Dividing a Commons under Tight Guarantees

Anna Bogomolnaia[*] and Hervé Moulin[†‡]

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Abstract

The exploitation of common property resources (to divide private commodities, share a cooperative production function, cover jointly liable costs etc..) continues to inspire multiple context-dependent definitions of fairness lacking the generality and reach of concepts like efficiency and incentive compatibility.

We introduce a new principle to manage any sort of commons, building on the profoundly appealing Lockean prescription that each agent should receive "the fruit of their own labor", aka the "self-ownership" viewpoint. What could this mean when the participants' interactions when they consume the resources makes it difficult to disentangle their individual contributions?

Our (multivalued) answer is to look for tight approximations of the decentralised Lockean ideal, limiting for each agent, from above and below, the impact of other agents on their own allocation.

We work out the consequences of this approach in a context-free model of the commons as a production function from individual inputs to an output that we must divide. Our approximation viewpoint is mathematically tractable because we assume a freely transferable output and a simple one dimensional input from each stakeholder. Choosing one pair in the infinite menu of tight decentralised approximations puts sharp but far from deterministic bounds on the final distribution of the output: it is a precise normative position that still leaves room for direct negotiations or the choice of a deterministic sharing rule.

We describe in detail these design options in several iconic problems where each tight pair of guarantees has a clear normative meaning: the allocation of indivisible goods or costly chores, cost sharing of a public facility and the exploitation of a commons with substitute or complementary inputs. The corresponding benefit or cost functions are all sub- or super-modular, and for this class we characterise the set of minimal upper and maximal lower guarantees in all two agent problems.

JEL classification codes: D6, D63

1 Introduction

The exploitation of common property resources (to divide private commodities, share a cooperative production function, assign costs to jointly liable agents, etc..) continues to inspire multiple context-dependent tests of fairness lacking the generality and reach of the two other concepts at the heart of the mechanism design program, efficiency and incentive compatibility.

[*]University of Glasgow and CNRS, Université Paris 1

[†]University of Glasgow

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Such properties include No Envy ([11], [36]), immunity to objections in an appropriate cooperative game ([9], [30]), a virtual decentralisation property ([13], [12]), solidarity in response to an improvement of the resources ([21], [22], [34]), Consistency with respect to the departure of some participants, ([38], [33]), and Welfare Bounds ([35], [26], [10], [6]).

Taking a step toward a general context-free definition of fairness, we introduce a new principle for the management of a commons inspired by the appealing simple prescription that each agent should receive "the fruit of their own labor" ([17]), in modern terminology the "self-ownership" viewpoint. We assume that our agents are fully responsible for their own type[1] (representing skills, effort, endowment, preferences, etc..) and propose an interpretation of self-ownership with much bite, even if it only places lower and upper bounds on the shares an agent with a given type can receive, independently of other agents' types.

In the classic model of the commons as a production function where each agent enters an input (their type) and receive a share of the output, our starting observation is that the Lockean principle is operational *only if* we can unambiguously separate what each agent's input contributes to the total output, or vice versa what is the cost of each individual demand of output. This is feasible if, and in fact only if, the production function has constant returns to scale (CRS). Then it is as if each agent owns the technology "privately": by using it she does not affect how others can use it, so there is no need to keep track of the various inputs because they do not affect my fair share: whatever I produce, no more no less. Indeed this solution for the CRS commons is often taken as a primitive requirement in the axiomatic discussion of the cooperative production problem (e. g. [26]); it can also be deduced from its incentive properties ([25]).

The same argument applies to the allocation of any resources in common property: ensuring that my allocation depends only on my own type, is not affected by other stakeholders' types, eliminates the transactional costs of negotiating and implementing a particular rule with its own bias on the way it resolves complex interpersonal externalities. Taking this independence property as a normative goal, and combining it with the horizontal equity principle that equal should be treated equally, forces the Lockean solution for a CRS technology: *if* everyone else is my clone, equal treatment gives us all the same share, and that share does not change when their types are arbitrarily different.

But exploiting a commons is a problem only to the extent that the returns of the technology are *not* constant, therefore this desirable independence property can only be approximated: if the impact of other agents' types on my own share cannot be eliminated we will minimise it as much as possible.

We work out the consequences of this principle in a thoroughly abstract model of the commons as a function $\mathcal{W}$ inputting a n-profile x of individual *types* x_i and returning the output $\mathcal{W}(x)$ to be divided among the stakeholders. For tractability we assume that the output is freely transferable accross agents, typically by cash transfers in our examples. Apart from some regularity properties, the only critical assumption is that $\mathcal{W}$ is symmetric in its n variables, which makes sense of the equal treatment property.

The function $\mathcal{W}$ captures all we need to know about efficiency. We also set aside any issue of incentive compatibility to concentrate on fairness only.

Our model accommodates a great variety of familiar fair division puzzles with completely different interpretations of the types and the output $\mathcal{W}(x)$.

The agents may share the cost $\mathcal{W}(x) = \max_i\{x_i\}$ of a public capacity and x_i is the cost of the capacity agent i needs ([17]). If x_i is i's willingness to pay for an indivisible good in

[1] So we do not enter the political philosophy controversy around self-ownership, defended by Nozick ([27]) but criticised for its potentially libertarian implications by Roemer ([29]) and Cohen ([7]).

common property, the same function $\mathcal{W}$ captures the efficient surplus to be shared by means of cash compensations (section 2).

In the classic commons (section 5) the types x_i are inputs in the function F: the return is $\mathcal{W}(x) = F(\sum_i x_i)$ if they are substitutable like investments or efforts; or $F(\Pi_i x_i)$, $F(\min_i\{x_i\})$ if they are complementary. Alternatively the agents pool individual orders x_i purchased at the bulk rate $C(\sum_i x_i)$; or x_i is the size of the job agent i sends to a server and $C(\sum_i x_i)$ is the total delay they must share ([31]).

Other examples in section 6 include $\mathcal{W}(x) = \max_i\{x_i\} - \min_i\{x_i\}$: sharing the cost of delivering mail to all agents on a line; and $\mathcal{W}(x) = F(median\{x_i\})$: production of the output $F(z)$ requires at least a majority of agents to input the effort z.

The complete absence of interpersonal externalities in $\mathcal{W}$ is the property of *additive separability*: $\mathcal{W}(x) = \sum_i w(x_i)$ for all profiles x, for which the Lockean argument above makes $w(x_i)$ the correct share for agent i of type x_i. For any other function $\mathcal{W}$ the influence on agent i's share of some of the other agents' types x_j cannot be avoided. Fixing her type x_i and varying all other types, agent i's share varies in some interval $I(x_i)$: our goal is to choose these intervals as small as permitted by the feasibility constraint that the sum of the shares is $\mathcal{W}(x)$. Critically, this design question is independent not only of the interpretation of types and output but also of the particular sharing rule or bargaining process by which the agents will arrive at a precise division of the output.

We write $I(x_i) = [g^-(x_i), g^+(x_i)]$ and call $g^-(x_i)$ type x_i's *lower guarantee*: if $\mathcal{W}(x)$ represents a surplus $g^-(x_i)$ is agent i's worst case share, but her best case if $\mathcal{W}(x)$ is a cost; similarly $g^+(x_i)$ is type x_i's *upper guarantee.* The feasibility constraint is the following system of inequalities on the pair (g^-, g^+): for $x = (x_1, \cdots, x_n) \in \mathcal{X}^{[n]}$

$$\sum_1^n g^-(x_i) \leq \mathcal{W}(x) \leq \sum_1^n g^+(x_i) \tag{1}$$

The lower guarantee g^- is *tight* if we can not increase $g^-(x_i)$ at any type x_i without bringing a violation of the left side (LS) of system (1) at some n-profile x containing x_i. Similarly g^+ on the right side (RS) of (1) is a tight upper guarantee if decreasing $g^+(x_i)$ at any x_i violates these inequalities at some profile x containing x_i.

We run into a hard and, as far as we know, original mathematical question: what are its tight additively separable approximations, from below and above, of a given function $\mathcal{W}$? This question is difficult, even when the types x_i vary in a real interval, which is the case in all the examples in section 2, 5 and 6 as well as in the general representation Theorem 7.1 for two person problems.

We find that the choice of a particular pair (g^-, g^+) of tight guarantees is very consequential: although for most types x_i the inequality $g^-(x_i) < g^+(x_i)$ leaves room for choice, a typical but not systematic pattern, first illustrated in section 2 for the simple function $\mathcal{W}(x) = \max_i\{x_i\}$, is that each tight pair has a single *benchmark* type $\widehat{x}_i$ for which $g^-(\widehat{x}_i) = g^+(\widehat{x}_i)$, the share of this type is independent of other agents' types.

Tight guarantees are a coarse form of mechanism design, compatible with unscripted negotiations where participants commit to the decision taken by the manager inside these bounds if no compromise is reached. If instead the division of $\mathcal{W}(x)$ is the outcome of playing in a mechanism with fully scripted messages, enforcing tight bounds on individual shares promotes participation by minimising the risk of playing badly.

A deterministic sharing rule speciies at each profile of types $x \in \mathcal{X}^{[n]}$ every agent i's share $\varphi_i(x)$ of $\mathcal{W}(x)$. The rule generates its own guarantees that may or may not be tight: selecting a rule that does eliminates many familiar rules and support others. In the classic model of the commons where $\mathcal{W}(x) = F(x_N)$ (with the notation $x_S = \sum_{i \in S} x_i$) and F is either convex or concave, this tests is failed by three well known rules: Average return (shares proportional to types), Shapley value (of the stand alone game $v(S) = F(x_S)$ for $S \subseteq N$) and Marginal pricing (i's share is $\frac{dF}{dx}(x_N)x_i$ plus a term independent of i); only the Serial rule passes the test, even for the more general class of modular functions $\mathcal{W}$: Proposition 4.3.

However, given any pair of tight guarantees it is easy to construct ad hoc sharing rules to implementing it by simple extrapolations of the guarantees, or by "trimming" an arbitrary sharing rule when its shares violate the guarantees (Lemma 3.3).

contents of the next sections Section 3 introduces the model for a general function $\mathcal{W}$ and domain $\mathcal{X}$, lists various topological properties, and two critical Lipschitz and differentiability properties that tight guarantees inherit from the function $\mathcal{W}$. The long list of technical Lemmas can be skipped by the reader impatient to discover the implications of system (1) in the concrete fair division problems of sections 2, 4, 5 and 6.

Section 4 discusses general super or submodular functions $\mathcal{W}$.[2] We identify three tight guarantees common to all modular functions. First the unanimity share $una(x_i) = \frac{1}{n}\mathcal{W}(x_i, \cdots, x_i)$ is the only tight guarantee on one side of (1) (Proposition 4.1). On the other side the two *incremental* shares adapt to our model the *stand alone* share of an agent using the commons without without sharing it with anyone else.

The commons $\mathcal{W}(x) = F(x_N)$ where F is convex or concave is the subject of section 5. We do not crack the full set of tight guarantees on the other side of the unanimity one but identify two interesting subsets, both of them linking the two incremental guarantees. The first one has only $(n-2)$ guarantees of the *stand alone* type: Proposition 5.1. The second set is a continuous line, containing most of the tangents to the graph of the unanimity function: Propositions 5.2.

Sections 6 introduces the rich class of *rank-separable* functions $\mathcal{W}$ of the form $\mathcal{W}(x) = \sum_{[n]} w_k(x^k)$ where $(x^k)_1^n$ is the order statistics of $(x_i)_1^n$. Theorem 6.1 fully characterise the solutions of (1) for the modular and rank separable functions: on the other side of the unanimity guarantee, all tight guarantees take a generalised stand alone form, and their set has dimension $(n-1)$. Applications include sharing the cost of connecting the agents distributed on the line.

Theorem 7.1 in section 7 solves system (1) for all *two agent* problems with a strictly super or submodular function $\mathcal{W}$. The dimension of the set of solutions is a large infinite.

Section 8 collects two open questions and some take home points; section 9 is an Appendix collecting several long or minor proofs.

fair shares in the literature For the fair division of private commodities, the role of endogenous fair shares is central to the cake cutting model ([32], [14]), the division of a bundle ω of private Arrow Debreu (AD) commodities ([36], [34]) or of indivisible items with cash transfers ([33], [2]). Like here the *unanimity* welfare or utility plays a key role: with convex AD preferences it is the welfare at the allocation $\frac{1}{n}\omega$ and with additive utilities over the cake it is that of a share worth $\frac{1}{n}$-th of the whole cake. But very little is known about tight guarantees in cake division with non additive utilities and for AD commodities with non convex preferences ([4]).

When the shared resource is a production function the oldest concept of fair share is the stand alone utility mentioned above: depending upon the returns to scale it can be a lower or an upper bound on welfare. The joint discussion of the unanimity and stand alone bounds on either side of the Pareto frontier is one of the first themes in the axiomatic discussion of cooperative production: for instance [23], Chapter 5 in [24], [26], and [37].

[2]Loosely speaking this means that the sign of $\partial_{ij}\mathcal{W}$ is constant. So these functions meet "one half" of additive separability characterised by $\partial_{ij}\mathcal{W}(x) \equiv 0$.

To understand how difficult the search for tight guarantees can be a case in point is the allocation of indivisible items (good or bad) with no cash transfers or randomisation, a favourite fair division topic in the last fifteen years or so ([16]). The *MaxMinShare* (MMS) adapts the concept of unanimity welfare to the indivisibility constraints: it is the utility of my worst share in the best n-partition of the items I can choose ([6]). But even with additive utilities over the items the MMS is hard to compute as well as unfeasible in some (very rare) utility profiles ([28]). A blunt $\frac{3}{4}$ fraction of the MMS is a feasible lower guarantee on welfare ([1]), but it is not tight; it is not clear that the exact largest feasible fraction of MMS is a tight guarantee. Alternative concepts of fair share in the additive case include the *any price share* ([3]) even less feasible than the MMS but much easier to compute, and, if we upper bound the relative weight of any item, Hill's bound both feasible and easy to compute ([8], [20], [15]) and more.

Probabilistic voting is another problem where we have looked for a "fair share of welfare". There is an easy canonical tight lower guarantee when preferences are dichotomic: an outcome I like is chosen with a probability no less than $\max\{\frac{1}{n}, \frac{1}{m}\}$ where m is the number of deterministic outcomes, but the search is much harder with general preferences and [5] offers only partial results.

Contrasting with these challenging fair division models, our model is mathematically simpler and allows much versatile examples and interpretations. But we rely heavily on the assumption, typically absent in the literature just reviewed, that utility is transferable via some numeraire like cash payments.

2 A canonical example

Example 2.1 *sharing the cost of a capacity* ([17])

The n agents share a public facility (canal, runway...) adjusted to their different needs (for a canal more or less wide or deep, for a short or long runway...). The cost of building enough capacity to serve the needs of agent i is x_i; the cost of serving everyone is $\mathcal{W}(x) = \max_{i\in[n]}\{x_i\}$, that must be divided in n shares y_i s. t. $\sum_{[n]} y_i = \max_{i\in[n]}\{x_i\}$. The range of possible individual needs x_i is the interval $[L, H]$ where $0 < L < H$.

The unanimity share $una(x_i) = \frac{1}{n}x_i$ meets the LS of system (1) ($\sum_{[n]} \frac{1}{n}x_i \leq \max_{i\in[n]}\{x_i\}$) and every lower guarantee g^- is bounded above by una (apply (1) to a unanimous profile). So una is the *only* tight guarantee on that side: agent i should never pay less than her fair share of the capacity she needs.

We generate two solutions of the RS in (1), not necessarily tight, by computing the worst case (largest cost share) of two very simple rules: Equal-Split $\varphi_i^{egal}(x) = \frac{1}{n}\max_{j\in[n]}\{x_j\}$ ignoring difference in individual needs, and Proportional $\varphi_i^{pro}(x) = \frac{x_i}{x_N}\max_{j\in[n]}\{x_j\}$ (recall the notation $x_N = \sum_{j\in[n]} x_j$) focusing on these needs in the way already suggested by Aristotle (and well defined everywhere because $L > 0$).

Computing $g^-(x_i) = \min_{x_{-i}}\{\varphi_i(x_i; x_{-i})\}$ and $g^+(x_i) = \min_{x_{-i}}\{\varphi_i(x_i; x_{-i})\}$ is easy for the former and takes a little longer for the latter

$$g^-_{egal}(x_i) = \frac{1}{n}x_i \;;\; g^+_{egal}(x_i) = \frac{1}{n}H$$

$$g^-_{pro}(x_i) = \frac{x_i^2}{x_i + (n-1)H}; g^+_{pro}(x_i) = \max\{\frac{x_i^2}{x_i + (n-1)L}, \frac{Hx_i}{x_i + H + (n-2)L}\} \tag{2}$$

So $g^-_{egal} = una$ is tight; this is true as well for g^+_{egal}: if the function f is s.t. $f(x_i) \leq \frac{1}{n}H$ for all x_i and this inequality is strict at some x_i^*, then it violates the RS of (1) at the x_i^* unanimity profile.

By contrast neither g^-_{pro} nor $g^+_{pro}$is tight. This is clear for g^-_{pro} as $g^-_{pro}(x_i) < g^-_{egal}(x_i)$ for all x_i except H. Now g^+_{pro} is not bounded below by g^+_{egal} but by another tight upper guarantee see Remark 2.1 after Proposition 2.1 below.

In our first result we use the notation $z_+ = \max\{z, 0\}$.

Proposition 2.1: *The minimal upper guarantees g^+_p of $\mathcal{W}(x) = \max_{1\leq i\leq n}\{x_i\}$ are parametrised by a benchmark type $p \in [L, H]$ as follows:* for $x_i \in [L, H]$

$$g^+_p(x_i) = \frac{1}{n}p + (x_i - p)_+ \tag{3}$$

Proof[3]: Assume g^+ is a tight upper guarantee and set $p = ng^+(L)$. At the unanimous profile of L-s the RS in (1) implies $p \geq L$. Tightness implies that g^+ increases weakly (Lemma 3.4) so $g^+(x_i) \geq \frac{1}{n}p$ for all x_i; moreover if $p > H$ we have $g^+(x_i) > g^+_{egal}(x_i)$ everywhere and g^+ is not tight; so $p \leq H$.

Inequality (1) applied to x_i and $n-1$ types L gives $g^+(x_i) \geq x_i - \frac{n-1}{n}p$; combining this with $g^+(x_i) \geq \frac{1}{n}p$ gives $g^+ \geq g^+_p$. To check finally that g^+_p meets the right inequalities in (1) is routine. ■

Remark 2.1. *We let the reader check that the upper guarantee g^+_{pro}* in (2) is dominated by $g^+_{\overline{p}}$ for $\overline{p} = \frac{nHL}{H+(n-1)L}$.

Remark 2.2. *The fact that in this example the Equal-Split rule generates a tight pair of guarantees is an anomaly. In general this rule is not even compatible with our interpretation of self-ownership for separably additive functions $\mathcal{W}$. Moreover if $\mathcal{W}$ is strictly increasing and modular, it is easy to check that g^-_{egal} and g^+_{egal} are dominated respectively by the left and right incremental guarantees (section 4.2).*

The end-points at $p = H$ and $p = L$ are the egalitarian $g^+_H = g^+_{egal}$ and the following guarantee g^+_L denoted g^+_{inc} and called *incremental* in section 4.2:

$$g^+_{inc}(x_i) = x_i - \frac{n-1}{n}L$$

It computes type x_i's worst cost share by assuming that every other agent demands the benchmark capacity L and charging i the full incremental cost $x_i - L$ on top of his fair share of L.

The normative choice between g^+_{inc} and g^+_{egal} is stark. The pair of guarantees (una, g^+_{inc}) implies that a type L *always* pays $\frac{1}{n}L$ (as $una(L) = g^+_{inc}(L)$), while under (una, g^+_{egal}) she can pay as much as $\frac{1}{n}H$; vice versa a type H *always* pays $\frac{1}{n}H$ under the latter pair and as much as H (if $L = 0$) under the former. Agents with small needs prefer g^+_{inc} to g^+_{egal}, and vice versa for agents with large needs.

The *serial sharing rule* (proposed by ([17]) for this problem) implements the tight pair (una, g^+_{inc}). Order the agents by increasing type, then charge $y_1 = \frac{1}{n}L + \frac{1}{n}(x_i - L), y_2 = y_1 + \frac{1}{n-1}(x_2 - x_1)$, etc.. We omit the easy proof, a special case of Proposition 4.3.[4]

The pair (una, g^+_p) is a moving compromise between the two previous ones. The benchmark type p always pays $\frac{1}{n}p$, the worst cost share for each type below p, similar to the egalitarian upper guarantee, while types abobe p can pay as much as $x_i - \frac{n-1}{n}p$, like an incremental share starting from p.

[3] The result is a special case of Theorem 6.1, but this redundant quick proof is much easier to follow.

[4] In section 4.2 we generalise the guarantee g^+_{inc} to the entire class of modular functions $\mathcal{W}$, and in section 4.3 the serial rule for one-dimensional types,

Note that In certain profiles of types the pair of guarantee determines the entire set of shares: say $x_{i^*} \geq p$ and $x_j \leq p$ for all $j \neq i^*$, then $y_{i^*} = x_{i^*} - \frac{n-1}{n}p$ and $y_j = \frac{1}{n}p$ for $j \neq i^*$.

Remark 2.3 *A true convex combination of g^+_{inc} and g^+_{egal} is another upper guarantee, but it is never tight. One checks easily that for any $\lambda \in]0,1[$ we have:* for $x_1 \in]L, H[$

$$\{\lambda g^+_{inc} + (1-\lambda) g^+_{ega}\}(x_1) > g^+_{\lambda L + (1-\lambda)H}(x_1)$$

The Corollary to Lemma 3.9 generalises this observation.

Example 2.1 revisited: *assigning an indivisible good or bad*

In this new interpretation of the same function $\mathcal{W}$ the n agents must assign an indivisible item that could be desirable (a good) or not (a bad, e. g., a chore). A positive type x_i is agent i's willingness to pay for the item, a negative x_i means that agent i must be paid at least $|x_i|$ to accept it (do the chore). Utilities vary in the real interval $[L, H]$, so if $L < 0 < H$ the item can be a good for some agents and a bad for others.

One of the efficient agents i^* (with the largest type), gets the item and pays a cash transfer y_j (that can be positive or negative) to each other agent j, so that i^*'s net utility is $y_{i^*} = x_{i^*} - \sum_{j \neq i^*} y_j$.

If $L \geq 0$ the item is good for everyone. The tight pair (una, g^+_p) give to each i at least a $\frac{1}{n}$-th share of what the good is worth to her. The parameter p is a reference (market?) price of the good: those who are not willing to pay that price ($x_i \leq p$) may receive up to a $\frac{1}{n}$-th share of p, those who would pay more than p may keep the incremental value $x_i - p$, and will do so for sure if no other type exceeds p.

If $H \leq 0$ the item is an objective and individible chore. Now the inefficient agents will pay the efficient agent for being spared the task. Among inefficient agents, those who would ask more than the reference wage $|p|$ to do the chore pay at least $\frac{1}{n}|p|$ (because $\frac{1}{n}x_i \leq y_i \leq \frac{1}{n}p$), others pay less; the agent i^* selected to do the chose may well end up with a net benefit, for instance if x_{i^*} is very small.

A simple auction-like sharing rule to implement (una, g^+_p) for any interval $[L, H]$ gives the item to an efficient agent i^* who in turn makes a (personalised) cash transfer $y_i = \frac{1}{n}(\max\{x_i, \min\{x_{i^*}, p\}\})$ to each other agent. By Lemma 3.3 the claim follows by checking that the share of any type x_i is in $[una(x_i), g^+_p(x_i)]$.

3 General model

The set of agents is $[n] = \{1, \cdots, n\}$ and $\mathcal{X}$ is the common set of types. All properties in this section apply if $\mathcal{X}$ is a compact subset of a general euclidian space $\mathbb{R}^A$ partially ordered in the usual way, an assumption maintained in this section except for statement *ii*) in Lemma 3.9 where $\mathcal{X}$ is a compact interval in $\mathbb{R}^A$.

At the profile $x = (x_i)_{i \in [n]} \in \mathcal{X}^{[n]}$ we must divide the benefit or cost $\mathcal{W}(x)$. A division of $\mathcal{W}(x)$ is $y = (y_i)_{i \in [n]} \in \mathbb{R}^{[n]}$ such that $\sum_{[n]} y_i = \mathcal{W}(x)$, and y_i is agent i's share. The function $\mathcal{W}$ is symmetric in the n variables x_i and continuous.

3.1 lower and upper guarantees

Definition 3.1: *The functions g^- and g^+ from $\mathcal{X}$ into $\mathbb{R}$ are respectively a lower and an upper guarantee of $\mathcal{W}$ if and only if they satisfy the inequalities:* for $x \in \mathcal{X}^{[n]}$

$$\sum_{i \in [n]} g^-(x_i) \leq \mathcal{W}(x) \leq \sum_{i \in [n]} g^+(x_i) \tag{4}$$

We write $\boldsymbol{G}^{-}, \boldsymbol{G}^{+}$ the sets of such guarantees.

Given two lower guarantees $g^1, g^2 \in \boldsymbol{G}^{-}$ we say that g^1 dominates g^2 if $g^1(x_i) \geq g^2(x_i)$ for $x_i \in \mathcal{X}$ and $g^1 \neq g^2$. The guarantee $g \in \boldsymbol{G}^{-}$ is *tight* if increasing its value at a single $x_1 \in \mathcal{X}$ creates a violation of the LS inequality in (4) for some $x_{-1} \in \mathcal{X}^{[n-1]}$.

The isomorphic statement for upper guarantees in $\boldsymbol{G}^{+}$ flips the domination inequality around and for tightness replaces increasing by decreasing and LS by RS.

We write $\mathcal{G}^{-}$ and $\mathcal{G}^{+}$ for the subsets of tight guarantees in $\boldsymbol{G}^{-}$ and $\boldsymbol{G}^{+}$.

Lemma 3.1 *For $\varepsilon = +, -$ every guarantee $g \in \boldsymbol{G}^{\varepsilon} \diagdown \mathcal{G}^{\varepsilon}$ is dominated by a tight one.*

The omitted proof is a simple application of Zorn's Lemma.

The restriction of $\mathcal{W}$ to the diagonal of $\mathcal{X}^{[n]}$ define the *unanimity* share of agent i:

$$una(x_i) = \frac{1}{n}\mathcal{W}(\overset{n}{x_i}) \tag{5}$$

where $(\overset{m}{z})$ is the m-vector with z in each coordinate.

Lemma 3.2
i) For any $(g^{-}, g^{+}) \in \boldsymbol{G}^{-} \times \boldsymbol{G}^{+}$ and for $x_i \in \mathcal{X}$

$$g^{-}(x_i) \leq una(x_i) \leq g^{+}(x_i) \tag{6}$$

ii) If una is a lower guarantee it dominates each lower guarantee; this is also true for upper guarantees. For $\varepsilon = +, -$:

$$una \in \boldsymbol{G}^{\varepsilon} \Longrightarrow \mathcal{G}^{\varepsilon} = \{una\}$$

The easy proof is again omitted.

If $\mathcal{W}$ is additively separable it takes the form $\mathcal{W}(x) = \sum_{[n]} una(x_i)$ and statement *ii)* implies $\mathcal{G}^{\varepsilon} = \{una\}$ for $\varepsilon = +, -$: tight guarantees imply the compelling interpretation of self-ownership discussed in section 1. Conversely if $\mathcal{G}^{\varepsilon} = \{una\}$ for $\varepsilon = +, -$ then una satisfies both sides of (4) so that $\mathcal{W}$ is additively separable.

In any other case there is a real choice of at least one type of tight guarantees. Moreover for any pair of tight guarantees the choice of a sharing rule φ to implement it is very open. Recall that φ maps $\mathcal{X}^{[n]}$ into $\mathbb{R}^{[n]}$ and $\sum_{[n]} \varphi_i(x) = \mathcal{W}(x)$ for all x.

Lemma 3 3. *Fix the function $\mathcal{W}$ and a tight pair $(g^{-}, g^{+}) \in \mathcal{G}^{-} \times \mathcal{G}^{+}$. If the sharing rule φ is such that $g^{-}(x_i) \leq \varphi_i(x) \leq g^{+}(x_i)$ for all i and x then it implements (g^{-}, g^{+}):* for all i and x

$$\min_{x_{-i}}\{\varphi_i(x_i, x_{-i})\} = g^{-}(x_i) \ ; \ \max_{x_{-i}}\{\varphi_i(x_i, x_{-i})\} = g^{+}(x_i)$$

A simple consequence of the tightness of g^{-} and g^{+}.

The moving average of g^{-} and g^{+} is the simplest sharing rule implementing any pair in $\mathcal{G}^{-} \times \mathcal{G}^{+}$:

$$\varphi_i(x) = \lambda g^{-}(x_i) + (1 - \lambda) g^{+}(x_i)$$

where λ is chosen s. t.

$$\lambda \sum_{[n]} g^{-}(x_i) + (1 - \lambda) \sum_{[n]} g^{+}(x_i) = \mathcal{W}(x)$$

Also, for any given sharing rule φ that does not implement (g^{-}, g^{+}) it is easy to adjust it *only* at those profiles where if fails at least one of these bounds so that the adjusted rule $\widetilde{\varphi}$ does implement the pair of guarantees and preserves the choices of φ as much as possible.

3.2 regularity and topological properties

Lemma 3.4 *If $\mathcal{X}$ is ordered by $\succ$ and $\mathcal{W}$ is weakly increasing in x, so is every tight guarantee in* $\mathcal{G}^{\varepsilon}$, for $\varepsilon = +, -$.

Proof Fix $g \in \mathcal{G}^{-}$. If $x_i \succ x_i'$ and $g(x_i) < g(x_i')$ define $\widetilde{g}(x_i) = g(x_i')$ and $\widetilde{g} = g$ otherwise, then check that $\widetilde{g}$ is still in $\boldsymbol{G}^{-}$. But g is tight so $\widetilde{g} = g$, a contradiction. ■

Lemma 3.5 *For $\varepsilon = +, -$ fix an equi-continuous function $\mathcal{W}$ in $\mathcal{X}^{[n]}$.*
i) A tight guarantee $g \in \mathcal{G}^{\varepsilon}$ is continuous in $\mathcal{X}$.
ii) A guarantee g in $\boldsymbol{G}^{\varepsilon}$ is tight if and only if: for all $x_i \in \mathcal{X}$ there exists $x_{-i} \in \mathcal{X}^{[n-1]}$ s. t.

$$g(x_i) + \sum_{j \in [n-1]} g(x_j) = \mathcal{W}(x_i, x_{-i}) \tag{7}$$

If equality (7) holds we call (x_i, x_{-i}) a contact profile of g at x_i ; the set of such profiles is the contact set $\mathcal{C}(g)$ of g.

Proof in the Appendix section 9.1.

Lemma 3.6 *For $\varepsilon = +, -$,*
i) For any $x_1 \in \mathcal{X}$ there is an tight guarantee $g \in \mathcal{G}^{\varepsilon}$ s.t. $g(x_1) = una(x_1)$.
ii) The set $\mathcal{G}^{\varepsilon}$ is a singleton if and only if it contains una.

Proof in the Appendix section 9.2.

By statement *ii*) we see that $\mathcal{G}^{-}$ and $\mathcal{G}^{+}$ are both singletons if and only if $\mathcal{W}$ is additively separable.

Note that the equality $g(x_1) = una(x_1)$ in statement *i*) holds if and only if the contact set $\mathcal{C}(g)$ intersects the diagonal of $\mathcal{X}^{[n]}$.

Next we state without proof two useful invariance properties.

Lemma 3.7 *For $\varepsilon = +, -$,*
i) If $\mathcal{W}_0$ is additively separable, $\mathcal{W}_0(x) = \sum_{[n]} w_0(x_i)$, and $\mathcal{W}$ an arbitrary symmetric function on $\mathcal{X}^{[n]}$ we have

$$\mathcal{G}^{\varepsilon}(\mathcal{W} + \mathcal{W}_0) = \mathcal{G}^{\varepsilon}(\mathcal{W}) + \{w_0\}$$

ii) Change of the type variable. If θ is a bicontinuous bijection $x_i = \theta(z_i)$ from $\mathcal{Z}$ into $\mathcal{X}$, $\mathcal{W}$ is defined on $\mathcal{X}^{[n]}$ and $g \in \mathcal{G}^{\varepsilon}(\mathcal{W})$, then $g \circ \theta \in \mathcal{G}^{\varepsilon}(\widetilde{\mathcal{W}})$ where $\widetilde{\mathcal{W}}(z) = \mathcal{W}(\theta(z))$ and $\theta(z)_i = \theta(z_i)$.

For instance the problem $\mathcal{W}(x) = F(\max_{i \in [n]}\{x_i\})$ reduces to $\widetilde{\mathcal{W}}(z) = \max_{i \in [n]}\{z_i\}$ by the change $x_i = F^{-1}(z_i)$; and $\mathcal{W}(x) = \min_{i \in [n]}\{x_i\}$ reduces to $\widetilde{\mathcal{W}}(z) = \max_{i \in [n]}\{z_i\}$ by the change of variable $x_i = -z_i$.

3.3 Lipschitz and differentiability properties

They are key to the characterisation results in sections 6,7, 8.

Lemma 3.8 *Fix $g \in \mathcal{G}^{+}$. For any x_i, x_i' and any contact profile x_{-i} of g at x_i we have*

$$g(x_i') - g(x_i) \geq \mathcal{W}(x_i', x_{-i}) - \mathcal{W}(x_i, x_{-i}) \tag{8}$$

and the opposite inequality if $g \in \mathcal{G}^{-}$.

Proof In the inequality

$$g(x_i') + \sum_{i=2}^{n} g(x_i) \geq \mathcal{W}(x_i', x_{-i})$$

we replace each term $g(x_i)$ by $\mathcal{W}(x_i,x_{-i})-\sum_{j\neq i} g(x_j)$ and rearrange it as follows

$$(n-1)(\mathcal{W}(x_i,x_{-i})-g(x_i))-(n-2)\sum_{i=2}^{n} g(x_i) \geq \mathcal{W}(x_i',x_{-i})-g(x_i')$$

$$\Longleftrightarrow \mathcal{W}(x_i,x_{-i})-g(x_i)+(n-2)(\mathcal{W}(x)-\sum_{[n]} g(x_i)) \geq \mathcal{W}(x_i',x_{-i})-g(x_i')$$

The claim follows because x_{-i} is a contact profile for g at x_i. ∎

Our last general result is critical to our two difficult Theorems 6.1 and 7.1, where it is only used in a one-dimensional interval of types. But its statement and proof are just as easy when $\mathcal{X}$ is a multidimensional interval.

Lemma 3.9
i) Suppose K is a positive constant, $\mathcal{X}\subset\mathbb{R}^A$ and the function $\mathcal{W}$ is K-Lipschitz in each x_i, uniformly in $x_{-i}\in\mathcal{X}^{[n-1]}$. Then so is each tight guarantee $g\in\mathcal{G}^-\cup\mathcal{G}^+$.
ii) Suppose $\mathcal{X}=[\boldsymbol{L},\boldsymbol{H}]$ is the interval $\boldsymbol{L}\leq x\leq\boldsymbol{H}$ in $\mathbb{R}^A$. We fix $x_i\in\mathcal{X}$, an tight guarantee $g\in\mathcal{G}^-\cup\mathcal{G}^+$ and a contact profile $x_{-i}\in\mathcal{X}^{[n-1]}$ of g at x_i. If for some $a\in A$, g and $\mathcal{W}(\cdot,x_{-i})$ are both differentiable in x_{ia} at x_i, we have
if $\boldsymbol{L}_a<x_{ia}<\boldsymbol{H}_a$

$$\frac{dg}{dx_{ia}}(x_{ia})=\frac{\partial\mathcal{W}}{\partial x_{ia}}(x_i,x_{-i}) \tag{9}$$

if $x_i=\boldsymbol{L}_a$ and $g\in\mathcal{G}^-$ or $x_i=\boldsymbol{H}_a$ and $g\in\mathcal{G}^+$

$$\frac{dg}{dx_{ia}}(x_{ia})\leq\frac{\partial\mathcal{W}}{\partial x_{ia}}(x_i,x_{-i})$$

if $x_i=\boldsymbol{H}_a$ and $g\in\mathcal{G}^-$ or $x_i=\boldsymbol{L}_a$ and $g\in\mathcal{G}^+$

$$\frac{dg}{dx_{ia}}(x_{ia})\geq\frac{\partial\mathcal{W}}{\partial x_{ia}}(x_i,x_{-i})$$

Proof *Statement i)* If $g\in\mathcal{G}^-$ property (8) and the Lipschitz assumption imply $g(x_i)-g(x_i')\leq K\|x_i-x_i'\|$ (where $\|\cdot\|$ is the norm w. r. t. which $\mathcal{W}$ is Lipschitz). Exchanging the roles of x_i and x_i' gives $g(x_i')-g(x_i)\leq K\|x_i'-x_i\|$ and the conclusion.

Statement ii) Note that if the functions f,g of one real variable z are differentiable at some z_0 in the interior of their common domain and the inequality $f(z)-f(z_0)\geq g(z)-g(z_0)$ holds for z close enough to z_0, then their derivatives at z_0 coincide. Apply this to the functions $x_{ia}\to g(x_i)$ and $x_{ia}\to\mathcal{W}(x_i,x_{-i})$ and the inequalities (8) to deduce the equality (9). The last two inequalities are equally easy to check. ∎

For a fixed coordinate $a\in A$ the Lipschitz property in statement i), that we call *uniformly Lipschitz* by a slight abuse of terminology[5], implies that g is differentiable in x_{ia} almost everywhere in $[\boldsymbol{L}_a,\boldsymbol{H}_a]$. All our examples in sections 5,6,7 involve functions $\mathcal{W}$ uniformly Lipschitz in this sense, therefore all corresponding tight guarantees are differentiable almost everywhere in each coordinate of x_i.

Corollary to Lemma 3.9 *Suppose $\mathcal{X}=[L,H]\subset\mathbb{R}$, $\mathcal{W}$ is differentiable in $[L,H]^{[n]}$ and for $\varepsilon=+,-$, the tight guarantees in $\mathcal{G}^\varepsilon$ are a. e. differentiable. Then tight guarantees are characterised*

[5] Because we only require the Lipschitz property in each coordinate.

by their contact set $\mathcal{C}(g)$. Moreover any (true)convex combination of two or more guarantees in $\mathcal{G}^\varepsilon$ stays in $\boldsymbol{G}^\varepsilon$ but leaves $\mathcal{G}^\varepsilon$. Formally: for $g, h \in \mathcal{G}^\varepsilon$

$$g \neq h \Longrightarrow \{\mathcal{C}(g) \neq \mathcal{C}(h),]g, h[\cap \mathcal{G}^\varepsilon = \varnothing\}$$

Proof. We prove the contraposition of the first statement with the help of statement *ii*). If $\mathcal{C}(g) = \mathcal{C}(h)$ we get $\frac{dg}{dx} = \frac{dh}{dx}$ in the interval $]L, H[$ so they differ by a constant, and if the latter is not zero one of g, h is not tight.

Check now the second statement by contradiction: say that $\mathcal{G}^-$ contains g, h and $\frac{1}{2}(g+h)$, all different. Fix $x_i \in]L, H[$ and a contact profile $\widetilde{x}_{-i}$ of $\frac{1}{2}(g+h)$ at x_i. Clearly $\widetilde{x}_{-i}$ is also a contact profile of g and of h at x_i. Therefore by statement *ii*) in Lemma 7, almost surely in $x_i \in]L, H[$ we have $\frac{dg}{dx_i}(x_i) = \frac{dh}{dx_i}(x_i) = \partial_i \mathcal{W}(x_i, \widetilde{x}_{-1})$. We conclude that $g - h$ is a constant and get a contradiction of $g \neq h$. The argument for larger convex combinations with general weights is entirely similar. ■

4 Sub- and super-modular functions $\mathcal{W}$

In this class of benefit and cost functions that includes most of our examples, the analysis of tight guarantees greatly simplifies and allows our two main characterisation results Theorems 6.1 and 7.1.

The type space $\mathcal{X}$ is a compact subset of $\mathbb{R}^A$ for Proposition 4.1, a compact interval in $\mathbb{R}^A$ for Propositions 4.2, and a one-dimensional interval in Proposition 4.3.

Definition 4.1 *We call $\mathcal{W}$ supermodular* if for $i, j \in [n]$ and x, x' in $\mathcal{X}^{[n]}$ such that $x_k = x'_k$ for all $k \neq i, j$ we have

$$\{x_i \leq x'_i, x_j \leq x'_j\} \Longrightarrow \mathcal{W}(x'_i, x_j; x_{-i,j}) + \mathcal{W}(x_i, x'_j; x_{-i,j}) \leq \mathcal{W}(x) + \mathcal{W}(x') \tag{10}$$

We say that $\mathcal{W}$ is strictly supermodular if whenever $(x_i, x_j) \ll (x'_i, x'_j)$ the RS of (10) is strict. And $\mathcal{W}$ is submodular or strictly so if the opposite inequalities holds under the same premises. A modular function is one that is either supermodular or submodular.

An equivalent definition of supermodularity is useful too: for $i, j \in [n]$ and x, x' in $\mathcal{X}^{[n]}$ s. t. $x_j \leq x'_j$ for all j

$$\mathcal{W}(x'_i, x_{-i}) - \mathcal{W}(x_i, x_{-i}) \leq \mathcal{W}(x'_i, x'_{-i}) - \mathcal{W}(x_i, x'_{-i}) \tag{11}$$

The Appendix 9.3 lists other well known properties of the partial derivatives of modular functions that are useful in some of the long proofs.

4.1 the unanimity guarantee of modular functions

The unanimity share of a modular function is a tight guarantee on one side of the system (4).

Proposition 4.1 *If $\mathcal{W}$ is supermodular the unanimity function (5) is the single tight upper guarantee: $\mathcal{G}^+ = \{una\}$. It is the single tight lower guarantee if $\mathcal{W}$ is submodular: $\mathcal{G}^- = \{una\}$.*

Notation: when the ordering of the coordinates does not matter $(z; \overset{k}{y})$ represents the $(k+1)$-vector where one coordinate is z and k coordinates are y.

Proof Fix $\mathcal{W}$ supermodular. For $n = 2$ the statement $una \in \mathcal{G}^+$ amounts to

$$\mathcal{W}(x_1, x_2) \leq \frac{1}{2}(\mathcal{W}(x_1, x_1) + \mathcal{W}(x_2, x_2))$$

a direct consequence of supermodularity and $\mathcal{W}(x_1, x_2) = \mathcal{W}(x_2, x_1)$.

Proceeding by induction we assume the statement is true up to $(n-1)$ agents and fix a n-person supermodular function $\mathcal{W}$ and a profile $x \in \mathcal{X}^{[n]}$. As una is an upper guarantee of the $(n-1)$-benefit function $\mathcal{W}(\cdot; x_i)$ we have: for all i and x_i

$$\mathcal{W}(x) \leq \frac{1}{n-1} \sum_{j \in [n] \setminus \{i\}} \mathcal{W}(x_i; \overset{n-1}{x_j}) \Longrightarrow n\mathcal{W}(x) \leq \frac{1}{n-1} \sum_{(i,j) \in P} \mathcal{W}(x_i; \overset{n-1}{x_j}) \tag{12}$$

where P is the set of ordered pairs (i,j) in $[n]$.

Apply next the same property of una for $\mathcal{W}(\cdot; x_j)$ but at each profile of the form $(x_i, \overset{n-2}{x_j})$:

$$\mathcal{W}(x_i; \overset{n-1}{x_j}) \leq \frac{1}{n-1}((n-2)\mathcal{W}(\overset{n}{x_j}) + \mathcal{W}(x_j; \overset{n-1}{x_i}))$$

Summing up both sides over $(i,j) \in P$ and writing S for the summation in the most RS of (12) gives

$$S \leq (n-2) \sum_{j=1}^{n} \mathcal{W}(\overset{n}{x_j}) + \frac{1}{n-1} S \Longrightarrow S \leq (n-1) \sum_{j=1}^{n} \mathcal{W}(\overset{n}{x_j})$$

Combining (12) with the latter inequality concludes the proof.

The proof for a submodular $\mathcal{W}$ exchanges a few signs. ■

Lemma 4.1 *Suppose $\mathcal{W}$ is supermodular and a tight lower guarantee $g \in \mathcal{G}^-$ has two different unanimous contact point: $g(x_i) = una(x_i)$ for some $x_1 \neq x_2$. Then $\mathcal{W}$ is separably additive in the interval $[(\overset{n}{x_1}), (\overset{n}{x_2})]$; in particular it is not strictly supermodular.The same is true if $\mathcal{W}$ is submodular for a tight upper guarantees.*

Proof Fix $\mathcal{W}, g \in \mathcal{G}^-$ as in the first statement and suppose g has two unanimous contact profiles $(\overset{n}{x_1})$ and $(\overset{n}{x_2})$ such that $x_1 < x_2$. Property (11) implies:

$$\mathcal{W}(x_2; \overset{n-1}{x_1}) - \mathcal{W}(\overset{n}{x_1}) \leq \mathcal{W}(\overset{n}{x_2}) - \mathcal{W}(x_1; \overset{n-1}{x_2})$$

$$\Longleftrightarrow \mathcal{W}(x_2; \overset{n-1}{x_1}) + \mathcal{W}(x_1; \overset{n-1}{x_2}) \leq \mathcal{W}(\overset{n}{x_1}) + \mathcal{W}(\overset{n}{x_2})$$

By our choice of x_1, x_2 the RS in the last inequality is

$$ng(x_1) + ng(x_2) = (g(x_2) + (n-1)g(x_1)) + (g(x_1) + (n-1)g(x_2)) \leq$$

$$\leq \mathcal{W}(x_2; \overset{n-1}{x_1}) + \mathcal{W}(x_1; \overset{n-1}{x_2})$$

so the supermodularity inequality (4) between $(\overset{n}{x_1})$ and $(\overset{n}{x_2})$ is in fact an equality. Its additive separability consequence is routine (Appendix 9.3). ■

If $\mathcal{W}$ is supermodular function, for each unanimity profile $(\overset{n}{x_i})$ there is a non empty set of tight lower guarantees for which $(\overset{n}{x_i})$ is a contact point (Lemma 3.6)[6], and those subsets of $\mathcal{G}^-$ are mutually disjoint (Lemma 4.1). Ditto for submodular $\mathcal{W}$ and tight upper guarantees.

But tight guarantees with no unanimous contact point are not a pathological occurrence: large sets of such guarantees are described in Proposition 5.1 and Theorem 6.1.

[6] Exactly one in Example 2.1, but infinitely many for strictly supermodular two person problems (Theorem 7.1).

4.2 two canonical incremental guarantees

All modular functions share two simple tight guarantees on the other side of the unanimity one.

Proposition 4.2 *Suppose $\mathcal{X}$ is an interval $[\boldsymbol{L},\boldsymbol{H}]\subseteq\mathbb{R}^A$ and $\mathcal{W}$ is supermodular. Then the equations*

$$\begin{aligned} g_{inc}(x_i) &= \mathcal{W}(x_i;\overset{n-1}{\boldsymbol{L}}) - \frac{n-1}{n}\mathcal{W}(\overset{n}{\boldsymbol{L}}) \\ g^{inc}(x_i) &= \mathcal{W}(x_i;\overset{n-1}{\boldsymbol{H}}) - \frac{n-1}{n}\mathcal{W}(\overset{n}{\boldsymbol{H}}) \end{aligned} \tag{13}$$

for $x_i\in[\boldsymbol{L},\boldsymbol{H}]$, *define two tight lower guarantees called the left-incremental g_{inc} and right-incremental g^{inc}. Their unanimous contact points are at $\boldsymbol{L}$ and $\boldsymbol{H}$ respectively. If $\mathcal{W}$ is submodular g_{inc} and g^{inc} are tight upper guarantees.*

In Example 2.1 these two guarantees are the end-points of $\mathcal{G}^+$ and g^{inc} comes from the egalitarian sharing rule.

Under the guarantees (una, g_{inc}), types $\boldsymbol{L}$ always get their unanimity share; and $g_{inc}(x_i)$ is what i must pay for sure if everyone else has type $\boldsymbol{L}$, which explains our terminology.

If $\mathcal{W}$ is supermodular and $\mathcal{W}(x)$ is a surplus, the left-incremental g_{inc} favors the types x_i close to $\boldsymbol{L}$ who get a share close to their best case $una(x_i)$, while g^{inc} favors those close to $\boldsymbol{H}$. Isomorphic comments obtain if $\mathcal{W}(x)$ is a cost and/or $\mathcal{W}$ is submodular.

Proof of Proposition 4.2

Fix $\mathcal{W}$ supermodular and check first that g_{inc} is a feasible lower guarantee. If $n=2$ this follows at once from (10). If $n=3$ we must show the following inequality for any x:

$$\mathcal{W}(x_1,\boldsymbol{L},\boldsymbol{L})+\mathcal{W}(x_2,\boldsymbol{L},\boldsymbol{L})+\mathcal{W}(x_3,\boldsymbol{L},\boldsymbol{L})\leq\mathcal{W}(x_1,x_2,x_3)+2\mathcal{W}(\boldsymbol{L},\boldsymbol{L},\boldsymbol{L})$$

We use the symmetry of W to apply successively (10) and (11):

$$\mathcal{W}(x_1,\boldsymbol{L},\boldsymbol{L})+\mathcal{W}(\boldsymbol{L},x_2,\boldsymbol{L})\leq\mathcal{W}(x_1,x_2,\boldsymbol{L})+\mathcal{W}(\boldsymbol{L},\boldsymbol{L},\boldsymbol{L})$$

$$\mathcal{W}(\boldsymbol{L},\boldsymbol{L},x_3)-\mathcal{W}(\boldsymbol{L},\boldsymbol{L},\boldsymbol{L})\leq\mathcal{W}(x_1,x_2,x_3)-\mathcal{W}(x_1,x_2,\boldsymbol{L})$$

and sum up these two inequalities.

The argument for any n is now clear: in the desired inequality

$$\sum_{[n]}\mathcal{W}(x_i;\overset{n-1}{\boldsymbol{L}})\leq\mathcal{W}(x)+(n-1)\mathcal{W}(\overset{n}{\boldsymbol{L}})$$

we replace the the first two terms on the LS by $\mathcal{W}(x_1,x_2;\overset{n-2}{\boldsymbol{L}})+\mathcal{W}(\overset{n}{\boldsymbol{L}})$: by (*10*) this increases weakly the LS so it is enough to check

$$\mathcal{W}(x_1,x_2;\overset{n-2}{\boldsymbol{L}})+\sum_3^n\mathcal{W}(x_i;\overset{n-1}{\boldsymbol{L}})\leq\mathcal{W}(x)+(n-2)\mathcal{W}(\overset{n}{\boldsymbol{L}})$$

Next by (11) we replace the two first terms on the LS by $\mathcal{W}(x_1,x_2,x_3;\overset{n-3}{\boldsymbol{L}})+\mathcal{W}(\overset{n}{\boldsymbol{L}})$ and so on.

To show finally that g_{inc} is tight we use (13) to check that $(x_i;\overset{n-1}{\boldsymbol{L}})$ is a contact profile of g_{inc} at x_i, then apply Lemma 3.5 . The proofs for g^{inc} and/or submodular $\mathcal{W}$ are identical up to switching the relevant signs. ■

4.3 implementing the incremental guarantees: the serial rules

We adapt to our model these well known sharing rules, originally introduced for the commons problem with substitutable inputs ([26], [31]), the object of the next section.

Definition 4.3 *Suppose $\mathcal{X}$ is an interval $[L, H] \subseteq \mathbb{R}$. The Serial↑ sharing rule φ from $[L, H]^{[n]}$ into $\mathbb{R}^{[n]}$ is defined by the combination of two properties a) it is symmetric in its variables and b) the share of agent i with type x_i is independent of other agents' larger shares.*[7]

When the agents are labelled by increasing types as $x_1 \leq x_2 \leq \cdots \leq x_n$ agent i's share is:

$$\varphi_i^{ser\uparrow}(x) = \frac{\mathcal{W}(x_1, \cdots, x_{i-1}, \overset{n-i+1}{x_i})}{n-i+1} - \sum_{j=1}^{i-1} \frac{\mathcal{W}(x_1, \cdots, x_{j-1}, \overset{n-j+1}{x_j})}{(n-j+1)(n-j)} \tag{14}$$

We omit this computation for brevity: see the details in ([25]) where this is equation (6).

The Serial↓ sharing rule is defined symmetrically property a) and b)* agent i' share is independent of other agents' smaller shares. It is given by the same expression (14) if we label the agents by decreasing types.

Proposition 4.3 *Fix a supermodular function $\mathcal{W}$ in $[L, H]^{[n]}$.*
The Serial↑ sharing rule implements both the left-incremental and unanimity guarantees g_{inc}, una; the Serial↓ rule implements both g^{inc} and una.

The isomorphic statement for submodular functions exchanges left- and right- incremental guarantees.

Proof in the Appendix 9.4.

5 Substitutable inputs

The continuous production function F transforms a profile of non negative one-dimensional inputs $x_i \in [L, H]$ $(L \geq 0)$ into the output $\mathcal{W}(x) = F(x_N)$. This problem is supermodular if (and only if) F is convex and submodular if F is concave. In the familiar interpretations of this model (section 1where F is typically increasing; this assumption is not needed for any of the results in this section, and in fact is not satisfied in our Example 5.2.

5.1 stand alone guarantees

For a general function $\mathcal{W}$ a *stand alone* guarantee is one that takes the form $g(x_i) = \mathcal{W}(x_i, c) - \gamma$ where $c \in \mathcal{X}^{[n-1]}$ and $\gamma \in \mathbb{R}$ are constant. If $\mathcal{W}$ is modular the left and right incremental guarantees are prime examples of tight stand alone guarantees (Proposition 4.2), and we will find many more in Theorem 6.1.

In the substitutable inputs model we find $n-2$ additional tight stand alone guarantees linking the two incremental ones.

Proposition 5.1
i) If F is convex in $[L, H]$ the supermodular commons $\mathcal{W}(x) = F(x_N)$ admits the following sequence of n tight lower guarantees $g_{\ell,h}$, where $\ell, h \in \mathbb{N} \cup \{0\}$ are s. t. $\ell + h = n-1$: for $x_i \in [L, H]$

$$g_{\ell,h}(x_i) = F(x_i + (\ell L + hH)) - \frac{1}{n}\{\ell F((\ell+1)L + hH) + hF(\ell L + (h+1)H)\} \tag{15}$$

and $g_{n-1,0} = g_{inc}$, $g_{0,n-1} = g^{inc}$ ((13)).

[7] The share $\varphi_i(x)$ does not change if agent j's type changes from x_j to x'_j both weakly larger than x_i.

ii) The gap $una(x_i) - g_{\ell,h}(x_i)$ *is minimal at the benchmark type* $x_i = \frac{1}{n-1}(\ell L + hH)$.
iii) If F *is strictly convex only* g_{inc} *and* g^{inc} *have a unanimous contact point.*

If F *is concave (15) defines* n *tight upper guarantees with the same properties for the gap* $g_{\ell,h} - una$ *and contact points.*

Proof in the Appendix 9.5.

We illustrate this result for a commons with an interesting type of input complementarity.

Example 5.1: *Commons with complementary inputs*

A project wil return one unit of surplus if and only if all agents succeed in completing their own part. Agent i's effort x_i is also the probability that i is successful, therefore the expected return is, for $x \in [L, H]^{[n]}$

$$\mathcal{W}(x) = x_1 x_2 \cdots x_n$$

where $[L, H] \subset [0, 1]$. We must to divide the expected return between the workers.

The function $\mathcal{W}$ is supermodular so $una(x_i) = \frac{1}{n} x_i^n$ is the single tight upper bound on type x_i's share.

By Lemma 3.7 the change of variable $x_i = e^{z_i}$ transforms $\mathcal{W}$ into $\widetilde{\mathcal{W}}(z) = e^{z_N}$ and the guarantees (15) for $\widetilde{\mathcal{W}}$ correspond for $\mathcal{W}$ to n tight lower guarantees linear in type:

$$g_{\ell,h}(x_i) = L^\ell H^h (x_i - \frac{1}{n}(\ell L + hH))$$

$$\begin{aligned} g_{inc}(x_i) &= L^{n-1}(x_i - \frac{n-1}{n} L) \\ g^{inc}(x_i) &= H^{n-1}(x - \frac{n-1}{n} H) \end{aligned}$$

Note that $g_{inc}(x_i) \geq \frac{1}{n} L^n$: even providing the minimal effort L guarantees the share $una(L) = \frac{1}{n} L^n$ (irrespective of other types). Contrast with g^{inc} that rewards high effort much more, even guarantees $una(H) = \frac{1}{n} H^n$ to the maximal effort H: this is feasible by charging cash penalties to all "slackers", defined as those with $x_i < \frac{n-1}{n} H$; for instance type L pays out $|g^{inc}(L)| = H^{n-1}(\frac{n-1}{n} H - L)$ in the worst case where all others provide maximal effort H.

The $n-2$ other guarantees $g_{\ell,h}$ allow the manager to adjust, along a grid increasingly fine as n grows, the critical effort level $\frac{1}{n}(\ell L + hH)$ guaranteeing a positive share.

5.2 tangent and hybrid guarantees

Observe that if the general function $\mathcal{W}$ is globally convex and differentiable in $[L, H]^{[n]}$ the tangent at any point $(\alpha, una(\alpha))$ of its unanimity graph defines a feasible but not necessarily tight lower guarantee $g_\alpha \in \mathbf{G}^-$: for $x_i \in [L, H]$

$$g_\alpha(x_i) = \frac{1}{n} \mathcal{W}(\overset{n}{\alpha}) + \partial_1 \mathcal{W}(\overset{n}{\alpha})(x_i - \alpha)$$

Indeed the LS of (4) reads

$$\mathcal{W}(\overset{n}{\alpha}) + \partial_1 \mathcal{W}(\overset{n}{\alpha})(x_N - n\alpha)) \leq \mathcal{W}(x)$$

precisely the tangent hyperplane inequality at $(\overset{n}{\alpha})$ because $\mathcal{W}$ is symmetric in all variables.

For the globally convex $\mathcal{W}(x) = F(x_N)$ we find that many of the tangents to the unanimity graph are *tight* lower guarantees: those touching that graph inside the subinterval of $[L, H]$ left after deleting $\frac{1}{n}$-th at each end. And on the deleted intervals we construct guarantees concatenating (parts of) a tangent and a stand alone guarantee. We obtain in this way a continuous line of tight guarantees with the two incremental ones at its endpoints.

Proposition 5.2: *If F is convex in $[nL, nH]$ the supermodular commons $\mathcal{W}(x) = F(x_N)$ admits the following tight lower guarantees g_α, where $\alpha \in [L, H]$ and $g_L = g_{inc}$, $g_H = g^{inc}$.*

i) If $\frac{n-1}{n}L + \frac{1}{n}H \leq \alpha \leq \frac{1}{n}L + \frac{n-1}{n}H$ the graph of g_α is tangent to that of una at $n\alpha$: for $L \leq x_i \leq H$

$$g_\alpha(x_i) = \frac{1}{n}F(n\alpha) + \frac{dF}{dx}(n\alpha)(x_i - \alpha) \tag{16}$$

ii) If $L \leq \alpha \leq \frac{n-1}{n}L + \frac{1}{n}H$ the graph starts as a tangent then takes a stand alone shape: for $L \leq x_i \leq n\alpha - (n-1)L$

$$g_\alpha(x_i) = \frac{1}{n}F(n\alpha) + \frac{dF}{dx}(n\alpha)(x_i - \alpha)$$

for $n\alpha - (n-1)L \leq x_i \leq H$

$$= F(x_i + (n-1)L) - \frac{(n-1)}{n}F(n\alpha) + (n-1)\frac{dF}{dx}(n\alpha)(\alpha - L) \tag{17}$$

iii) If $\frac{1}{n}L + \frac{n-1}{n}H \leq \alpha \leq H$ *the graph starts as a stand alone then turn into a tangent:* for $L \leq x_i \leq n\alpha - (n-1)H$

$$g_\alpha(x_i) = F(x_i + (n-1)H) - \frac{n-1}{n}F(n\alpha) - (n-1)\frac{dF}{dx}(n\alpha)(H - \alpha)$$

for $n\alpha - (n-1)H \leq x_i \leq H$

$$= \frac{1}{n}F(n\alpha) + \frac{dF}{dx}(n\alpha)(x_i - \alpha)$$

Proof

Statement i) We already noted that g_α is in $\mathbf{G}^-$. For tightness we fix a type x_i and look for a vector x_{-i} such that $x_i + x_{N \smallsetminus i} = n\alpha$: then (16) implies $\sum_{[n]} g_\alpha(x_j) = F(n\alpha)$ and (x_i, x_{-i}) is a contact profile of g_α at x_i (Lemma 3.5). Such x_{-i} exists if and only if $x_i + (n-1)L \leq n\alpha \leq x_i + (n-1)H$, precisely the bounds on α we assume.

Statement ii) At a profile x where $x_i \leq n\alpha - (n-1)L$ for all i, we just saw that g meets the LS of (4). We check now this inequality for a profile x where the first t types are above $n\alpha - (n-1)L$, $t \geq 1$, and the other $n - t$ types (possibly zero) are below that bound.

In the LS of (4) a type x_i for $i \leq t$ affects the difference $F(x_i + x_{N \smallsetminus i}) - F(x_i + (n-1)L)$; as $x_{N \smallsetminus i} \geq (n-1)L$ the inequality in question is most demanding (the difference is smallest) if $x_i = n\alpha - (n-1)L$. Similarly a type x_j for $j > t$, if any, affects $\Delta = F(x_j + x_{N \smallsetminus j}) - \frac{dF}{dx}(n\alpha)x_i$. Now $t \geq 1$ implies $x_{N \smallsetminus j} \geq n\alpha - (n-1)L + (n-2)L = n\alpha - L$, therefore the derivative of Δ w.r.t. x_j is non negative at $x_j = L$ and weakly increasing: the inequality in question is most demanding if $x_j = L$. It is then enough to check

$$tg_\alpha(n\alpha - (n-1)L) + (n-t)g_\alpha(L) \leq F(tn\alpha - (t-1)nL)$$

$$\iff \frac{dF}{dx}(n\alpha)(t-1)n(\alpha - L) \leq F(tn\alpha - (t-1)nL) - F(n\alpha)$$

which follows at once from the convexity of F.

Checking tightness. At a type $x_i \leq n\alpha - (n-1)L$ we have

$$x_i + (n-1)L \leq n\alpha \leq x_i + (n-1)(n\alpha - (n-1)L)$$

(replace x_i by L on the RS and rearrange). As in the proof of statement $i)$ this implies the existence of a contact profile (x_i, x_{-i}) entirely inside $[L, n\alpha - (n-1)L]$. And at a type $x_i \geq n\alpha - (n-1)L$ we see that $(x_i, \overset{n-1}{L})$ is a contact profile of g_α.

We omit the symmetric proof of statement $iii)$. ■

Example 5.2 *sharing the cost of the variance*

Agents choose a type x_i in $[0,1]$ and must share (n times) the variance of their distribution:

$$\mathcal{W}(x) = \sum_{[n]} x_i^2 - \frac{1}{n}(\sum_{[n]} x_i)^2 \tag{18}$$

For instance x_i is i's location in $[0,1]$ and a public facility is located at the mean $\frac{1}{n}x_N$ of this distribution, to minimise the *quadratic* transportation costs to the facility; the total cost $\mathcal{W}(x)$ is precisely (18).

The problem is submodular and $una(x_i) \equiv 0$ so the only tight lower guarantee (best case) is to pay nothing: no one should get a net profit but everyone can hope that his type is adopted by everyone else. By statement $i)$ in Lemma 3.7 and a change of sign, every tight upper guarantee g^+ of $\mathcal{W}$ obtains from a tight lower guarantee g^* of $\mathcal{W}^*(x) = (x_N)^2$ as $g^+(x_i) = x_i^2 - \frac{1}{n}g^*(x_i)$.

The tangent lower guarantees of $\mathcal{W}_*$ (statement $i)$ in Proposition 5.2) are $g^*_\alpha(x_i) = n\alpha(2x_i - \alpha)$ and in turn deliver the simple tight upper guarantees $g^+_\alpha(x_i) = (x_i - \alpha)^2$ of $\mathcal{W}$ for $\alpha \in [\frac{1}{n}, \frac{n-1}{n}]$

Here the location α is "free": a type α never pays, and the worst cost share at other locations is precisely the travel cost to the benchmark.

The stand alone guarantees in Proposition 5.1, denoted g^+_h and indexed by the integer h are: for $h = 0, 1, \cdots, n-1$

$$g^+_h(x_i) = \frac{n-1}{n}(x_i - \frac{h}{n-1})^2 + \delta_h$$

in particular $g_{inc}(x_i) = \frac{n-1}{n}x_i^2$ and $g^{inc}(x_i) = \frac{n-1}{n}(1-x_i)^2$ (also the endpoints in Proposition 5.2).

For any h the non negative constant $\delta_h = \frac{h(n-1-h)}{n^2(n-1)}$ is below $\frac{1}{4n}$, so if n is large and $\alpha \simeq \frac{h}{n-1}$ the guarantees g^+_α and g^+_h are similar: g^+_h is $\frac{n-1}{n}$flater than g^+_α and smaller at 0 and 1, but g^+_h never vanishes.

For $\alpha \leq \frac{1}{n}$ the hybrid guarantees g^+_α (statement $ii)$ in Proposition 5.2) concatenate smoothly $(x_i - \alpha)^2$ for $x_i \leq n\alpha$ with $\frac{n-1}{n}x_i^2 - (n-1)\alpha^2$ for $x_i \geq n\alpha$; the location α is still the "free" benchmark.

5.3 three familiar sharing rules and their guarantees

While Proposition 4.3 shows that each Serial sharing rule implements one of the the two (tight) incremental guarantees we show next that the guarantees implemented by three other familiar rules are mostly not tight.

We fix F strictly concave on $[0, H]$ and such that $F(0) = 0$. The three sharing rules under scrutiny are:

Average Cost: $\varphi_i^{avg}(x) = x_i AF(x_N)$, with the notation $AF(z) = \frac{F(z)}{z}$;[8]

[8] At the profile $(\overset{n}{0})$ the definition needs adjusting, e. g. to equal split, but this does not affect the computations of worst and best cases.

Shapley value: $\varphi_i^{Sha}(x) = \mathbb{E}_S(F(x_i + x_S) - F(x_S))$, where the expectation is over $S, \varnothing \subseteq S \subseteq N\diagdown\{i\}$, uniformly distributed;

Marginal pricing: $\varphi_i^{mrg}(x) = \frac{1}{n}F(x_N) + \frac{dF}{dx}(x_N)(x_i - \frac{1}{n}x_N)$ (always a non negative share as $F(z) \geq \frac{dF}{dx}(z)z$ for $z \geq 0$).

Lemma 5.1

$i)$ For the Average and Shapley rules: for $x_i \in [0, H[$

$$g^-_{avg}(x_i), g^-_{Sha}(x_i) < una(x_i) = \frac{1}{n}F(nx_i)$$

(with equality at H) and for $x_i \in [0, H]$

$$g^+_{avg}(x_i) = g^+_{Sha}(x_i) = g_{inc}(x_i) = F(x_i)$$

$ii)$ For the Marginal pricing rule a reverse pattern may hold: for $x_i \in [0, H]$

$$g^-_{mrg}(x_i) = una(x_i) = \frac{1}{n}F(nx_i)$$

We conjecture that g^+_{mrg} is not a tight upper guarantee for all the functions F we allow in the Lemma.

Proof of statement $i)$ For the Average rule note that the average return AF decreases strictly so that $g^{--}_{avg}(x_i) = x_i AF(x_i + (n-1)H)$. For the same reason $g^+_{avg}(x_i) = x_i AF(x_i) = F(x_i)$.

For the Shapley rule we have

$$\begin{aligned} g^-_{Sha}(x_i) &= \frac{1}{n}\sum_{k=0}^{n-1}(F(x_i + kH) - F(kH)) \\ &< \frac{1}{n}\sum_{k=0}^{n-1}(F(x_i + kx_i) - F(kx_i)) = \frac{1}{n}F(nx) \end{aligned}$$

and $g^+_{Sha}(x_i) = F(x_i)$ after replacing H by 0 in the expression of g^-_{Sha} above.

Remark 5.1 If the domain of types $[L, H]$ starts at $L > 0$, it is easy to check that neither of g^+_{avg} or g^+_{Sha} is tight.

Proof of statement $ii)$ The derivative of $\varphi_i^{mrg}(x)$ w. r. t. $x_{N\diagdown i}$ is $\frac{n-1}{n}\frac{d^2F}{dx^2}(x_N)(x_i - \frac{x_{N\diagdown i}}{n-1})$, so when $x_{N\diagdown i}$ varies from 0 to $(n-1)H$ the share $\varphi_i^{mrg}(x)$ decreases until $x_{N\diagdown i} = (n-1)x_i$ then increases. This implies that for a fixed x_i, the share $\varphi_i^{mrg}(x)$ is minimal at the profile $(\overset{n}{x_i})$ and $g^-_{mrg}(x_i) = \varphi_i^{mrg}(nx_i) = una(x_i)$. Moreover that share $\varphi_i^{mrg}(x)$ is maximal either for $x_{N\diagdown i} = 0$ or for $x_{N\diagdown i} = (n-1)H$, which gives a closed form for $g^+_{mrg}(x_i)$ and some hope to decide whether or not it is tight. ■

6 Rank separable functions

In this section like the previous and next ones the domain of types $\mathcal{X}$ is an interval $[L, H] \subseteq \mathbb{R}$. Given a profile $x \in [L, H]^{[n]}$ we write its *decreasing order statistics* as $(x^k)_{k=1}^n$, so $x^1 = \max_i\{x_i\}$ and $x^n = \min_i\{x_i\}$. The statement "x_i is of rank k in profile x" is unambiguous if x_i is different from every other coordinate; otherwise we mean that x_i appears at rank k for some weakly increasing ordering of the coordinates of x.

Definition 6.1 *The function $\mathcal{W}$ on $[L,H]^{[n]}$ is called rank-separable if there exist n equicontinuous real valued functions w_k on $[L,H]$ s. t. $w_k(L) = w_\ell(L)$ for $k,\ell \in [n]$* and for $x \in [L,H]^{[n]}$

$$\mathcal{W}(x) = \sum_{k=1}^{n} w_k(x^k) \tag{19}$$

From the mathematical angle a rank-separable function is almost everywhere separably additive: this is true in the open cone of $[L,H]^{[n]}$ defined by the strict inequalities $x_1 < x_2 < \cdots < x_n$ and in the $n!$ isomorphic cones obtained by permuting the coordinates.[9]

Note that the equicontinuous functions w_k are differentiable almost everywhere (a. e.) in $[L,H]$.

Lemma 6.1 *The rank-separable function (19) is supermodular if and only if we have:* for $k \in [n-1]$ and a. e. in $x_i \in [L,H]$

$$\frac{dw_k}{dx}(x_i) \leq \frac{dw_{k+1}}{dx}(x_i) \tag{20}$$

and is submodular iff the opposite inequalities hold.

Proof in Appendix 9.6.

For instance $\max_{[n]}\{x_i\} = x^1$ is submodular while $\min_{[n]}\{x_i\} = x^n$ is supermodular.

To introduce our next result we go back to Example 2.1 and write the tight upper guarantees g_p^+ of the function $\mathcal{W}(x) = x^1$ (Proposition 2.1) in a different way

$$g_p^+(x_i) = (x_i - p)_+ + \frac{1}{n}p = \mathcal{W}(x_i, \overset{n-1}{p}) - \frac{n-1}{n}\mathcal{W}(\overset{n}{p})$$

At the beginning of section 5.1 we called this the *stand alone* form $g_{c,\gamma}(x_i) = \mathcal{W}(x_i, c) - \gamma$ where c is a $(n-1)$-profile of types and $\gamma \in \mathbb{R}$. Other examples are the incremental guarantees (Proposition 4.2) and the n guarantees in Proposition 5.1.

In all of these c is a contact profile of *every type* x_i: applying this to the $n-1$ types c_k determines γ as a function of c and $\mathcal{W}$.

Definition 6.2 *Fix a function $\mathcal{W}$ and $c \in [L,H]^{[n-1]}$. If the function g_c* on $[L,H]$

$$g_c(x_i) = \mathcal{W}(x_i, c) - \frac{1}{n}\Big(\sum_{k\in[n-1]} \mathcal{W}(c_k, c)\Big) \tag{21}$$

is a feasible (upper or lower) guarantee in $\mathbf{G}^\varepsilon$, we call it a general stand alone guarantee.

Then g_c is tight and (x_i, c) is a contact profile for all x_i.

Verifying the contact property $g_c(x_i) + \sum_1^{n-1} g_c(c_k) = \mathcal{W}(x_i, c)$ is straightforward.

Theorem 6.1 *Fix a rank-separable and supermodular function $\mathcal{W}$. The set of its tight lower guarantees is given by (21) for all possible choices of c: $\mathcal{G}^-(\mathcal{W}) = \{g_c; c \in [L,H]^{[n-1]}\}$. If instead $\mathcal{W}$ is submodular, this is the set of its tight upper guarantees.*

Proof in Appendix 9.7.

If the parameter $c = (\overset{n-1}{c_0})$ is unanimous the tight guarantee $g_c(x_i) = \mathcal{W}(x_i; \overset{n-1}{c_0}) - \frac{n-1}{n}\mathcal{W}(\overset{n}{c_0})$ "touches" the unanimity guarantee at c_0, $g_c(c_0) = una(c_0)$, and c_0 is the familiar benchmark type. But if c is not unanimous, we do not expect the graph of these two functions to share a point.[10]

[9] For a continuous and symmetric function $\mathcal{W}$ Rank Separability is characterised by the property that $\partial_{ij}\mathcal{W}(x) = 0$ for all $i,j \in [n]$ and x such that $x_i \neq x_j$.

[10] Take for instance $n = 3$, $\mathcal{W}(x) = 3x^1 + 2x^2 + x^3$ in the domain $[0,1]$ and check that the two graphs do not meet if $c_1 > 0$ and $2c_1 + c_2 > 1$.

Example 6.1 *sharing the connection cost*

After each agent i chooses a location x_i in the interval $[L, H]$ the manager must cover the cost of connecting them (e g by building a road) which we assume linear in the largest distance between agents: for $x \in [L, H]^{[n]}$

$$\mathcal{W}(x) = x^1 - x^n \tag{22}$$

Should an agent be penalised (pay more than the average) for being far away at the periphery of the distribution of agents, and if so, by how much? As in Example 2.1, enforcing a pair of tight guarantees does not require to take a position on this issue: it allows to completely disregrard differences in location, or on the contrary to reward steeply proximity to an arbitrary location, and a two-dimensional family of intermediate positions between these two.

The cost function $\mathcal{W}$ is submodular and the tight lower guarantee is $una(x_i) \equiv 0$: everyone's best case is to pay nothing. By Theorem 6.1 a tight upper guarantee involves the choice of $n-1$ variables c_k but it is easy to check in equation (21) that for any $n \geq 3$ only the largest and smallest values c^+ and c^- matter:

$$g_c(x_i) = (\max\{x_i, c^+\} - \min\{x_i, c^-\}) - \frac{n-1}{n}(c^+ - c^-)$$

Setting $\mu = \frac{1}{n}(c^+ - c^-)$ we develop this equation as follows: $g_c(x_i) = \mu$ if $c^- \leq x_i \leq c^+$; $g_c(x_i) = \mu + (c^- - x)$ if $L \leq x_i \leq c^-$; $g_c(x_i) = \mu + (x - c^+)$ if $c^+ \leq x_i \leq H$.

All types in the benchmark interval $[c^-, c^+]$ have the same worst cost share μ; a type outside this interval could pay, in addition to μ, the full connecting cost to the benchmark.

If $c^- = c^+ = c^*$ an agent locating at c^* pays nothing (irrespective of other agents' location) and $g_c(x_i) = |x - c^*|$. While if $(c^-, c^+) = (L, H)$ the worst cost share is $\frac{1}{n}(H - L)$ for everybody, compatible with the egalitarian sharing rule, and many others.

Remark 6.1 *A facility location problem with linear transportation costs (instead of quadratic in Example 5.2) generates a cost function similar to (22): the optimal location of the facility is at the median of the individual locations x_i in $[L, H]$. If $n = 2m + 1$ is odd the median is x^{q+1} and the total cost to share is*

$$\mathcal{W}(x) = \sum_{k=1}^{q} x^k - \sum_{\ell=q+2}^{2q+1} x^\ell$$

still submodular. The tight upper guarantees g_c resembles those just discussed but with more flexibility in the design because tgheir set is of dimension $n-1$.

Example 6.2 *ranked commons*

Fix a rank $k \in [n]$. Agent i inputs the effort x_i: to achieve the output $y = F(z)$ we need at least $n - k + 1$ agents contributing an effort at least z: for $x \in [L, H]^{n]}$

$$\mathcal{W}_k(x) = F(x^k) \tag{23}$$

The function $\mathcal{W}_k$ is *neither sub nor supermodular*, except submodular for $k = 1$ (Example 2.1) and supermodular for $k = n$, two cases we exclude below.

We see that $una(x_i) = \frac{1}{n}F(x_i)$ is neither a lower guarantee nor an upper guarantee: there is now a one dimensional choice of tight guarantees on both sides of (4). It is in fact easy to describe the sets $\mathcal{G}_k^{\pm}$: the proof given in Appendix 9.8 mimicks that of Proposition 1.

The set $\mathcal{G}_k^+$ is parametrised by $p \in [L, H]$:

$$g_{k,p}^+(x_i) = \frac{1}{n}F(p) + \frac{1}{k}(F(x_i) - F(p))_+$$

and $\mathcal{G}_k^-$ is similarly parametrised by $q \in [L, H]$:

$$g_{k,q}^-(x_i) = \frac{1}{n}F(q) + \frac{1}{n-k+1}(F(x_i) - F(q))$$

If $p = q = z^*$ this "standard" level of effort guarantees the share $\frac{1}{n}F(z^*)$. If the actual output x^k is below z^* the agents inputting a sub-standard effort must subsidize those who input at least z^* because the worst share of the latter is $\frac{1}{n}F(z^*)$; conversely if x^k is above z^* the "slackers" cannot get more than the standard share $\frac{1}{n}F(z^*)$, and may get less.

7 Two person modular problems

In two person strictly modular problems with one-dimensional types we give a general representation of the tight solutions of system (4) on the other side of the unanimity (Proposition 4.1) by means of their contact set. For a tight guarantee g this set has the simple shape of a decreasing and occasionally multivalued function φ described in the next two Lemmas. Conversely we can pick any such function φ and integrate the critical differential equation $\frac{dg}{dx_i}(x_i) = \frac{\partial \mathcal{W}}{\partial x_i}(x_i, \varphi(x_i))$ (Lemma 3.9) to get an integral representation of a tight guarantee, and in turn a complete resolution of the functional inequalities (4).

For any modular function $\mathcal{W}$ on $[L, H]^2$ and $g \in \mathcal{G}^{\pm}$, a tight guarantee on either side of (4), we define the contact correspondence φ:

$$\varphi(x_1) = \{x_2 \in [L, H] | g(x_1) + g(x_2) = \mathcal{W}(x_1, x_2)\} \tag{24}$$

(non empty by Lemma 3.5). Its graph is $\Gamma(\varphi)$.

Lemma 7.1 *If* $\mathcal{W}$ is supermodular, $g \in \mathcal{G}^-$ and $\Gamma(\varphi)$ *contains* (x_1, x_2) *and* (x_1', x_2') *s.t.* $(x_1, x_2) \ll (x_1', x_2')$, *then* $(x_1, x_2'), (x_1', x_2) \in \Gamma(\varphi)$ *as well, and* $\mathcal{W}$ *is not strictly supermodular.*

For a submodular function $\mathcal{W}$ *simply replace* $\mathcal{G}^-$ *by* $\mathcal{G}^+$.

Proof We sum up the two equalities in (24) for (x_1, x_2) and $(x_1\prime, x_2\prime)$:

$$\mathcal{W}(x_1, x_2) + \mathcal{W}(x_1', x_2') = \{g(x_1) + g(x_2')\} + \{g(x_1') + g(x_2)\} \le \mathcal{W}(x_1, x_2') + \mathcal{W}(x_1', x_2)$$

Combined with the supermodular inequality (10) this gives an equality and the conclusion by Definition 4.1. As explained in Appendix 9.3 this also means that $\mathcal{W}$ is locally additive. ∎

Lemma 7.2 *Fix and a strictly supermodular function* $\mathcal{W}$ *and a tight guarantee* $g \in \mathcal{G}^-$ *– or a submodular* $\mathcal{W}$ *and* $g \in \mathcal{G}^+$ *– with contact correspondence* φ.

i) $\Gamma(\varphi)$ *is symmetric:* $x_2 \in \varphi(x_1) \iff x_1 \in \varphi(x_2)$ *for all* x_1, x_2.

ii) φ *is convex valued:* $\varphi(x_1) = [\varphi^-(x_1), \varphi^+(x_1)]$, *single-valued a.e., and upper-hemi-continuous (its graph is closed).*

iii) φ^- *and* φ^+ *are weakly decreasing and* $x_1 \le x_1' \Longrightarrow \varphi^-(x_1) \ge \varphi^+(x_1')$; φ *is the u.h.c. closure of both* φ^- and φ^+.

iv) $\varphi(L)$ *contains* H *and* $\varphi(H)$ *contains* L.

v) φ *has a unique fixed point* a: $a \in \varphi(a)$, *and* a *is an end-point of* $\varphi(a)$.

Proof in the Appendix 9.9.

Theorem 7.1 *Fix a strictly super- (resp. sub-) modular function* $\mathcal{W}$, *continuously differentiable in* $[L, H]^2$.

i) For any correspondence φ *as in Lemma 7.2, the following equation*

$$g(x_1) = \int_a^{x_1} \partial_1 \mathcal{W}(t, \varphi(t))dt + una(a) \tag{25}$$

defines a tight lower guarantee $g \in \mathcal{G}^-$ (resp. $\mathcal{G}^+$).
ii) Conversely if g is a guarantee in $\mathcal{G}^-$ (resp. $\mathcal{G}^+$) with contact correspondence φ (as in Lemma 7.2) then g takes the form (25).

Proof in the Appendix 9.10.

Theorem 2 shows that the sets $\mathcal{G}^{\pm}$ on the other side of unanimity are parametrised by a large set of functions φ. After choosing the benchmark type a which guarantees the share $una(a)$ we can pick any decreasing single-valued function $\overline{\varphi}$ from $[L,a]$ into $[a,H]$ mapping L to H, then fill the (countably many) jumps down to create the correspondence φ of which the graph connects (L,H) to (a,a), and finally extend φ to $[a,H]$ where its graph is the symmetric around the diagonal of $[L,H]^2$ of its graph in $[L,a]$, so that φ maps H to L.

We illustrate this embarrassement of riches in the problem of section 5.

Example 7.1 *Commons with substitutable inputs*

We have $\mathcal{W}(x) = F(x_1 + x_2)$ and F is strictly concave on $[0,1]$.

The contact correspondence of the incremental guarantees g_{inc} is $\varphi_{inc}(0) = [0,1]; \varphi_{inc}(x_1) = 0$ for $x_1 \in]0,1]$; and φ^{inc} simply exchange the role of L and H.

Proposition 5.1 has no bite for $n = 2$. Statement $i)$ in Proposition 5.2 delivers a single full tangent guarantee $g_{\frac{1}{2}}$ in (16) with the anti-diagonal contact function $\varphi_{\frac{1}{2}}(x_1) = 1 - x_1$. The contact functions of the guarantees in statements $ii)$ and $iii)$ are two-piece linear. For instance if $\alpha \in [0, \frac{1}{2}]$: $\varphi_\alpha(x_1) = 2\alpha - x_1$ on $[0, 2\alpha]$ and $\varphi_\alpha(x_1) = 0$ on $[2\alpha, 1]$.

To find new tight upper guarantees connecting g_{inc} and g^{inc} we pick φ with a similar piecewise constant graph. For $\beta \in [0,1]$ define $\varphi_\beta \equiv 1$ on $[0, \beta[$; $\varphi_\beta(\beta) = [\beta, 1]$; $\varphi_\beta \equiv \beta$ on $]\beta, 1]$. Equation (25) gives on $[0, \beta]$

$$g_\beta(x_1) = F(x_1 + 1) - F(\beta + 1) + \frac{1}{2}F(2\beta)$$

and on $[\beta, 1]$

$$g_\beta(x_1) = F(x_1 + \beta) - \frac{1}{2}F(2\beta)$$

concatenating parts of two different stand alone guarantees, connected at $x_1 = \beta$ where they touch the unanimity graph but, unlike with the guarantees g_α in Proposition 5.2, the connection is not smooth.

Exchanging the roles of the end types 0 and 1 we define for $\gamma \in [0,1]$: $\varphi_\gamma \equiv \gamma$ on $[0, \gamma[$; $\varphi_\gamma(\gamma) = [0, \gamma]$; $\varphi_\gamma \equiv 0$ on $]\gamma, 1]$. Then (25) delivers a new non smooth concatenation of partial stand alone guarantees:

$$g_\gamma(x_1) = F(x_1 + \gamma) - \frac{1}{2}F(2\gamma)$$

on $[0, \gamma]$ and on $[\gamma, 1]$:

$$g_\gamma(x_1) = F(x_1) - F(\gamma) + \frac{1}{2}F(2\gamma)$$

8 Concluding comments

We start with two open questions.

extending Theorem 7.1 for $n \geq 3$ The key for two agent problems is the deep understanding of the contact correspondence of any tight guarantee (Lemmas 7.1, 7.2). We could not gain a similar understanding of this correspondence with three or more agents. In particular Lemma 7.2 shows that in a *two agent* problem the contact set of *every* tight guarantee g in $\mathcal{G}^{\varepsilon}$ intersects the diagonal (and g touches una): this gives the crucial starting point of the integral equation (25). But we saw in Proposition 5.1 and Theorem 6.1 many problems with $n \geq 3$ where the contact set of some of tight guarantees does not intersect the diagonal.

multi-dimensional types The general results in section 3 apply to functions $\mathcal{W}$ of m real variables x_i for any m, and so do the Propositions 4.1 and 4.2 for general modular functions. On the way to further develop the multidimensional analysis we run into an extremely challenging *decentralisation* question.

The following claim is obvious from the definitions and Lemma 3.5. Suppose each type has two components $x_i = (x_i^1, x_i^2) \in \mathcal{X}^1 \times \mathcal{X}^2 = \mathcal{X}$ and we have two functions: $\mathcal{W}_1$ defined on $\mathcal{X}^{1[n]}$ and $\mathcal{W}_2$ on $\mathcal{X}^{2[n]}$. If $g_i^\varepsilon \in \mathcal{G}^\varepsilon(\mathcal{W}_i)$ for some $\varepsilon = +, -$ and both $i = 1, 2$, then $g_1^\varepsilon + g_2^\varepsilon$ is a tight guarantee of the function $\mathcal{W}$ adding the two independent problems as $\mathcal{W}(x) = \mathcal{W}_1(x^1) + \mathcal{W}_2(x^2)$ for $x \in \mathcal{X}$.

We do not know for which domain of functions $\mathcal{W}$ the converse decentralisation property holds: *every tight guarantee g^ε of $\mathcal{W}_1 + \mathcal{W}_2$ (both in the domain) is the sum of two tight guarantees in the component problems.*

The answer eludes us even for the specific problem of assigning multiple indivisible objects and cash transfers when utilities are additive over objects (and linear in money), which is a sum of multiple one object problems as in Example 2.1 (second interpretation). With much sweat we showed that the decentralisation property holds for two agents and two objects!

some take-home points The generality of our approach is a *story-free* interpretation of individual rights in an abstract model of cooperative production.Each pair of tight guarantees severely restricts the range of feasible allocations, but is far from choosing a specific division of the commons; a frequent exception is one single "benchmark" type whose share is fixed, does not depend upon other agents' types.

The prominent role of the unanimity guarantee in the rich class of modular problems confirms its importance already recognised in other fair division problems (see the literature review in section 1).

We find that even if the unanimity guarantee is optimal for one side of system (3), the other side offers an choice of infinitely many guarantees.

When the function $\mathcal{W}$ takes a concrete interpretation in our examples, this large menu offers a range of options compatible with many sharply different normative positions the designer can take.

We can of course apply new tests to help our choice in the menu. For instance, in line with the Lockean interpretation, we can choose $g^\pm$ so that the pair $(una, g^\pm)$ minimises the largest gap over all types, or the expected gap w.r.t. some given distribution of types.

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9 Appendix: missing proofs

9.1 Lemma 3.5

Step 1: upper-hemi-continuity We fix $g \in \mathcal{G}^-$ and check that it is u.h.c.. If it is not, there is in $\mathcal{X}$ some x_1, a sequence $\{x_1^t\}$ converging to x_1, and some $\delta > 0$ such that $g(x_1^t) \geq g(x_1) + \delta$ for all t. Then we have, for any $x_{-1} \in \mathcal{X}^{[n-1]}$

$$\mathcal{W}(x_1^t, x_{-1}) \geq g(x_1^t) + \sum_{i=2}^{n} g(x_i) \geq (g(x_1) + \delta) + \sum_{i=2}^{n} g(x_i)$$

Taking the limit in t of $\mathcal{W}(x_1^t, x_{-1})$ and ignoring the middle term we see that we can increase g at x_1 without violating (4), a contradiction of our assumption $g \in \mathcal{G}^-$.

Step 2: statement ii) "If" is clear. For "only if" we fix $g \in \mathcal{G}^-$ and show that it meets property (*7*). For any $x_1 \in \mathcal{X}$ define

$$\delta(x_1) = \min_{x_{-1} \in \mathcal{X}^{[n-1]}} \{\mathcal{W}(x_1, x_{-1}) - \sum_{[n]} g^-(x_i)\}$$

and note that this minimum is achieved at some $\overline{x}_{-1}$ because the function $x_{-1} \to \sum_{i=2}^{n} g^-(x_i)$ is u.h.c. (step 1). Moreover $\delta(x_1)$ is non negative.

If $\delta(x_1) = 0$ property (7) holds at $\overline{x}_{-1}$. If $\delta(x_1) > 0$ we can increase g at x_1 to $g(x_1) + \delta(x_1)$, everything else equal, to get a guarantee dominating g.

Step 3: lower-hemi-continuity We fix $g \in \mathcal{G}^-$ and check that it is l.h.c.. By assumption $\mathcal{W}$ is equi-continuous in its first variable, uniformly in the others:

$$\forall \eta > 0, \exists \theta > 0, \forall x_1, x_1^*, x_{-1} : ||x_1 - x_1^*|| \leq \theta \Rightarrow \mathcal{W}(x_1, x_{-1}) \leq \mathcal{W}(x_1^*, x_{-1}) + \eta \quad (26)$$

If g is not l.h.c. there is some x_1 and $\{x_1^t\}$ converging to x_1 and $\delta > 0$ s.t. $g(x_1^t) \leq g(x_1) - \delta$ for all t. Pick θ for which (26) holds with $\eta = \frac{1}{2}\delta$ and t large enough that $||x_1^t - x_1|| \leq \theta$: then for any x_{-1} we have

$$g(x_1) + \sum_{i=2}^{n} g(x_i) \leq \mathcal{W}(x_1, x_{-1}) \leq \mathcal{W}(x_1^t, x_{-1}) + \frac{1}{2}\delta$$

Replacing $g(x_1)$ with $g(x_1^t) + \delta$ gives $g(x_1^t) + \sum_{i=2}^{n} g(x_i) \leq \mathcal{W}(x_1^t, x_{-1}) - \frac{1}{2}\delta$ for any x_{-1}: this contradicts the contact property (7) for x_1^t.

9.2 Lemma 3.6

Proof *Statement i)* Fix $\varepsilon = -$, an arbitrary $\widetilde{x}_1 \in \mathcal{X}$ and write $B(\widetilde{x}_1, r)$ for the closed ball of center $\widetilde{x}_1$ and radius r. Use the notation $\Delta(x) = \sum_1^n una(x_i) - \mathcal{W}(x)$ to define the function

$$\delta(x_1) = \max\{\Delta(x_1, x_{-1}) : \forall i \geq 2, x_i \in B(\widetilde{x}_1, d(x_1, \widetilde{x}_1))\}$$

It is clearly continuous, non negative because $\Delta(x_1, x_{-1}) = 0$ if $x_i = x_1$ for $i \geq 2$, and $\delta(\widetilde{x}_1) = 0$. Define $g = una - \delta$ and check that g is the desired lower guarantee of $\mathcal{W}$. At an arbitrary profile $x = (x_i)_1^n$ choose x_{i^*} s.t. $d(\widetilde{x}_1, x_{i^*})$ is the largest: this implies $\delta(x_{i^*}) \geq \Delta(x)$. Combining this with $\delta(x_i) \geq 0$ for $i \neq i^*$ gives $\sum_1^n \delta(x_i) \geq \Delta(x)$ which, in turn, is the LS inequality in (4) for g. As g is in $\boldsymbol{G}^-$, it is dominated by some $\widetilde{g}$ in $\mathcal{G}^-$(Lemma 3.1) and $\widetilde{g}(x_1) = una(\widetilde{x}_1)$ by inequality (6).

Statement ii) The *if* part is statement *ii)* in Lemma 3.2. For *only if* we assume that $\boldsymbol{G}^-$ does not contain una and check that $\boldsymbol{G}^-$ is not a singleton. This assumption and the continuity of $\mathcal{W}$ imply that for an open set of profiles $x \in \mathcal{X}^{[n]}$ we have $\sum_{[n]} una(x_i) > \mathcal{W}(x)$. Fix such an x and (by Lemma 3.1) pick for each i a tight guarantee g_i equal to una at x_i: these n guarantees are not identical.

9.3 Some properties of modular functions

Whenever the partial derivative $\partial_i \mathcal{W}(x)$ is defined in a neighborhood of x, supermodularity implies that it is weakly increasing in x_j for $j \neq i$. And if $\partial_i \mathcal{W}(x)$ is strictly increasing in x_j then $\mathcal{W}$ is strictly supermodular. The isomorphic statements for submodularity replaces increasing by decreasing.

Whenever $\partial_i \mathcal{W}(x)$ is differentiable almost everywhere, the supermodularity property can be written: for $i, j \in [n], i \neq j$, $\partial_{ij} \mathcal{W}(x) \geq 0$ a. e. in $x \in [L, H]^{[n]}$. For submodularity reverse the inequality.

A well known consequence of modularity is this: if $(x_i, x_j) \ll (x'_i, x'_j)$ and the RS of (*10*) is an equality, then in the interval $[(x_i, x_j), (x'_i, x'_j)]$ the function $(z_i, z_j) \to \mathcal{W}(z_i, z_j; x_{-i,j})$ is separably additive, and its cross derivative $\partial_{ij} \mathcal{W}(\cdot, \cdot; x_{-i,j})$ is identically zero. We say that $\mathcal{W}$ is *locally* i, j-*additive* at the profile x if there is a rectangular neighborhood of (x_i, x_j) in which $\partial_{ij} \mathcal{W}(\cdot; x_{-i,j})$ is zero.

A strictly modular function like, in section 5, $\mathcal{W}(x) = F(\sum_{[n]} x_i)$ with F strictly convex or strictly concave, is not i, j-additive anywhere. But the submodular function $\mathcal{W}(x) = \max_i \{x_i\}$ (Example 2.1) is locally i, j-additive whenever $x_i \neq x_j$, hence almost everywhere (although $\mathcal{W}$ is not globally i, j-additive!).

Submodularity is preserved by positive linear combinations, but not by the maximum or minimum operation. For instance if n is odd, the median of profile x is $\min_S \max_{i \in S}\{x_i\}$ where the min is over all majority coalitions S: it is the minimum of several submodular functions but is neither sub- nor super-modular.

9.4 Proposition 4.3

We prove the statement for the serial↑ rule (14). By Remark 3.1 it is enough to check the inequality $g_{inc}(x_i) \leq \varphi_i^{ser\uparrow}(x) \leq una(x_i)$ for all x.

Step 1. We show that $\varphi_i^{ser\uparrow}(x)$ increases (weakly) in all variables x_j such that $x_j \leq x_i$, i. e., for $j \leq i - 1$. This generalises Lemma 1 in [25].

If $\mathcal{W}$ is differentiable in $[L,H]^n$ we check this by computing the derivative $\partial_k \varphi_i^{ser\uparrow}$ for $k \le i-1$ in the LS of equation (14) and using the symmetry of $\mathcal{W}$:

$$\partial_k \varphi_i^{ser\uparrow}(x) = \frac{\partial_k \mathcal{W}(x_1,\cdots,x_{i-1},\overset{n-i+1}{x_i})}{n-i+1} - \frac{\partial_k \mathcal{W}(x_1,\cdots,x_{k-1},\overset{n-k+1}{x_k})}{n-k} - \sum_{j=k+1}^{i-1} \frac{\partial_k \mathcal{W}(x_1,\cdots,x_{j-1},\overset{n-j+1}{x_i})}{(n-j+1)(n-j)}$$

Because $\partial_k \mathcal{W}$ increases weakly in $x_j, j \neq k$, the numerator of each negative fraction is not larger than that of the first fraction. The identity $\frac{1}{n-i+1} = \frac{1}{n-k} + \sum_{j=k+1}^{i-1} \frac{1}{(n-j+1)(n-j)}$ concludes the proof.

Without the differentiability assumption the only step that requires an additional argument is the following consequence of supermodularity. If the coordinates of the profile x are weakly increasing then $\mathcal{W}(x) - \frac{1}{n-k+1}\mathcal{W}(x_1,\cdots,x_{k-1},\overset{n-k+1}{x_k})$ increases weakly in x_k for each $k \le n-1$. We omit the straightforward proof.

Step 2. By construction of $\varphi^{ser\uparrow}$ we have $\varphi_i^{ser\uparrow}(x) = \varphi_i^{ser\uparrow}(x_1,\cdots,x_{i-1},\overset{n-i+1}{x_i})$ and by Step 1 it is enough to check that $g_{inc}(x_i)$ lower bounds $\varphi_i^{ser\uparrow}(x)$ at the profile $(\overset{i-1}{L},\overset{n-i+1}{x_i})$ while *una* upper bounds it at $(\overset{n}{x_i})$. The latter follows from $\varphi_i^{ser\uparrow}(\overset{n}{x_i}) = una(x_i)$. Applying (14) we see that the desired lower bound reduces to

$$\mathcal{W}(\overset{n-1}{L},x_i) \le \frac{1}{n-i+1}\mathcal{W}(\overset{i-1}{L},\overset{n-i+1}{x_i}) + \frac{n-i}{n-i+1}\mathcal{W}(\overset{n}{L})$$

$$\Longleftrightarrow (n-i)(\mathcal{W}(\overset{n-1}{L},x_i) - \mathcal{W}(\overset{n}{L})) \le \mathcal{W}(\overset{i-1}{L},\overset{n-i+1}{x_i}) - \mathcal{W}(\overset{n-1}{L},x_i)$$

Finally we apply (11) to successively lower bound $\mathcal{W}(\overset{k}{L},\overset{n-k}{x_i}) - \mathcal{W}(\overset{k+1}{L},\overset{n-k-1}{x_i})$ by $\mathcal{W}(\overset{n-1}{L},x_i) - \mathcal{W}(\overset{n}{L})$ for $k = (n-2),\cdots,(i-1)$ and sum up these inequalities.

9.5 Proposition 5.1

Step 1 The function $g_{\ell,h}$ defined by (15) is a lower guarantee: $g_{\ell,h} \in \mathbf{G}^-$.

We set $Z = \ell L + hH$ for easier reading. The feasibility inequality (4) applied to $g_{\ell,h}$ reads: for $x \in [L,H]^{[n]}$

$$\sum_{[n]} F(x_i + Z) \le F(x_N) + \ell F(Z+L) + hF(Z+H) \tag{27}$$

We proceed by induction on n. There is nothing to prove if $n=2$. For $n=3$ we already know that $g_{2,0}$ and $g_{0,2}$ are in $\mathcal{G}^-$; for $g_{1,1}$ the inequality (27) is

$$\sum_{[3]} F(x_i + L + H) \le F(x_{123}) + F(2L+H) + F(L+2H) \tag{28}$$

Suppose $x_{12} \ge L+H$: then the convexity of F implies

$$F(x_3 + L + H) - F(2L+H) \le F(x_{123}) - F(x_{12}+L)$$

Replacing $F(x_3+L+H)$ in (28) by this upper bound and rearranging gives a more demanding inequality

$$F(x_1+L+H) + F(x_2+L+H) \le F(x_{12}+L) + F(L+2H)$$

following again from the convexity of F.

So we are done if $x_{ij} \geq L + H$ for any pair i, j. Suppose next $x_{ij} \leq L + H$ for all three pairs. Then we have for $i = 1, 2, 3$

$$x_{123}, 2L + H \leq x_i + L + H \leq L + 2H$$

and the uniform distribution on the triple $x_{123}, 2L + H, L + 2H$ is a mean-preserving spread of that on $(x_i + L + H)_{i \in [3]}$, which proves (28).

For the inductive argument we fix $n \geq 4$ and $g_{\ell,h}$ s. t. $\ell + h = n - 1$ and $\ell \geq 1$. We assume that (27) holds for $n - 1$ agent problems and prove it for (ℓ, h).

Suppose $x_{N \setminus \{n\}} \geq Z$ for some agent labeled n without loss of generality. Then the convexity of F implies

$$F(x_n + Z) - F(Z + L) \leq F(x_N) - F(x_{N \setminus \{n\}} + L)$$

As before we replace $F(x_n + Z)$ by this upper bound and rearrange (27) to the more demanding

$$\sum_{[n-1]} F(x_i + Z) \leq F(x_{N \setminus \{n\}} + L) + (\ell - 1)F(Z + L) + hF(Z + H)$$

which for the convex function $\widetilde{F}(y) = F(y + L)$ and $\widetilde{Z} = (\ell - 1)L + hH$ is exactly (27) at x_{-n} for the guarantee $g_{(\ell-1),h}$.

We are left with the case where $x_{N \setminus \{i\}} \leq Z$ for all i for which the different terms under F in (27) are ranked as follows:

$$x_N, Z + L \leq x_i + Z \leq Z + H$$

and the distribution $(\frac{1}{n}, \frac{\ell}{n}, \frac{h}{n})$ on the support $x, Z + L, Z + H$ is a mean-preserving spread of the uniform distribution on the n inputs $x_i + Z$. So $g_{\ell,h}$ meets (27).

If $h \geq 1$ the symmetric proof starts by assuming $x_{N \setminus \{n\}} \leq Z$ and using the convexity inequality

$$F(x_{N \setminus \{n\}} + H) - F(x_N) \leq F(Z + H) - F(x_n + Z)$$

to obtain a more demanding inequality that is in fact (27) for $g_{\ell,h-1}$ and the function $\overleftrightarrow{F}(y) = F(y + H)$.

Step 2 The guarantee $g_{\ell,h}$ is tight. We fix x_i and compute

$$g_{\ell,h}(x_i) + \ell g_{\ell,h}(L) + h g_{\ell,h}(H) = F(x_i + \ell L + hH)$$

We see that the profile $(x_i, \overset{\ell}{L}, \overset{h}{H})$ is in the contact set of $g_{\ell,h}$ at x_i and conclude by Lemma 3.5.

Statement ii) The derivative of the gap function is $\frac{dF}{dx}(nx_i) - \frac{dF}{dx}(x_i + Z)$ which changes from negative to positive at $\frac{1}{n-1}Z$.

Statement iii) The equality $g_{\ell,h}(x_i) = una(x_i)$ is rearranged as_i:

$$F(x_i + Z) = \frac{1}{n}F(nx_i) + \frac{\ell}{n}F(Z + L) + \frac{h}{n}F(Z + H))$$

This contradicts the strict convexity of F if ℓ, h are both positive. If ℓ or h is zero we are dealing with g_{inc} or g^{inc} with unanimous contact points at L and H respectively.

9.6 Lemma 6.1

Fix $\mathcal{W}$ defined by (19) and the equicontinuous functions w_k. For "only if" we assume that $\mathcal{W}$ is supermodular. Fix two agents i, j and a $(n-2)$-profile $x_{-ij} \in [L, H]^{[n]\setminus i,j}$. For any 4-tuple x_i, y_i, x_j, y_j such that $x_i > y_i$ and $x_j > y_j$ supermodularity means

$$\mathcal{W}(x_i, x_j; x_{-ij}) - \mathcal{W}(y_i, x_j; x_{-ij}) \geq \mathcal{W}(x_i, y_j; x_{-ij}) - \mathcal{W}(y_i, y_j; x_{-ij})$$

Suppose $L < y_i < x_i < H$ and pick an arbitrary rank $k, k \leq n-1$: we can choose x_{-ij}, x_j and y_j s. t. in the profiles on the RH x_i and y_i are of rank k, while after increasing y_j to x_j they are of rank $k+1$ in the profiles on the LH. Then the inequality above reads

$$w_{k+1}(x_i) - w_{k+1}(y_i) \geq w_k(x_i) - w_k(y_i)$$

As x_i, y_i can be chosen arbitrary close to each other, this proves (20) at any interior point of $[L, H]$ where w_k is differentiable (that is, a. e.).

For "if" we assume (20) and fix x_{-ij}. For any x_i, y_j s. t. x_i has rank k in $(x_i, y_j; x_{-ij})$ we have $\partial_i \mathcal{W}(x_i, y_j; x_{-ij}) = \frac{dw_k}{dx}(x_i)$ (a. e.): if y_j is below x_i and jumps up to x_j above x_i then by (20) $\partial_i \mathcal{W}(x_i, x_j; x_{-ij})$ also increases (weakly) to $\frac{dw_{k+1}}{dx}(x_i)$. If x_i is not isolated in the profile $(x_i, y_j; x_{-ij})$ the same argument applies to the left and right derivatives of $\mathcal{W}$ in x_i.

9.7 Theorem 6.1

We fix $\mathcal{W}$ given by (19) and supermodular, so $\frac{dw_k}{dx}(\cdot)$ increases weakly with k.

Step 1. *For any c the function g_c defined by (21) is in $\mathcal{G}^-$.* We saw in Definition 6.2 that it is enough to show $g_c \in \mathbf{G}^-$.

Because $g_c(x_i)$ and $\mathcal{W}(x_i; c)$ are continuous in x_i, c it is enough to prove the LS inequality (4) for strictly decreasing sequences $\{x_\ell\}_1^n$ and $\{c_k\}_1^{n-1}$ such that $H > c_1$ and $c_{n-1} > L$ and moreover $x_\ell \neq c_k$ for all ℓ, k. These assumptions hold for all the sequences x, c below.

Step 1.1 Call the profile of types x^* *regular* if

$$x_1^* > c_1 > x_2^* > c_2 > \cdots > c_{k-1} > x_k^* > c_k > \cdots > c_{n-1} > x_n^* \tag{29}$$

then compute

$$\sum_1^n g_c(x_k^*) = \sum_1^n \mathcal{W}(x_k^*, c) - \sum_1^{n-1} \mathcal{W}(c_k, c) = \sum_1^{n-1} (w_k(x_k^*) - w_k(c_k)) + \mathcal{W}(x_n^*, c) = \mathcal{W}(x^*)$$

so that x^* is a contact profile of g_c.

Step 1.2 For any three sequences x, x' and c we say that x' is reached from x by an *elementary jump up above* c_k if there is some ℓ such that $x_{-\ell} = x'_{-\ell}$; c_k is adjacent to x_ℓ in x from above and adjacent to x'_ℓ in x' from below. In other words: $x'_\ell > c_k > x_\ell$ and there is no other element of x or c between x_ℓ and x'_ℓ. The definition of an elementary jump down below c_k is exactly symmetrical.

We claim that for any sequence $\widetilde{x}$ we can find a regular profile x^* and a path (a sequence of sequences) $\sigma = \{\widetilde{x} = x^1, \cdots, x^t, \cdots, x^T = x^*\}$ such that
1) each step from x^t to x^{t+1} is an elementary jump up or down of some x_ℓ^t over some c_k
2) $\ell \leq k$ if x_ℓ^t jumps up above c_k, and $\ell \geq k+1$ if x_ℓ^t jumps down below c_k.

The proof by induction on n starts by distinguishing

Case 1: $\widetilde{x}_1 > c_1$. Then $\widetilde{x}_1$ never moves and $\widetilde{x}_1 = x_1^*$; if $\widetilde{x}_2, \cdots, \widetilde{x}_\ell$ are above c_1 then $\ell - 1$ successive elementary jumps down of these below c_1 defines the first $\ell - 1$ steps of the desired path;

continuing until there are none, it remains to construct a path from the shorter sequence $\widetilde{x}_{-1}$ into a one regular w. r. t. the sequence c_{-1} by invoking the inductive assumption.

Case 2: $c_1 > \widetilde{x}_1$. Then the successive elementary jumps up of $\widetilde{x}_1$ over the closest c_k then $c_{k-1}, \cdots, c_1$ define the first k steps of the desired path until $x^{k+1} = x_1^*$ that never moves again; then we proceed with the shorter sequences $\widetilde{x}_{-1}$ and c_{-1} by the inductive assumption.

Step 1.3 We pick an arbitrary profile $\widetilde{x}$ and construct a sequence σ from $\widetilde{x}$ to some regular x^*, and check that in each step of the sequence the sum $\sum_1^n g_c(x_\ell) - \mathcal{W}(x)$ cannot decrease, which together with Step 1.1 concludes the proof that $g_c \in \mathbf{G}^-$. This sum develops as

$$\overbrace{(\sum_{\ell=1}^{n} \mathcal{W}(x_\ell, c))}^{B} - \overbrace{\mathcal{W}(x)}^{C} - \overbrace{\sum_{k=1}^{n-1} \mathcal{W}(c_k, c)}^{D}$$

Consider a jump up of x_ℓ^t above c_k: $x_\ell^{t+1} > c_k > x_\ell^t$. The net changes to the sum are

$$\begin{aligned} \Delta B &= w_k(x_\ell^{t+1}) - w_{k+1}(x_\ell^t) + w_{k+1}(c_k) - w_k(c_k) \\ \Delta C &= w_\ell(x_\ell^{t+1}) - w_\ell(x_\ell^t) \ ; \ \Delta D = 0 \end{aligned}$$

With the notation $\Delta f(a \to b) = f(b) - f(a)$ and some rearranging this gives

$$\Delta B - \Delta C + \Delta D = \Delta(w_k - w_\ell)(c_k \to x_\ell^{t+1}) + \Delta(w_{k+1} - w_\ell)(x_\ell^t \to c_k)$$

where both final Δ terms are non negative because $\ell \leq k$ and by (20) $w_k - w_\ell$ and $w_{k+1} - w_\ell$ increase weakly.

The proof for a jump down step is quite similar by computing the variation of $\sum_1^n g_c(x_\ell) - \mathcal{W}(x)$ to be $\Delta(w_\ell - w_k)(c_k \to x_\ell^t) + \Delta(w_\ell - w_{k+1})(x_\ell^{t+1} \to c_k)$ and recalling that in this case we have $\ell \geq k + 1$.

Step 2 *A tight guarantee $g \in \mathcal{G}^-$ takes the form g_c in (21).*

Recall the notation $\mathcal{C}(g)$ for the set of contact profiles of g defined by (7). For each $k \in [n]$ its projection $\mathcal{C}_k(g)$ is the set of those $x_i \in [L, H]$ appearing in some profile $x \in \mathcal{C}(g)$ with the rank k; it is closed because $\mathcal{C}(g)$ is closed and we call its lower bound c_k. The sequence $\{c_k\}$ decreases weakly because in a contact profile where c_k is k-th the type x_{k+1} ranked $k+1$ is weakly below c_k. And $c_n = L$ because c_n is in some contact profile of g.

Check first that $\mathcal{C}_1(g) = [c_1, H]$ with the help of Lemma 3.9. For each $x_1 \in [c_1, H[$ where g is differentiable and x_1 appears with rank k in some contact profile we have $\frac{dg}{dx}(x_1) = \frac{dw_k}{dx}(x_1) \geq \frac{dw_1}{dx}(x_1)$ because $\mathcal{W}$ is supermodular. This implies $g(x_1) - g(c_1) \geq w_1(x_1) - w_1(c_1)$ everywhere in $[c_1, H]$.

Pick a profile $(c_1, x_{-1}) \in \mathcal{C}(g)$ where c_1 is ranked first and combine the latter inequality with this contact equation:

$$g(c_1) - w_1(c_1) = \sum_2^n (w_k(x_k) - g(x_k)) \leq g(x_1) - w_1(x_1)$$

The inequality above must be an equality because g is a lower guarantee therefore $\frac{dg}{dx}(x_1) = \frac{dw_1}{dx}(x_1)$ a.e. in $[c_1, H]$ and $[c_1, H] = \mathcal{C}_1(g)$.

We repeat this argument for $x_2 \in [c_2, c_1[$. In any of its contact profiles its rank is at least 2 by definition of c_1, so when g is differentiable at x_2 we have $\frac{dg}{dx}(x_2) = \frac{dw_k}{dx}(x_2) \leq \frac{dw_2}{dx}(x_2)$ by submodularity of $\mathcal{W}$. Then $g(x_2) \leq g(c_2) + w_2(x_2) - w_2(c_2)$ holds in $[c_2, c_1]$ and by plugging as above this inequality at a contact profile where c_2 is ranked second, we see that it is an equality and conclude that first, $\frac{dg}{dx}(x_2) = \frac{dw_2}{dx}(x_2)$ a.e. in $[c_2, c_1]$ and second, $[c_2, c_1] \subseteq \mathcal{C}_2(g)$.[11]

The clear induction argument gives $\frac{dg}{dx}(x_k) = \frac{dw_k}{dx}(x_k)$ a.e. in $[c_k, c_{k-1}]$; together with the continuity of g it implies that g is entirely determined by the value $g(L)$. But for $c = (c_1, \cdots, c_{n-1})$ the tight lower guarantee g_c ((21)) meets precisely the same differential system, therefore g and g_c differ by a constant; if they don't coincide g is either not a lower guarantee or not tight.

9.8 Example 6.2

We can without loss assume that F is the identity because the change of variable $y_i = F(x_i)$ reaches precisely that problem (exactly like in Example 2.1).

The proof resembles that of Proposition 2.1. Fix a tight upper guarantee $g^+ \in \mathcal{G}_k^+$ and recall that g^+ is weakly increasing (Lemma 3.4). Define $p = ng^+(L)$: from $una(x_i) = \frac{1}{n}x_i$ and inequality (6) (Lemma 3.2) we get $p \geq L$. Observe next that $g_H(x_i) \equiv \frac{1}{n}H$ is in $\mathbf{G}_k^+$ (in fact also in $\mathcal{G}_k^+$ as we show below); if $p > H$ then g^+ is everywhere larger than g_H, a contradiction. So $p \in [L, H]$.

Apply now the feasibility inequality (4) to g^+ at the profile $(\overset{n-k}{L}, \overset{k}{x_i})$:

$$\frac{n-k}{n}p + kg^+(x_i) \geq x_i$$

If $k = n$ this gives $g^+(x_i) \geq una(x_i)$: as $una \in g^+$ we conclude $g^+ = una$. For $k \leq n-1$ we combine the inequality above with $g^+(x_i) \geq \frac{1}{n}p$ and obtain

$$g^+(x_i) \geq \max\{\frac{1}{n}p, \frac{1}{k}(x_i - \frac{n-k}{n}p)\} = \frac{1}{n}p + \frac{1}{k}(x_i - p)_+$$

It remains to check that the function on the right, which we write g_p^+, is itself an upper guarantee. Pick an arbitrary profile $x \in [L, H]^{[n]}$ and suppose that p is s. t. $x^\ell \geq p \geq x^{\ell+1}$. We must show

$$\sum_{[n]} g_p^+(x_i) = p + \frac{1}{k}((\sum_{t=1}^{\ell} x^t) - \ell p) \geq x^k$$

If $p \geq x^k$ we are done because the term in parenthesis is non negative. Assume now $p < x^k$ so that $x^k \geq \cdots \geq x^\ell \geq p \geq x^{\ell+1}$, then note that $(\sum_{t=1}^{\ell} x^t) - \ell p \geq k(x^k - p)$ and we are done.

The proof that for $k \geq 2$ the set $\mathcal{G}_k^-$ is also parametrised by $q \in [L, H]$ as

$$g_p^-(x_i) \geq \frac{1}{n}q + \frac{1}{n-k+1}(x_i - q)_-$$

and for $k = 1$ contains only una, is entirely similar.

9.9 Lemma 7.2

Statement i) is clear because $\mathcal{W}$ is symmetric. In *Statement ii)* upper-hemi-continuity of φ is clear because $\mathcal{W}$ and g are both continous (Lemma 3.5).

To check that φ is convex valued we fix $(x_1, x_2), (x_1, x_2') \in \Gamma(\varphi)$ and z s. t. $x_2 < z < x_2'$, and check that $\Gamma(\varphi)$ contains (x_1, z) too. Pick some $w \in \varphi(z)$: if $w > x_1$ we see that $\Gamma(\varphi)$ contains

[11] Note that $\mathcal{C}_2(g)$ can extend beyond c_1 but this can only happen if $\frac{dw_2}{dx} = \frac{dw_1}{dx}$ in the overlap interval. To see this compare two contact profiles x and y such that $x^1 \geq x^2 > y^1 \geq y^2$ and use the LS of (4) at the two profiles where x^2 and y^2 have been swapped plus supermodularity of $\mathcal{W}$ to deduce that they are contact profiles as well.

(x_1, x_2) and (w, z) s.t. $(x_1, x_2) \ll (w, z)$ which is a contradiction by Lemma 7.1. If $w < x_1$ we use instead (w, z) and (x_1, x_2') to reach a similar contradiction, and we conclude $w = x_1$.

The proof below that φ is single-valued a. e. will complete that of statement *ii*).

Statement iii) If $x_1 < x_1'$ in $\mathcal{X}$ and $\varphi^-(x_1) < \varphi^+(x_1')$ we again contradict the strict supermodularity of $\mathcal{W}$ (Lemma 7.1) . So $x_1 < x_1' \implies \varphi^-(x_1) \geq \varphi^+(x_1')$ and φ^- and φ^+ are weakly decreasing.

If $\varphi(x_1)$ is not a singleton, $\varphi^+(x_1) > \varphi^-(x_1)$, then φ^+ jumps down at x_1; a weakly decreasing function can only do this a countable number of times. That the u.h.c. closure of φ^+ contains $[\varphi^-(x_1), \varphi^+(x_1)]$ follows from $\varphi^-(x_1) \geq \varphi^+(x_1 + \delta)$ for any $\delta > 0$.

Statement iv) If $\varphi(L)$ does not contain H we pick some x_1 in $\varphi(H)$: by statement *i*) $\varphi(x_1)$ contains H therefore $x_1 > L$; we reach a contradiction again from Lemma 10 because $\Gamma(\varphi)$ contains $(L, \varphi^+(L))$ and the strictly larger (x_1, H).

Statement v) Kakutani's theorem implies that at least one fixed point exists. If $\Gamma(\varphi)$ contains both (a, a) and (b, b) we contradicts again Lemma 10. Check finally that the inequalities $\varphi^-(a) < a < \varphi^+(a)$ are not compatible. Pick $\delta > 0$ s.t. $\varphi(a)$ contains $a - \delta$ and $a + \delta$: then $\Gamma(\varphi)$ contains $(a, a + \delta)$ and $(a - \delta, a)$ (by symmetry) and we invoke Lemma 7.1 again.

9.10 Theorem 7.1

Step 0: the integral in (25) is well defined.

For any correspondence φ as in Lemma 7.2 the integral $\int_a^{x_1} \partial_1 \mathcal{W}(t, \varphi(t))dt$ is the value of $\int_a^{x_1} \partial_1 \mathcal{W}(t, f(t))dt$ for any single-valued selection f of φ: this is independent of the choice of f because φ is multi-valued only at a countable number of points and every single-valued selection of $\varphi(x_1)$ is a measurable function.

Statement ii) Fix $g \in \mathcal{G}^-$ and its contact correspondence φ. The function $\mathcal{W}$ is uniformly Lipschitz in $[L, H]^2$ so by Lemma 3.8 g is Lipschitz as well, hence differentiable a. e.. The derivative $\frac{dg}{dx}$ is given by property (9) in Lemma 3.9: given x_1 for any $x_2 \in \varphi(x_1)$ we have $\frac{dg}{dx}(x_1) = \partial_1 \mathcal{W}(x_1, x_2)$, therefore we can write the RH as $\partial_1 \mathcal{W}(x_1, \varphi(x_1))$ without specifying a particular selection of $\varphi(x_1)$.

Note that $g(a) = una(a)$ because $(a, a) \in \Gamma(\varphi)$. Now integrating the differential equation above with this initial condition at a gives the desired representation (25).

Statement *i*)

Step 1 Lemma 7.2 implies that $\Gamma(\varphi)$ is a one-dimensional line connecting (L, H) and (H, L) that we can parametrise by a smooth mapping $s \to (\xi_1(s), \xi_2(s))$ from $[0, 1]$ into $[L, H]^2$ s.t. $\xi_1(\cdot)$ increases weakly from L to H and $\xi_2(\cdot)$ decreases weakly from H to L. We can also choose this mapping so that $\xi_1(\frac{1}{2}) = \xi_2(\frac{1}{2}) = a$, the fixed point of φ.[12]

We fix an arbitrary selection γ of φ, an arbitrary $\overline{x}_1$ in $[L, H]$, and check the identity

$$\int_a^{\overline{x}_1} \partial_1 \mathcal{W}(t, \varphi(t))dt + \int_a^{\gamma(\overline{x}_1)} \partial_1 \mathcal{W}(t, \varphi(t))dt = \mathcal{W}(\overline{x}_1, \gamma(\overline{x}_1)) - \mathcal{W}(a, a) \tag{30}$$

We change the variable t to s by $t = \xi_1(s)$ in the former and by $t = \xi_2(s)$ in the latter. Next $\overline{s}$ is the parameter at which $(\xi_1(\overline{s}), \xi_2(\overline{s})) = (\overline{x}_1, \gamma(\overline{x}_1))$ and we rewrite the LH above as

$$\int_{\frac{1}{2}}^{\overline{s}} \partial_1 \mathcal{W}(\xi_1(s), \xi_2(s)) \frac{\partial \xi_1}{\partial s}(s)ds + \int_{\frac{1}{2}}^{\overline{s}} \partial_1 \mathcal{W}(\xi_2(s), \xi_1(s)) \frac{\partial \xi_2}{\partial s}(s)ds$$

[12] If a is 0,or 1 we check that (25) defines the two canonical incremental guarantees in Proposition 4.2.

where in each term $\partial_1\mathcal{W}(t,\varphi(t))$ we can select a proper selection of the (possible) interval because $(\xi_1(s),\xi_2(s))\in\Gamma(\varphi)$. As $\mathcal{W}(x_1,x_2)$ is symmetric in x_1,x_2, we can replace the second integral by $\int_{\frac{1}{2}}^{\overline{s}}\partial_2\mathcal{W}(\xi_1(s),\xi_2(s))\frac{\partial\xi_2}{\partial s}(s)ds$ and conclude that the sum is precisely

$$\mathcal{W}(\xi_1(\overline{s}),\xi_2(\overline{s}))-\mathcal{W}(\xi_1(\frac{1}{2}),\xi_2(\frac{1}{2}))=\mathcal{W}(\overline{x}_1,\gamma(\overline{x}_1))-\mathcal{W}(a,a)$$

Step 2 We show that (25) defines a bona fide guarantee g: $g(x_1)+g(x_2)\le\mathcal{W}(x_1,x_2)$ for $x_1,x_2\in[L,H]$.

The identity (30) amounts to $g(x_1)+g(\gamma(x_1))=\mathcal{W}(x_1,\gamma(x_1))$ for all x_1. If we prove that $g\in\boldsymbol{G}^-$ this will imply it is tight. Compute

$$g(x_1)+g(x_2)=\mathcal{W}(x_1,\gamma(x_1))+g(x_2)-g(\gamma(x_1))=\mathcal{W}(x_1,\gamma(x_1))+\int_{\gamma(x_1)}^{x_2}\partial_1\mathcal{W}(t,\varphi(t))dt$$

We are left to show

$$\int_{\gamma(x_1)}^{x_2}\partial_1\mathcal{W}(t,\varphi(t))dt\le\mathcal{W}(x_1,x_2)-\mathcal{W}(x_1,\gamma(x_1)) \tag{31}$$

We assume without loss $x_1\le x_2$ and distinguish several cases by the relative positions of a and x_1,x_2 .

Case 1: $a\le x_1\le x_2$, so that $\gamma(x_1)\le a$. For every $t\ge\gamma(x_1)$ property *iii)* in Lemma 7.2 implies $\varphi^+(t)\le\varphi^-(\gamma(x_1))$ and $\varphi(\gamma(x_1))$ contains x_1: therefore submodularity of $\mathcal{W}$ implies $\partial_1\mathcal{W}(t,\varphi(t))\le\partial_1\mathcal{W}(t,x_1)$ and

$$\int_{\gamma(x_1)}^{x_2}\partial_1\mathcal{W}(t,\varphi(t))dt\le\int_{\gamma(x_1)}^{x_2}\partial_1\mathcal{W}(t,x_1)dt=\mathcal{W}(x_2,x_1)-\mathcal{W}(\gamma(x_1),x_1)$$

Case 2: $x_1\le a\le\gamma(x_1)\le x_2$. Similarly for $t\ge\gamma(x_1)$ we have $\varphi^+(t)\le\varphi^-(\gamma(x_1))$ and conclude as in Case 1.

Case 3: $x_1\le x_2\le a$, so that $\gamma(x_1)\ge a$. For all $t\le\gamma(x_1)$ we have $\varphi^-(t)\ge\varphi^+(\gamma(x_1))$ and $\varphi(\gamma(x_1))$ contains x_1: now submodularity of $\mathcal{W}$ gives $\partial_1\mathcal{W}(t,z)\ge\partial_1\mathcal{W}(t,x_2)$ for z between x_2 and $\gamma(x_1)$ and the desired inequality because the integral in (31) goes from high to low.

Case 4: $x_1\le a\le x_2\le\gamma(x_1)$. Same argument as in Case 3.