orderings allows for comparison of different choice behaviour. Existence of such an ordering derives the unique orderly representation for any aggregate choice behavior consistent with a model. However, one can question if there exist choice models rendering unique orderly representations independent of the primitive orderings. We show that a choice model satisfies this stringent requirement–called *universal self-progressiveness*–if and only if it corresponds to the maximizer set of an additively separable value function.

when they choose from complex environments. See Chernev et al. (2015) for a recent meta-analysis.

[4]Here, a choice function is analogous to a "worldview" as described by Bernheim et al. (2021) who offer a dynamic model of endogenous preference formation.

2 Self-progressive choice models

Let X be the (grand) **alternative set** with n elements. A **choice set** S is a subset of X containing at least two alternatives. The **choice domain** Ω is a nonempty collection of choice sets allowing for limited data sets. A **choice function** is a mapping $c : \Omega \to X$ such that for each $S \in \Omega$, we have $c(S) \in S$. Let $\mathcal{C}$ denote the set of all choice functions. A **choice model** $\mu \subset \mathcal{C}$ is a set of choice functions. We consider two choice procedures with possibly different formulations as equivalent if these procedures are observationally indistinguishable in the revealed preference framework, that is, two choice procedures rationalize the same set of choice functions.

A **random choice function** (RCF) ρ assigns each choice set $S \in \Omega$ a probability measure over S. We denote by $\rho_x(S)$ the probability that alternative x is chosen from choice set S. A (deterministic) choice function can be represented by an $|\Omega| \times |X|$ matrix with rows indexed by the choice sets and columns indexed by the alternatives, and entries in $\{0, 1\}$ such that each row has exactly one 1. For each $(S, x) \in \Omega \times X$, having 1 in the entry corresponding to row S and column x indicates that x is chosen in S. Similarly, a RCF can be represented by an $|\Omega| \times |X|$ matrix having entries in $[0, 1]$ such that the sum of the entries in each row is 1. For each RCF and each pair $(S, x) \in \Omega \times X$, the associated entry indicates the probability that x is chosen in S. Then, it follows from Birkhoff-von Neumann Theorem (Birkhoff 1946, Von Neumann 1953) that each RCF can be represented as a probability distribution over a set of deterministic choice functions. However, this representation is not necessarily unique. For each choice model μ, let $\Delta(\mu)$ be the set of RCFs that can be represented via a probability distribution over choice functions contained in μ.

For each choice set $S \in \Omega$, a **primitive ordering** $>_S$ is a complete, transitive, and antisymmetric binary relation over S. We write $\geq_S$ for its union with the equality relation. Then, we obtain the partial order $\rhd$ from the primitive orderings such that for each pair of choice functions c and c', we have $c \rhd c'$ if and only if $c(S) \geq_S c'(S)$ for each $S \in \Omega$, and there exists $S \in \Omega$ with $c(S) \neq c'(S)$. We write $c \trianglerighteq c'$ if $c \rhd c'$ or $c = c'$.

Definition. Let $\rhd$ be the partial order over choice functions obtained from the primitive orderings $\{>_S\}_{S\in\Omega}$. Then, a choice model μ is **self-progressive** with respect to $\rhd$ if each RCF $\rho \in \Delta(\mu)$ can be uniquely represented as a probability distribution over a set of choice functions $\{c^1, \ldots, c^k\} \subset \mu$ such that $c^1 \rhd c^2 \cdots \rhd c^k$.

If such a "progressive representation" exists, then it must be unique. To see this, consider the $\rhd$-best choice function c^1 in a progressive representation. Note that c^1 chooses the $>_S$-highest-ranked alternative that is assigned positive probability by ρ in each $S \in \Omega$. Therefore, the weight of c^1 must be equal to the minimum of $\rho_{c(S)}(S)$ over $S \in \Omega$. Thus, c^1 and its probability weight is uniquely determined. Iterating this reasoning yields the uniqueness of the progressive representation.

3 Self-progressive choice models and lattices

Let $\{>_S\}_{S\in\Omega}$ be the primitive orderings and $\rhd$ be the associated partial order over choice functions. For each pair of choice functions c and c', their *join* (*meet*) is the choice function $c \vee c'$ $(c \wedge c')$ that chooses from each choice set S, the $>_S$-best(worst) alternative among the ones chosen by c and c' at S. Then, for each choice model μ, the pair $\langle \mu, \rhd \rangle$ is a **lattice** if for each pair of choice functions c and c' in μ, their join $c \vee c'$ and meet $c \wedge c'$ are in μ as well.

Theorem 1. *Let μ be a choice model and $\triangleright$ be the partial order over choice functions that is obtained from the primitive orderings $\{>_S\}_{S\in\Omega}$. Then, μ is self-progressive with respect to $\triangleright$ if and only if the pair $\langle\mu,\triangleright\rangle$ is a lattice.*

To see that the *only if* part holds, let $c, c' \in \mu$. Then, consider the RCF ρ such that for each $S \in \Omega$, $c(S)$ or $c'(S)$ is chosen evenly. Note that ρ has a unique progressive representation in which only $c \vee c'$ and $c \wedge c'$ receive positive probability. Since μ is self-progressive, it follows that $c \vee c' \in \mu$ and $c \wedge c' \in \mu$.

As for the *if* part, suppose that $\langle\mu,\triangleright\rangle$ is a lattice, and let $\rho \in \Delta(\mu)$. Next, we present our **uniform decomposition procedure**, which yields the progressive random choice representation for ρ with respect to $\triangleright$. Figure 1 demonstrates the procedure.

Step 1: For each choice set S, let $\rho^+(S) = \{x \in S : \rho(x,S) > 0\}$, and partition the $(0,1]$ interval into $|\rho^+(S)|$ intervals $\{I_{Sx}\}_{\{x\in\rho^+(S)\}}$ such that each interval I_{Sx} is half open of the type $(l_{Sx}, u_{Sx}]$ with length $\rho(x,S)$, and for each $x, y \in \rho^+(S)$ if $x >_S y$, then l_{Sx} is less than l_{Sy}.

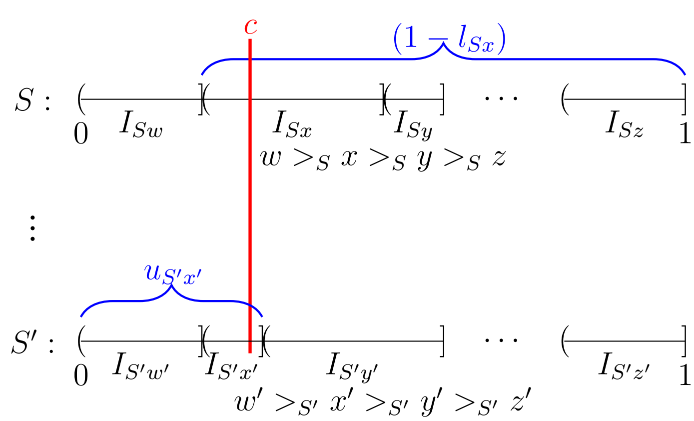

Figure 1

Step 2: Pick a real number $r \in (0,1]$ according to the Uniform distribution on $(0,1]$. Then, for each choice set and alternative pair (S,x), let $c(S)=x$ if and only if $r \in I_{Sx}$. It is clear that as a result of the procedure we obtain a unique probability distribution over a set of choice functions $\{c^1,\ldots,c^k\} \subset \mu$ such that $c^1 \rhd c^2 \cdots \rhd c^k$.[5] In proving the if part of Theorem 1, we will show that $\{c^1,\ldots,c^k\} \subset \mu$ by using Lemma 1.

Lemma 1. *Let μ be a choice model such that $\langle \mu, \rhd \rangle$ is a lattice, and let $c \in \mathcal{C}$. If for each $S, S' \in \Omega$, there exists $c^* \in \mu$ such that $c^*(S)=c(S)$ and $c^*(S')=c(S')$, then $c \in \mu$.*

Proof. The result is obtained by applying the following observation inductively. Consider any $\mathbb{S} \subset \Omega$ containing at least three choice sets. Let $c_1, c_2, c_3 \in \mu$ be such that for each $i \in \{1,2,3\}$, there exists at most one $S_i \in \mathbb{S}$ with $c_i(S_i) \neq c(S_i)$. Suppose that for each $i,j \in \{1,2,3\}$, if such S_i and S_j exist, then $S_i \neq S_j$. Now, for each $S \in \mathbb{S}$, we have $c(S)$ is chosen by the choice function $(c_1 \wedge c_2) \vee (c_1 \wedge c_3) \vee (c_2 \wedge c_3) \in \mu$. To see this, let $S \in \mathbb{S}$, and note that there exist at least two $i,j \in \{1,2,3\}$ such that $c_i(S)=c_j(S)=c(S)$. Assume without loss of generality that $i=1$ and $j=2$. Now, if $c(S) \geq_S c_3(S)$, then we get $c(S) \vee c_3(S) \vee c_3(S) = c(S)$; if $c_3(S) >_S c(S)$, then we get $c(S) \vee c(S) \vee c(S) = c(S)$. □

Proof of Theorem 1. We proved the *only if* part. For the *if* part, let $\{c^1,\ldots,c^k\}$ be the set of choice functions that are assigned positive probability in the uniform decomposition procedure. We show that $\{c^1,\ldots,c^k\} \subset \mu$ by using Lemma 1. For this, suppose that $S, S' \in \Omega$, where $x = c^i(S)$ and $x' = c^i(S')$ for some $i \in \{1,\ldots,k\}$. Since $\rho \in \Delta(\mu)$, it must be that μ contains a choice function choosing x from S; and possibly a different choice function choosing x' from S'. We show that there exists $c^* \in \mu$ such that both $c^*(S)=x$ and $c^*(S')=x'$. Then, it will directly follow from Lemma 1 that $c^i \in \mu$.

[5] See Theorem 1 by Filiz-Ozbay & Masatlioglu (2022) for an elaborate proof of this fact. It is easy to see that this procedure is applicable even if the choice space is infinite. In a contemporary study, Petri (2023) independently extends Theorem 1 by Filiz-Ozbay & Masatlioglu (2022) to infinite choice spaces.

First, as demonstrated in Figure 1, we have $(1 - l_{Sx}) + u_{S'x'} > 1$. Since $\rho \in \Delta(\mu)$, it follows that there exists $c_1 \in \mu$ such that $c_1(S) \geq_S x$ and $c_1(S') \leq_S x'$. Symmetrically, since $(1 - l_{S'x'}) + u_{Sx} > 1$, there exists $c_2 \in \mu$ such that $c_2(S) \leq_S x$ and $c_2(S') \geq_{S'} x'$. Next, consider $\{c \in \mu : x \geq_S c(S)\}$ and let c_x be its join. Similarly, consider $\{c \in \mu : x' \geq_{S'} c(S')\}$ and let $c_{x'}$ be its join. By construction, $c_x(S) = x$ and $c_{x'}(S') = x'$. Moreover, c_2 is a member of the former set, while c_1 is a member of the latter one. Now, let $c^* = c_x \wedge c_{x'}$. Then, $c^*(S) = x$, since $c_x(S) = x$ and $c_{x'}(S) \geq_S c_1(S) \geq_S x$. Similarly, $c^*(S') = x'$, since $c_{x'}(S') = x'$ and $c_x(S') \geq_{S'} c_2(S') \geq_{S'} x'$. □

4 Examples and discussion

4.1 Rational choice and chain lattices

To observe that the random choice model fails to be self-progressive, let $X = \{a, b, c\}$ and $\Omega = \{X, \{a, b\}, \{a, c\}, \{b, c\}\}$. Suppose that each primitive ordering is obtained by restricting the ordering $a > b > c$ to a choice set. Figure 2 demonstrates the associated choice functions lattice in which each array specifies the chosen alternatives respectively. The rational choice functions are highlighted (in red) and clearly fail to form a lattice, since each light-colored choice function is a join or meet of a rational choice function.

The equivalence to lattices guides us to restrict or extend rational choice model as to be self-progressive. In this vein, a particularly simple lattice is a *chain lattice,* which is a set of choice functions $\{c_i\}_{i=1}^k$ that are comparable: $c_1 \rhd c_2 \cdots \rhd c_n$. Suppose that each primitive ordering $>_S$ is obtained by restricting the ordering $>_X$ to the choice set S. Additionally, suppose that the choice domain Ω contains every choice set. Then, one

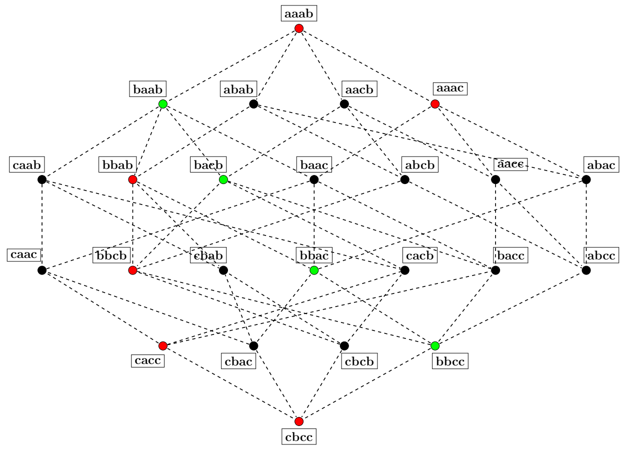

Figure 2: The choice functions lattice.

can easily observe that if $\langle \mu, \rhd \rangle$ is a lattice then it must be chain lattice.[6] Moreover, there is a one-to-one correspondence between the chain lattices of rational choice model and the preferences with *single-crossing property* introduced by Apesteguia, Ballester & Lu (2017). To see this, let $\mu = \{c_i\}_{i=1}^k$ be a choice model consisting of choice functions rationalized by preferences $\{\succ_i\}_{i=1}^k$. That is, for each choice set S, $c_i(S)$ maximizes the preference relation $\succ_i$ over S. Then, $\{\succ_i\}_{i=1}^k$ is *single-crossing with respect to* $>_X$ means: for each alternative pair $x >_X y$, if $x \succ_i y$, then $x \succ_j y$ for every $i > j$. It is easy to see that $\langle \mu, \rhd \rangle$ is a chain lattice if and only if $\{\succ_i\}_{i=1}^k$ is single-crossing with respect to $>_X$.[7] Apesteguia, Ballester & Lu (2017) present economic examples of rational choice functions that form chain lattices.[8] Going beyond rational choice, Filiz-Ozbay & Masatlioglu (2022) present examples in which the choice functions are ordered according to a single behavioral characteristic, thus these examples also form chain lattices.

[6]This is not true for a general domain of choice sets. For a simple example, suppose that the choice domain consists of disjoint binary choice sets. Then, every choice function is rational, thus every sublattice of choice functions is a set of rational choice functions.

[7]See also Lemma 1 by Filiz-Ozbay & Masatlioglu (2022).

[8]See also Curello & Sinander (2019) who characterize when a common primitive ordering over alternatives allows for preferences form a lattice according to single-crossing dominance, and provide several applications.

4.2 Beyond chain lattices

It follows from our Theorem 1 that self-progressive choice models are not limited to chain lattices, and thus allow for parametrization according to multiple behavioral characteristics. We will demonstrate this point via our examples. In our first example, we propose a choice model consistent with the choice overload phenomena indicating that agents tend to deviate from their accurate preferences more when they choose from more complex environments, such as larger choice sets.

Example 1. *(Smaller-is-better)* Let $\mathcal{P}$ be a set of faulty preferences that are single-crossing with respect to the accurate preference $>$. Then, a choice function $c \in \mu$ if for each choice set S, the alternative $c(S)$ is the $\succ_S$-maximal one in S for some $\succ_S \in \mathcal{P}$ such that $\succ_S$ is more aligned with $>$ (less faulty) than (or is identical to) $\succ_{S'}$, whenever S is a subset of S'. Note that μ is self-progressive with respect to the comparison relation obtained from $>$, since for each $c^i, c^j \in \mu$, their join and meet are the choice functions described by maximization of the preferences $max(\{\succ^i_S, \succ^j_S\}, \geq)$ and $min(\{\succ^i_S, \succ^j_S\}, \geq)$.

Example 2. *(Limited attention meets satisficing)*[9] Consider a population with primitive orderings $\{>_S\}_{S\in\Omega}$ in which each agent i has the same preference relation $\succ^*$, but a possibly different *threshold alternative* x^i_S for each choice set S. Then, for given choice set S, agent i chooses the $\succ^*$-best alternative in the consideration set $\{x \in S : x \geq_S x^i_S\}$. Let μ be the set of associated choice functions. Then, $\langle \mu, \rhd \rangle$–where $\rhd$ is obtained from $\succ^*$–is a lattice, since for each $c^i, c^j \in \mu$, their join and meet are the choice functions described by threshold alternatives $max(\{x^i_S, x^j_S\}, \geq_S)$ and $min(\{x^i_S, x^j_S\}, \geq_S)$.[10]

[9] See Simon (1955), Tyson (2008), Rubinstein & Salant (2008), and Masatlioglu et al. (2012).

[10] As a special case, consider agents who faces temptation with limited willpower formulated as by Masatlioglu et al. (2020). Each agent i chooses the alternative that maximizes the common *commitment ranking* u from the set of alternatives where agent i overcomes *temptation*, represented by v^i, with his

Example 3. *(Similarity-based choice)* Let (m, p) denote a *lottery* giving a monetary prize $m \in (0, 1]$ with probability $p \in (0, 1]$ and the prize 0 with the remaining probability. Consider a population consisting of agents choosing from binary lottery sets[11] such that each agent i has a *perception of similarity* described by (ϵ^i, δ^i) with $\delta^i \geq \epsilon^i$ as follows: for each $t_1, t_2 \in (0, 1]$, "t_1 is similar to t_2" if $|t_1 - t_2| < \epsilon^i$ and "t_1 is different from t_2" if $|t_1 - t_2| > \delta^i$. Then, in the vein of Rubinstein (1988), to choose between two lotteries (m_1, p_1) and (m_2, p_2), agent i first checks if "m_1 is similar to m_2 and p_1 is different from p_2", or vice versa.[12] If one of these two statements is true, for instance, m_1 is similar to m_2 and p_1 is different from p_2, then the probability dimension becomes the decisive factor, and i chooses the lottery with the higher probability. Otherwise, each agent chooses the higher-ranked lottery according to a given primitive ordering $>^*$. Let μ be the set of associated choice functions. Then, $\langle \mu, \rhd \rangle$–where $\rhd$ is generated from $>^*$–is a lattice, since for each $c^i, c^j \in \mu$, their join and meet are the choice functions whose perceptions of similarity are described by $(min(\epsilon^i, \epsilon^j), max(\delta^i, \delta^j))$ and $(max(\epsilon^i, \epsilon^j), min(\delta^i, \delta^j))$.[13]

Example 4. *(Maximization of set contingent utilities)* For each $S \in \Omega$ and $x \in S$, let $U(x, S)$ be the *set contingent utility* of choosing x. In addition to the intrinsic utility of alternative x that may be menu independent, $U(x, S)$ can accommodate the search cost of spotting x in S, the likelihood of S being available, self-perception considerations, or the temptation cost due to choosing x in the presence of more tempting alternatives.[14]

willpower stock w^i. Suppose that the primitive orderings are aligned with the commitment ranking u. Then, for each choice set S, let the threshold alternative x^i_S be the $>_S$-worst one such that $v^i(x) - max_{z \in S} v^i(z) \leq w^i$. As demonstrated by Filiz-Ozbay & Masatlioglu (2022) if we only allow agents' willpower stock to differ, then we obtain a choice model forming a chain lattice.

[11] One can assume that the monetary prizes and probability values have a finite domain.

[12] Rubinstein (1988) additionally requires one of these two statements be true. The slight difference is that our "t_1 is different from t_2" statement implies the negation of "t_1 is similar to t_2", while the converse does not necessarily hold. Both versions of the procedure provide explanations to the Allais paradox.

[13] Note that $\langle \mu, \rhd \rangle$ may not be a chain lattice since we can have $\epsilon^i > \epsilon^j$, while $\delta^i < \delta^j$.

[14] For example, in the vein of Gul & Pesendorfer (2001), one can set $U(x, S) = u(x) + v(x) - max_{z \in S} v(z)$, where u represents the *commitment* ranking and v represents the *temptation ranking*.

Let μ be the set of choice functions maximizing the sum of these set contingent utilities, that is $\mu = \operatorname{argmax}_{c \in \mathcal{C}} \sum_{S \in \Omega} U(c(S), S)$. Note that if c^* is obtained as a "mixture" of some $c, c' \in \mu$ in the sense that $c^*(S) \in \{c(S), c'(S)\}$ for every choice set $S \in \Omega$, then $c^* \in \mu$ as well. Since meet and join are special mixtures, $\langle \mu, \rhd \rangle$ is a lattice. Thus, by Theorem 1, μ is self-progressive. Notably, this is true for any specification of primitive orderings, since the set contingent utility functions are not necessarily linked to the primitive orderings.

5 Minimal self-progressive extension of rational choice

We will follow the guide provided by Theorem 1 to discover the "minimal" self-progressive extension of the rational choice functions assuming a single primitive ordering $>$ rankings of which reflect alternatives' "accurate values" and Ω contains every choice set. Let $\rhd$ be the comparison relation over choice functions obtained from $>$ as usual. An extension is minimal if we are parsimonious in adding nonrational choice functions so that each choice model containing rational choice functions and is contained in the minimal extension fails to be self-progressive with respect to $\rhd$. In Figure 3, we demonstrate–by using Theorem 1–that the minimal self-progressive extension of a model is unique. Next, we characterize the minimal self-progressive extension of the rational choice model in terms of two choice axioms.

Proposition 1. *Let μ^θ be the minimal self-progressive extension of the rational choice model with respect to $\rhd$. Then, a choice function $c \in \mu^\theta$ if and only if for each $S \in \Omega$ and $x \in S$,*

$\theta 1.$ *if $c(S) > x$ then $c(S \setminus x) \geq c(S)$, and*

$\theta 2.$ *if $x > c(S)$ then $c(S) \geq c(S \setminus x)$.*[15]

[15] *Independence from preferred alternative* formulated by Masatlioglu et al. (2020) similarly require choice remain unchanged whenever unchosen better options are removed.

Axioms $\theta 1$ and $\theta 2$ require a more valuable (or the same) alternative be chosen whenever we remove alternatives that are less valuable than the chosen one, or add alternatives that are more valuable than the chosen one. Along these lines–in an attempt to unravel the choice overload phenomena–Chernev & Hamilton (2009) experimentally demonstrate that consumers' selection among choice sets is driven by the value of the alternatives constituting the choice sets, in that the smaller choice set is more likely to be selected when the value of the alternatives is high than when it is low. Next, we present the proof of Proposition 1, which demonstrates that Theorem 1 and Lemma 1 may prove useful in obtaining similar results.

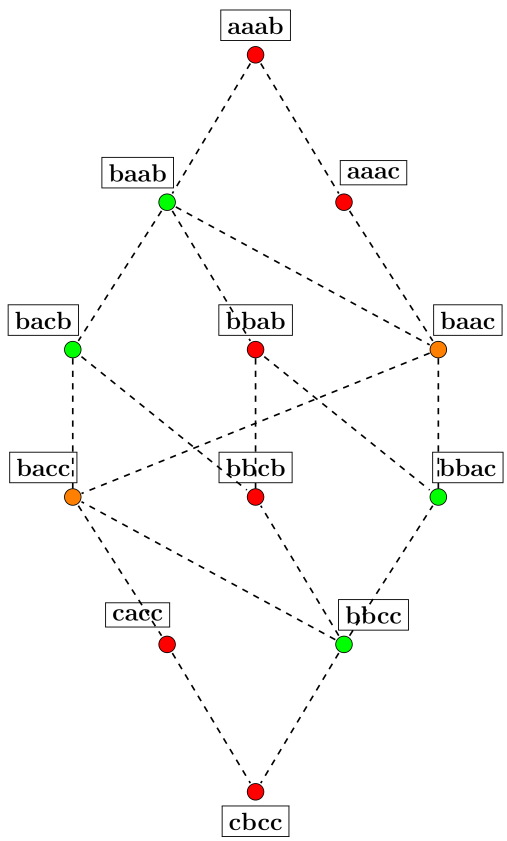

Figure 3: A demonstration of $\langle \mu^\theta, \rhd \rangle$, where $X = \{a, b, c\}$, $\Omega = \{X, \{a, b\}, \{a, c\}, \{b, c\}\}$, and each array specifies the respective choices. The rational choice functions are colored in red, their joins and meets are colored in green, and the additional ones–obtained as a join or meet of the previous ones–are colored in orange.

Proof of Proposition 1. Since μ^θ is self-progressive, it follows from Theorem 1 that $\langle \mu^\theta, \rhd \rangle$ is a lattice such that there is no $\mu \subsetneq \mu^\theta$ that contains every rational choice function and $\langle \mu, \rhd \rangle$ is a lattice. Let μ^* be the choice model comprising choice functions that satisfy $\theta 1$ and $\theta 2$.

We first show that $\mu^\theta \subset \mu^*$. To see this, first note that each rational choice function $c \in \mu^*$, since for each $S \in \Omega$ and $x \in S$, rationality of c implies that $c(S) \neq c(S \setminus x)$ only if $x = c(S)$. Next, we show that $\langle \mu^*, \rhd \rangle$ is a lattice. Let $c^1, c^2 \in \mu^*$ and $c = c^1 \vee c^2$. Then, to see that c satisfies $\mu 1$ and $\mu 2$, assume w.l.o.g. that $c(S) = c^1(S)$. Now, if $c^1(S) > x$ then, since c^1 satisfies $\mu 1$, we have $c^1(S \setminus x) \geq c^1(S)$. It follows that $c(S \setminus x) \geq c(S)$. If $x > c^1(S)$, then $x > c^2(S)$. Since c^1 and c^2 satisfy $\mu 2$, we have $c(S) \geq c(S \setminus x)$. Thus, we conclude that $c^1 \vee c^2 \in \mu^*$. Symmetric arguments show that $c^1 \wedge c^2 \in \mu^*$ as well.

Next, we show that $\mu^* \subset \mu^\theta$. To see this, let $c \in \mu^*$. Since $\langle \mu^\theta, \rhd \rangle$ is a lattice, by Lemma 1, it suffices to show that for each $S, S' \in \Omega$, there exists $c^* \in \mu^\theta$ such that $c^*(S) = c(S)$ and $c^*(S') = c(S')$. Let $S, S' \in \Omega$ such that $c(S) = a$ and $c(S') = a'$. If $a = a'$, then $c(S)$ and $c(S')$ are obtained by maximizing a preference relation that top-ranks a. If $a \neq a'$, then assume w.l.o.g. that $a > a'$. Now, there are two cases.

Case 1: Suppose that $\{a, a'\} \not\subset S \cap S'$. Then, let c_1 be a choice function maximizing a preference relation that top-ranks first a then a', and c_2 be a choice function maximizing a preference relation that top-ranks first a' then a. Next, if $a \notin S'$ then let $c^* = c_1 \vee c_2$, if $a' \notin S$ then let $c^* = c_1 \wedge c_2$. For both cases, $c^*(S) = a$ and $c^*(S') = a'$, and $c^* \in \mu^\theta$ since $\langle \mu^\theta, \rhd \rangle$ is a lattice containing every rational choice function.

Case 2: Suppose that $\{a, a'\} \subset S \cap S'$. First, we show that either (i) there exists $x \in S \setminus S'$ with $x > a$ or (ii) there exists $y \in S' \setminus S$ with $a' > y$. If not, then consider $S \cap S'$. Since $c \in \mu^\theta$, by applying $\theta 1$ for each $x \in S \setminus S'$, we conclude that $c(S \cap S') \geq c(S)$. Next, by applying $\theta 2$ for each $y \in S' \setminus S$, we conclude that $c(S') \geq c(S \cap S')$. Therefore, we must have $a' \geq a$, a contradiction. Thus, we conclude that (i) or (ii) holds.

Suppose that (i) holds. Then, let $c^* = c_1 \wedge c_2$, where c_1 maximizes a preference relation that top-ranks first x then a', and c_2 maximizes a preference relation that top-ranks a. Suppose that (ii) holds. Then, let $c^* = c_1 \vee c_2$, where c_1 maximizes a preference relation that top-ranks first y then a, and c_2 maximizes a preference relation that top-ranks a'. For both cases, $c^*(S) = a$ and $c^*(S') = a'$, and $c^* \in \mu^\theta$ since $\langle \mu^\theta, \rhd \rangle$ is a lattice containing every rational choice function. $\square$

6 Self-progressiveness and supermodular optimization

We provide an optimization-based foundation of self-progressive choice models. In the vein of Example 4, a choice function can be interpreted as the specification of a complete contingent plan that is to be implemented upon observing the set of available alternatives. More generally, imagine that agents in a population consider a lattice of choice functions L and evaluate these choice functions via a common value function V, which can be thought of as the *indirect utility function* associated to the problem of optimally selecting a choice function. Then, each agent adopts a choice function by maximizing V over L. The population is homogeneous in the sense that to adopt a choice function, each agent pays attention to same choice functions and maximizes the same value function. The only source of heterogeneity in the population comes from the presence of multiple maximizers.

Next, to establish a connection to self-progressive choice models, suppose that the value function $V : L \to \mathbb{R}$ is **supermodular**, that is for each $c, c' \in L$, we have

$$V(c \vee c') + V(c \wedge c') \geq V(c) + V(c'). \tag{1}$$

The maximizers of V form a lattice, and thus, by Theorem 1, the choice model consisting of the maximizers of V over L is a self-progressive choice model. In Example 4, L is the set of all choice functions and V is the sum of set contingent utilities, a modular (additive) value function. In our Theorem 2, we provide a converse result showing that every self-progressive choice model μ can be obtained as the maximizer set of a supermodular value function defined over a specific but fairly large domain of choice functions. As for the interpretation of a value function being supermodular, first, note that (1) can be rewritten as $V(c \vee c') - V(c') \geq V(c) - V(c \wedge c')$. Then, the simple intuition behind supermodularity is as follows: The change from $c \wedge c'$ to c corresponds to $>_S$-better alternatives chosen from a family of choice sets; the effect of this change should amplify if the change was made while in another family of choice sets, the chosen alternatives (by c') are $>_S$-better than the ones chosen by $c \wedge c'$.

Next, we describe the special domain of choice functions used in our Theorem 2. We first introduce the notion of *consecutiveness*. A pair of choice functions $c, c' \in \mu$ are **consecutive** if $c \triangleright c'$ and there is no $c'' \in \mu$ that *lies in between*, that is $c \triangleright c'' \triangleright c'$. Now, if two consecutive choice functions are considered via the agents in a population, then it seems reasonable for every choice function that lies in between be considered as well. To formalize this intuition, let $Cons(\mu)$ be the **consecutive hull** of μ that contains each choice function (by definition outside of the model) that lies between two consecutive choice functions in μ. Since $Cons(\mu)$ may not be a lattice in general, let $L(\mu)$ be the smallest lattice that contains $Cons(\mu)$, i.e. $\mu \subset L(\mu)$ and there is no other lattice of choice functions L' such that $Cons(\mu) \subset L' \subset L(\mu)$. Next, we state our result and prove it by referring to three lemmas relegated to the appendix.

Theorem 2. *Let μ be a self-progressive choice model. Then, there exists a supermodular function $V : L(\mu) \to \mathbb{R}$ such that μ is the set of choice functions that maximize V, that is $\mu = \mathrm{argmax}_{c \in L(\mu)} V(c)$.*

Proof. We will construct the desired supermodular function V. Since, by Theorem 1, μ is a lattice, let $\bar{c}(\underline{c})$ be the $\rhd$-best(worst) choice functions in μ. For each $c \in L(\mu)$, consider the associated *better-than set* $B_c = \{Sx : \bar{c}(S) >_S x \geq_S c(S)\}$. For each pair of consecutive choice functions $c, c' \in \mu$, define the associated *difference set* $\lambda = B_{c'} \setminus B_c$. Let Λ be the collection of all such difference sets associated with each consecutive $c, c' \in \mu$. Let $S \in \Omega$ and $x \in S$ such that $\bar{c}(S) \geq_S x \geq_S \underline{c}(S)$. Then, it follows from Lemma 3 that there exists unique $\lambda_{Sx} \in \Lambda$ that contains Sx. Note that $\lambda_{Sx} = \emptyset$ for every $Sx \in \bar{c}$.

Next, let $\lambda, \lambda' \in \Lambda$. Then, λ *precedes* λ', denoted by $\lambda \rightarrow \lambda'$, if for each $c \in \mu$, $\lambda' \subset B_c$ implies that $\lambda \subset B_c$. We define the *predecessor set* $\lambda^{\uparrow}_{Sx}$ of λ_{Sx} as the union of λ_{Sx} and all $\lambda \in \Lambda$ that precede λ_{Sx}, i.e. $\lambda^{\uparrow}_{Sx} = \lambda_{Sx} \cup \{\lambda \in \Lambda : \lambda \rightarrow \lambda_{Sx}\}$. Finally, for each $c \in L(\mu)$ we can define $V(c)$ as the number of pairs Sx in c such that the predecessor set $\lambda^{\uparrow}_{Sx}$ of λ_{Sx} is contained in the better-than set of c, i.e.

$$V(c) = |\{Sx \in c : \lambda^{\uparrow}_{Sx} \subset B_c\}|.$$

Note that $V(c) = \sum_{Sx \in c} \mathbb{1}_c(Sx)$, where for each $Sx \in c$, $\mathbb{1}_c(Sx) = 1$ if $\lambda^{\uparrow}_{Sx} \subset B_c$, and 0 otherwise. Then, for each $c \in L$, we have $V(c) \leq |\Omega|$, and it directly follows from Lemma 4 that $V(c) = |\Omega|$ if and only if $c \in \mu$.

To see that V is supermodular over $L(\mu)$, let $c, c' \in L(\mu)$, and let $Sx \in c \cup c'$. Then, it is sufficient to show that whenever Sx receives a value of 1 by either of $\mathbb{1}_c$ or $\mathbb{1}_{c'}$, then Sx receives a corresponding value of 1 by $\mathbb{1}_{c \vee c'}$ or $\mathbb{1}_{c \wedge c'}$.

Case 1: Suppose that $\mathbb{1}_c(Sx) = 1$ and $\mathbb{1}_{c'}(Sx) = 1$. Then, since $\lambda^{\uparrow}_{Sx} \subset B_c \cap B_{c'}$, we have $\mathbb{1}_{c \vee c'}(Sx) = 1$ and $\mathbb{1}_{c \wedge c'}(Sx) = 1$.

Case 2: Suppose that $\mathbb{1}_c(Sx) = 1$ and $Sx \in c'$, but $\mathbb{1}_{c'}(Sx) = 0$. Then, since $Sx \in c \wedge c'$ and $\lambda^{\uparrow}_{Sx} \subset B_c \subset B_{c \wedge c'}$, we have $\mathbb{1}_{c \wedge c'}(Sx) = 1$.

Case 3: Suppose that $\mathbb{1}_c(Sx) = 1$ and $Sx \notin c'$. Let $c'(S) = y$. If $y >_S x$, then $Sx \in c \wedge c'$, and since $\lambda^{\uparrow}_{Sx} \subset B_{c \wedge c'}$, we have $\mathbb{1}_{c \wedge c'}(Sx) = 1$. Next, suppose that $x >_S y$. Then, we have $Sx \in c \vee c'$, and to conclude that $\mathbb{1}_{c \vee c'}(Sx) = 1$, we show that $\lambda^{\uparrow}_{Sx} \subset B_{c'}$. First, we show that there exists $c^* \in \mu$ that contains Sx. To see this, let $c_1, c_2 \in \mu$ be consecutive choice functions such that $\lambda_{Sx} = B_{c_2} \setminus B_{c_1}$. Since $\lambda_{Sx} \subset B_c$ and $Sx \in c$, for each $Sz \in \lambda_{Sx}$, by definition of B_c, we must have $z \geq_S x$. Therefore, $\lambda_{Sx} = B_{c_2} \setminus B_{c_1}$ implies that $Sx \in c_2$ as well. Thus, we have $Sx \in c^*$ for some $c^* \in \mu$. Then, since $x >_S c'(S)$, it directly follows from Lemma 5 that $\lambda^{\uparrow}_{Sx} \subset B_{c'}$. □

7 Universally self-progressive choice models

We have assumed so far that a partial ordering $\triangleright$ derived from the primitive orderings $\{>_S\}_{S \in \Omega}$ allows for comparison of different choice behaviour, thus derives the unique orderly representation for any aggregate choice behavior consistent with a model. However, one can question if there exist choice models rendering unique orderly representations independent of the primitive orderings. We first formalize this rather stringent requirement, then provide a characterization of the choice models that satisfy it.

Definition. *A choice model μ is **universally self-progressive** if μ is self-progressive with respect to every partial order $\rhd$ obtained from a set of primitive orderings $\{>_S\}_{S\in\Omega}$.*

In our Example 4, we have seen that if a choice model μ can be represented as the set of choice functions maximizing the sum of set contingent utilities, then μ is universally self-progressive. In our next result, we simply observe that the converse is true as well. Then, we present Example 5, demonstrating that this result facilitates identifying universally self-progressive choice models.

Proposition 2. *A choice model μ is universally self-progressive if and only if for each $S \in \Omega$, there exist a set contingent utility function $U(\cdot, S)$ such that μ is the set of choice functions that maximize their sum, that is $\mu = \text{argmax}_{c\in\mathcal{C}} \sum_{S\in\Omega} U(c(S), S)$.*

Proof. As for the *only if* part, let μ be a universally–self-progressive choice model. Then, we first show that μ is *convex*: for each $c_1, c_2 \in \mu$, if $c(S) \in \{c_1(S), c_2(S)\}$ for every $S \in \Omega$, then $c \in \mu$. By contradiction, suppose that $c \notin \mu$. Then, for each $S \in \Omega$, define the primitive ordering $>_S$ such that $c(S)$ is highest-ranked. Thus, we have $c = c_1 \vee c_2$, and $\langle \mu, \rhd \rangle$ is a not a lattice. By Theorem 1, this contradicts that μ is universally self-progressive. We define the set contingent utilities as follows: for each $S \in \Omega$, $U(x, S) = 1$ if there exists $c \in \mu$ with $c(S) = x$, and $U(x, S) = 0$ otherwise. Since μ is convex, a choice function $c \in \mu$ if and only if for each $S \in \Omega$, there exists $c' \in \mu$ with $c'(S) = c(S)$. It follows that μ is the set of choice functions that maximize $\sum_{S\in\Omega} U(c(S), S)$. □

Example 5. *Kalai, Rubinstein & Spiegler (2002)* Let $\{\succ_k\}_{k=1}^K$ be a K-tuple of strict preference relations on X. Then, a choice function $c \in \mu$ if for each $S \in \Omega$, the alternative $c(S)$ is the $\succ_k$-maximal one in S for some k. To see that μ is universally self-progressive, define $U(x, S) = 1$ if x is the $\succ_k$-maximal alternative in S for some k; and $U(x, S) = 0$

otherwise. It follows that μ is the set of choice functions that maximize $\sum_{S\in\Omega} U(c(S), S)$. However, every universally self-progressive choice model is not representable in this way. To see this, suppose that $U(x, S) = 1$ and $U(x, T) = 0$ for a pair of choice sets S and T such that $x \in T \subset S$. Then, there exists a strict preference relation $\succ_k$ such that x is the $\succ_k$-maximal alternative in S. Therefore, there exists $c \in \mu$ with $c(T) = x$ contradicting that c maximizes $\sum_{S\in\Omega} U(c(S), S)$.

Theorem 2 provides for a recipe to restrict or extend a choice model for universal self-progression, while reflecting its demanding nature. To see this, consider a choice model μ consisting of two choice functions rationalized by maximizing preference relations $\succ_1$ and $\succ_2$. For given primitive orderings, we can extend μ into a self-progressive model by adding at most two choice functions. In contrast, to extend μ to be universally self-progressive we must add every choice function choosing the $\succ_1$- or $\succ_2$-maximal alternative in each choice set. More generally–assuming that the choice domain contains every choice set–in order to extend rational choice model into a universally self-progressive one, we must add every choice function.

8 Final comments

We have dealt with a different approach to analyze heterogeneity in the aggregate choice behavior of a population. We take choice models as our primitive objects, thus leaving the details of the agents' choices unspecified. Our analysis presents testable and optimization-based foundations of choice models that guarantee a unique orderly representation for each aggregate choice behavior consistent with the model. We are motivated by the conjecture that using the so-called self-progressive choice models would facilitate organization and analysis of random choice data. In this vein, we observe that they are in a one-to-one correspondence with familiar algebraic structures called lattices, indicating that self-progressive models are not limited to comparable choice functions, but allow for a generality that is economically significant. As an advantage of our model-free approach, our results provide guides to restrict or extend any choice model as to be (universally) self-progressive. We demonstrated this by characterizing the minimal self-progressive extension of the rational choice model.

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9 Appendix: Lemmas

We present and prove the three lemmas used in proving Theorem 2. We first prove an auxiliary result that we will use in proving these lemmas. For brevity, from now on, we denote $L(\mu)$ by L.

Lemma 2. *Let $c, c' \in \mu$ be a pair of consecutive choice functions and let $\lambda = B_{c'} \setminus B_c$. Then, for each $c'' \in \mu$, (i) if there exists $Sx \in B_{c''} \cap \lambda$, then $c' \trianglerighteq c \wedge c''$; (ii) if there exists $Sx \in \lambda \setminus B_{c''}$, then $c' \vee c'' \trianglerighteq c$.*

Proof. (i) Let $c'' \in \mu$ such that $Sx \in B_{c''} \cap \lambda$. Then, let $c^* = c \wedge (c' \vee c'')$. Since μ is a lattice, $c^* \in \mu$, and by definition, $c \trianglerighteq c^* \trianglerighteq c'$. Since c and c' are consecutive, it follows that $c^* \in \{c, c'\}$. Moreover, since $Sx \in B_{c''} \cap \lambda$, we have $c(S) >_S x \geq_S (c' \vee c'')(S)$. Since $c' \vee c'' \trianglerighteq c^*$, we conclude that $c^* = c'$. Then,by distributivity, $c' = c' \vee (c \wedge c'')$, which implies that $c' \trianglerighteq c \wedge c''$.

(ii) Let $c'' \in \mu$ such that there exists $Sx \in \lambda \setminus B_{c''}$. Then, let $c^* = c' \vee (c \wedge c'')$. Since μ is a lattice, $c^* \in \mu$, and by definition, $c \trianglerighteq c^* \trianglerighteq c'$. Since c and c' are consecutive, it follows that $c^* \in \{c, c'\}$. Since $Sx \in \lambda \setminus B_{c''}$, we have $(c \wedge c'')(S) >_S x \geq_S c'(S)$. Since $c^* \trianglerighteq c' \wedge c''$, we conclude that $c^* = c$. Then, by distributivity, we have $c = c \wedge (c' \vee c'')$, which implies that $c' \vee c'' \trianglerighteq c$. □

Lemma 3. *For each $\lambda_1, \lambda_2 \in \Lambda$, if $\lambda_1 \cap \lambda_2 \neq \emptyset$, then $\lambda_1 = \lambda_2$.*

Proof. Suppose that $Sx \in \lambda_1 \cap \lambda_2$, and let $c_1, c_1' \in \mu$ and $c_2, c_2' \in \mu$ be pairs of consecutive choice functions associated with λ_1 and λ_2. We conclude that $\lambda_1 = \lambda_2$ in two steps.

Step 1: We show that $\lambda_2 \setminus B_{c_1} = \lambda_1 \setminus B_{c_2}$. First, to see that $c_1 \wedge c_2' = c_1' \wedge c_2$, consider c_1, c_1' and c_2'. Since $Sx \in \lambda_1 \cap \lambda_2$, we have $Sx \in B_{c_2'} \cap \lambda_1$. Then, it follows from Lemma

2 (i) that $c_1' \trianglerighteq c_1 \wedge c_2'$. Now, if we take the meet of both sides with c_2, then we obtain that $c_1' \wedge c_2 \trianglerighteq c_1 \wedge c_2' \wedge c_2$. Since $c_2 \trianglerighteq c_2'$, it follows that $c_1' \wedge c_2 \trianglerighteq c_1 \wedge c_2'$. Symmetrically, when we start with c_2, c_2' and c_1', then we obtain that $c_1 \wedge c_2' \trianglerighteq c_1' \wedge c_2$. Thus, we conclude that $c_1 \wedge c_2' = c_1' \wedge c_2$. It follows that $B_{c_1} \cup B_{c_2'} = B_{c_1'} \cup B_{c_2}$. Finally, if we exclude the set $B_{c_1} \cup B_{c_2}$ from both sides of this equality, then we obtain $\lambda_2 \setminus B_{c_1} = \lambda_1 \setminus B_{c_2}$.

Step 2: We show that $\lambda_2 \cap B_{c_1} = \lambda_1 \cap B_{c_2} = \emptyset$. First, to see that $c_1' \vee c_2 = c_1 \vee c_2'$, consider c_1, c_1' and c_2. Since $Sx \in \lambda_1 \cap \lambda_2$, we have $Sx \in \lambda_1 \setminus B_{c_2}$. Then, it follows from Lemma 2 (ii) that $c_1' \vee c_2 \trianglerighteq c_1$. Now, if we take the join of both sides with c_2', then we obtain that $c_1' \vee c_2 \vee c_2' \trianglerighteq c_1 \vee c_2'$. Since $c_2 \trianglerighteq c_2'$, it follows that $c_1' \vee c_2 \trianglerighteq c_1 \vee c_2'$. Symmetrically, when we start with c_2, c_2' and c_1, we obtain that $c_1 \vee c_2' \trianglerighteq c_1' \vee c_2$. Thus, we conclude that $c_1' \vee c_2 = c_1 \vee c_2'$.

Since $c_1' \vee c_2 = c_1 \vee c_2'$, we have $B_{c_1'} \cap B_{c_2} = B_{c_1} \cap B_{c_2'}$. Next, when we exclude B_{c_1} from both sides of this equality, we obtain that $\lambda_1 \cap (B_{c_2} \setminus B_{c_1}) = \emptyset$. Since $\lambda_1 = B_{c_1'} \setminus B_{c_1}$, we also have $\lambda_1 \cap B_{c_1} = \emptyset$. It follows that $\lambda_1 \cap B_{c_2} = \emptyset$. Symmetrically, when we start by excluding B_{c_2} from both sides of $B_{c_1'} \cap B_{c_2} = B_{c_1} \cap B_{c_2'}$, we conclude that $\lambda_2 \cap B_{c_1} = \emptyset$. $\square$

Let $\lambda, \lambda' \in \Lambda$. Then, λ **precedes** λ', denoted by $\lambda \rightarrow \lambda'$, if for each $c \in \mu$, $\lambda' \subset B_c$ implies that $\lambda \subset B_c$. Let $S \in \Omega$ and $x \in X$ such that $\bar{c}(S) \geq_S x \geq_S \underline{c}(S)$. Then, we define $\lambda_{Sx}^{\uparrow}$ as the set of $\lambda \in \Lambda$ that precede λ_{Sx}, i.e. $\lambda_{Sx}^{\uparrow} = \lambda_{Sx} \cup \{\lambda \in \Lambda : \lambda \rightarrow \lambda_{Sx}\}$. We assume that $\lambda_{Sx}^{\uparrow} = \emptyset$ for each $Sx \in \bar{c}$.

Lemma 4. *Let $c \in L$. Then, $c \in \mu$ if and only if $\lambda_{Sx}^{\uparrow} \subset B_c$ for each $Sx \in c$.*

Proof. (*If part*) First, note that $B_c = \bigcup_{Sx \in c} \lambda_{Sx}^{\uparrow}$, since for each $Sz \in B_c$, $\lambda_{Sz} \rightarrow \lambda_{Sx}$ for $x = c(S)$ implies that $\lambda_{Sz} \subset \lambda_{Sx}^{\uparrow}$, which is contained in B_c. Next, let $Sx \in c$ and c^{Sx} be the $\triangleright$-best choice function in μ such that x is chosen in S. Then, by definition of $\lambda_{Sx}^{\uparrow}$, we have $B_{c^{Sx}} = \lambda^{\uparrow}_{Sx}$. Next, let $c^* = \bigwedge_{Sx \in c} c^{Sx}$. Since μ is a lattice, $c^* \in \mu$. Moreover, we have $B_{c^*} = \bigcup_{Sx \in c} \lambda^{\uparrow}_{Sx}$, which implies that $c^* = c$, and thus $c \in \mu$.

(*Only if part*) We first show that $\lambda_{Sx} \subset B_c$. To see this, let $Sx \in c$ and $c_1, c_2 \in \mu$ be consecutive choice functions such that $\lambda_{Sx} = B_{c_2} \setminus B_{c_1}$. Then, consider c_1, c_2 and c. Since $c \in \mu$ and $Sx \in B_c \cap \lambda_{Sx}$, it follows from Lemma 2 (i) that $c_2 \trianglerighteq c_1 \wedge c$. Therefore, $B_{c_2} \subset B_{c_1} \cup B_c$. Since $B_{c_1} \subset B_{c_2}$, it directly follows that $B_{c_2} \setminus B_{c_1} \subset B_c$. Thus, we conclude that $\lambda_{Sx} \subset B_c$. Next, since $Sx \in c$ and $c \in \mu$, it follows from the definition of $\lambda^{\uparrow}_{Sx}$, that $\lambda^{\uparrow}_{Sx} \subset B_c$. □

Lemma 5. *Let $c \in L$ and $Sx \in c^*$ for some $c^* \in \mu$. If $x >_S c(S)$, then $\lambda^{\uparrow}_{Sx} \subset B_c$.*

Proof. Step 1: We show that the assertion holds for each $c \in Cons(\mu)$. To see this, suppose that $c_1 \trianglerighteq c \trianglerighteq c_2$ for consecutive choice functions $c_1, c_2 \in \mu$. Then, we show that $x \geq_S c_1(S)$. By contradiction, suppose that $c_1(S) >_S x$. Then, let $\lambda = B_{c_2} \setminus B_{c_1}$, and consider c_1, c_2 and c^*. Since $Sx \in B_{c^*} \cap \lambda$, it follows from Lemma 2 (i) that $c_2 \trianglerighteq c_1 \wedge c^*$. However, this implies that $c_2(S) \geq_S x >_S c(S)$, contradicting that $c \trianglerighteq c_2$. Thus, we conclude that $x \geq_S c_1(S)$. Therefore, $(c^* \vee c_1)(S) = x$. Moreover, $c^* \vee c_1 \in \mu$, since μ is a lattice. Then, it follows from Lemma 4 that $\lambda^{\uparrow}_{Sx} \subset B_{c^* \vee c_1}$. Finally, $c^* \vee c_1 \triangleright c$ implies that $B_{c^* \vee c_1} \subset B_c$, and thus we conclude that $\lambda^{\uparrow}_{Sx} \subset B_c$.

Step 2: Let $J \subset L$ so that the assertion holds for each $c \in J$. Then, we show that the assertion holds for $c \vee c'$ and $c \wedge c'$ where $c, c' \in J$. First, consider $c \vee c'$. If $x >_S (c \vee c')(S)$, then $x >_S c(S)$ and $x >_S c'(S)$. Since the assertion holds for c and c', it follows that $\lambda^{\uparrow}_{Sx} \subset B_c \cap B_{c'} = B_{c \vee c'}$. Next, consider $c \wedge c'$. If $x >_S (c \wedge c')(S)$, then $x >_S c(S)$ or $x >_S c'(S)$. Since the assertion holds for c and c', it follows that $\lambda^{\uparrow}_{Sx} \subset B_c \cup B_{c'} = B_{c \wedge c'}$.
Step 3: By Step 1, the assertion holds for each $c \in Cons(\mu)$. Then, by Step 2, the assertion holds for each $c \in L$ that is obtained as a join or meet of the choice functions

in $Cons(\mu)$. By proceeding recursively to add the joins and meets, we obtain a finite sequence of sets that satisfy the assertion. This set sequence converges to L, since L is the smallest sublattice of $\mathcal{C}$ that contains $Cons(\mu)$. Thus, we conclude that the assertion holds for each choice function in L. $\square$