

# Persuasion with Ambiguous Communication\*

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## Abstract

We explore whether ambiguous communication can be beneficial to the sender in a persuasion problem, when the receiver (and possibly the sender) is ambiguity averse. Our analysis highlights the necessity of using a collection of experiments that form a splitting of an obedient experiment. Some experiments in the collection must be Pareto-ranked in that both players agree on their payoff ranking. If an optimal Bayesian persuasion experiment can be split in this way, then any not-too-ambiguity-averse sender as well as the receiver benefit. There are no benefits when the receiver has only two actions.

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# 1 Introduction

*“If I seem unduly clear to you, you must have misunderstood what I said.”*

Alan Greenspan, Speaking to a Senate Committee in 1987, as quoted in the Guardian Weekly, November 4, 2005.

This paper considers the problem of a sender who wishes to favorably influence, through strategic communication of information, the action taken by a receiver. As in the large literature on Bayesian persuasion following [Kamenica and Gentzkow \(2011\)](#) (see also [Rayo and Segal \(2010\)](#) and surveys by [Bergemann and Morris \(2019\)](#) and [Kamenica \(2019\)](#)), we model the sender as committing to a communication strategy and the receiver as best responding to that strategy. A communication strategy for the sender is usually described as a *statistical experiment*, a function mapping from payoff-relevant states to probability distributions over messages (or signals). Our first key departure from most of the literature is that we enlarge the set of the sender’s communication strategies to include ambiguous strategies. These are strategies generating, from the perspective of both players, subjective uncertainty about which statistical experiment will be used to generate the message. Our second key departure is that the receiver (and possibly the sender) treats this uncertainty as ambiguity and is ambiguity averse. Would the sender ever benefit from intentionally using an ambiguous communication strategy? We show that the answer can be yes, and provide understanding of the circumstances under which this can occur and the nature of these beneficial strategies. Thus, our theory suggests that one might want to communicate in a deliberately ambiguous manner even when it is possible to costlessly eliminate any ambiguity.

We illustrate our main results with the help of a simple introductory example. There is a sender and a receiver, three actions  $a_1, a_2$  and  $a_3$ , and two payoff-relevant states  $\omega_1$  and  $\omega_2$ , with equal prior probabilities  $p = (1/2, 1/2)$ .<sup>1</sup> The sender influences the action the receiver takes with the release of information. The payoffs are:

$(u_s, u_r)$	$a_1$	$a_2$	$a_3$
$\omega_1$	1, 1	−1, −1	−4, 2
$\omega_2$	0, 0	2, 2	−4, −4

Table 1: Payoff table (first coordinate is the sender’s payoff)

The receiver prefers  $a_3$  in state  $\omega_1$ , while the sender prefers  $a_1$  in that state. This is the conflict of interest in this example. The receiver prefers  $a_1$  when their beliefs about  $\omega_2$  are intermediate (i.e., in  $[1/5, 1/2]$ ),  $a_2$  when their beliefs are higher than  $1/2$ , and  $a_3$  when they are lower than  $1/5$ .

<sup>1</sup>The example needs at least three actions since we show (Corollary 1) there is no benefit from using ambiguous communication strategies when the receiver has only two actions.

An interpretation of this example in the context of stress testing and banking regulation is as follows: Think of the sender as a banking regulatory authority (“the regulator”) who must design, conduct and communicate the results of stress testing of the banking sector (“the bank”). Imagine the receiver as a representative investor (“the investor”) choosing among alternative investments,  $a_i$ , whose payoff depends on the realization of the state  $\omega$ . Think of the  $\omega$  as investment-relevant information about the health of the banking sector, with  $\omega_1$  and  $\omega_2$  associated with “bad” and “good” health, respectively. Actions  $a_1$  and  $a_2$  are socially-productive investments (i.e., productive from the viewpoint of the economy as a whole, a viewpoint that we assume the regulator adopts). Action  $a_3$  is a socially-detrimental, purely speculative investment. The regulator’s choice of communication strategy can be seen as their choice of rules/specifications for the stress tests.<sup>2</sup> The regulator’s challenge is to design and communicate stress tests so as to better coordinate investment behavior with the health of the banking sector, without diverting investments to the speculative activity, which is always socially detrimental but beneficial for the investor when the health of the banking sector is bad.

We first apply the seminal work of [Kamenica and Gentzkow \(2011\)](#) on Bayesian persuasion to this example. [Kamenica and Gentzkow \(2011\)](#) study a dynamic game between a sender and a receiver, where the sender first designs a statistical experiment  $\sigma : \{\omega_1, \omega_2\} \rightarrow \Delta(M)$ , the receiver observes the chosen experiment  $\sigma$  and the realized message  $m$ , and then takes an action. In our language, this information design is *unambiguous*, that is, the receiver knows the experiment that generates the message and, therefore, knows the likelihood of each message given each state  $\omega$ . [Kamenica and Gentzkow \(2011\)](#) show that the highest payoff the sender can achieve is the value of the concavification of their indirect utility at the prior  $p$ . In our example, this value is  $5/4$  as illustrated in Figure 1. In the figure, we plot the receiver’s expected payoff associated with each of the three actions as dotted lines – each line is labelled with its action. We plot the sender’s indirect utility, i.e., the expected payoff the sender obtains when the receiver best responds, as a thick solid curve, and its concavification as a thick dashed curve.

It is immediate to verify that the experiment  $\sigma^{BP}$  defined in Figure 1 attains this optimal payoff using the messages “ $a_1$ ” and “ $a_2$ ”. The message “ $a_2$ ” reveals that the state is  $\omega_2$ . Intuitively, since the preferences are perfectly aligned when the state is  $\omega_2$ , the sender wants the receiver to learn it. At the same time, the sender does not want the receiver to be too pessimistic about  $\omega_2$  when the state is  $\omega_1$ , as the receiver chooses  $a_3$  at all beliefs less than  $1/5$  on  $\omega_2$ . The optimal experiment  $\sigma^{BP}$  balances these two forces by making the likelihood of message “ $a_1$ ” in state  $\omega_2$  just large enough to induce action  $a_1$ . We note that the experiment  $\sigma^{BP}$  is *canonical*, that is, the messages are action recommendations, and, in the example, it is also *obedient*, that is, the receiver finds it optimal to obey the recommendations. In Bayesian persuasion, the restriction to canonical and

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<sup>2</sup>See [Bergemann and Morris \(2016\)](#) for another example of Bayesian persuasion in the context of stress testing.

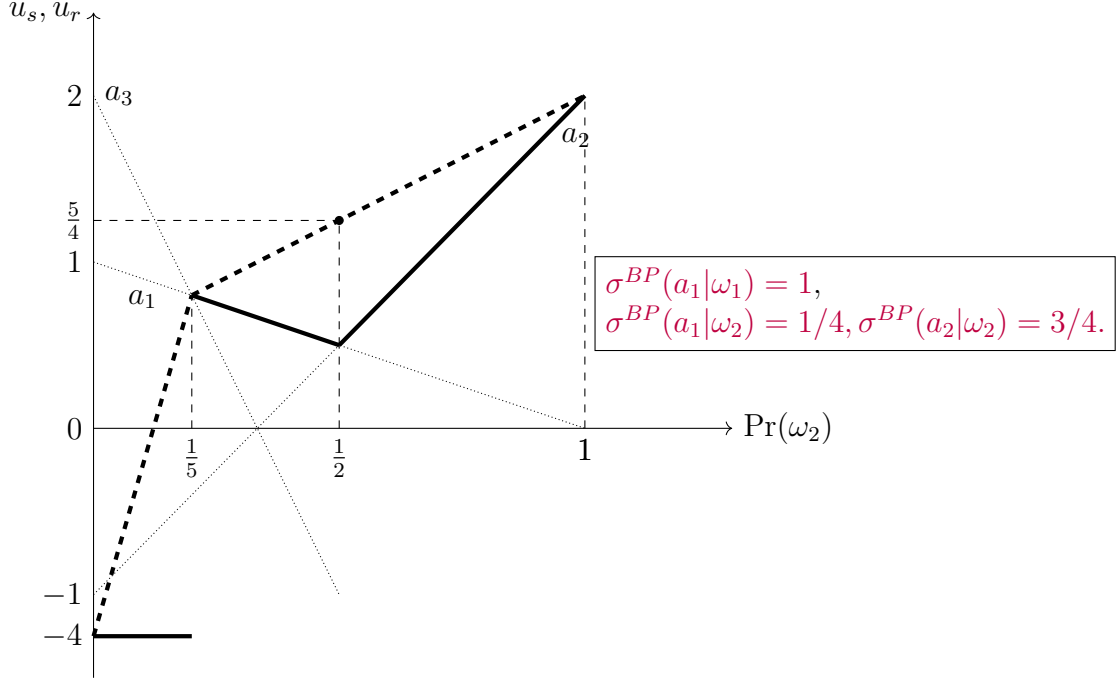


Figure 1: Sender's indirect utility (thick curve) and its concavification (thick dashed curve)

obedient experiments is without loss, and we prove (Proposition 1) that this remains true in our generalization.

Now, suppose that the sender can design *ambiguous* experiments. These are communication strategies that leave some uncertainty about which statistical experiment will generate the messages, and, in this sense, may not completely pin-down the message likelihoods. We model ambiguous experiments as generated by (finitely-ranged) mappings from payoff-irrelevant ambiguous events to statistical experiments. Examples of such ambiguous events include artificially generated ambiguity like draws from an Ellsberg urn provided by a third-party, or natural-event ambiguity derived from meteorological or other events. More broadly, the only requirements beyond payoff-irrelevance are that the sender and receiver share a common view of the subjective uncertainty about these events, and that the receiver (and possibly the sender as well) is ambiguity averse and treats the uncertainty about these events as ambiguity. Our theory is agnostic about why there is a common view of the uncertainty over these events. It could be this common view comes from shared (but limited) historical data or, as is particularly likely in the case of artificially generated ambiguity, from symmetry or other logical considerations, or the commonality could be viewed simply as a convenient baseline modeling assumption.

Formally, we model the source of such ambiguous events as a continuum,  $\mathcal{A}$ , of payoff-irrelevant states,  $\alpha$ , along with a continuous  $\rho \in \Delta(\mathcal{A})$  that represents the common view of uncertainty over  $\mathcal{A}$  that an ambiguity neutral player would use to compute their expected util-

ity. Let  $B$  be any finite partition of  $\mathcal{A}$  and define  $\tilde{\mu} \in \Delta(B)$  by  $\tilde{\mu}(b) \equiv \int_{\alpha \in b} \rho(\alpha) d\alpha$  for each  $b \in B$ . By richness of  $\mathcal{A}$  and continuity of  $\rho$ , any finitely-supported distribution over experiments can be induced by the choice of some finite partition  $B$  together with some mapping from  $B$  to experiments. Hence, as justified by payoff-irrelevance of  $\mathcal{A}$ , we model an *ambiguous experiment* as a pair  $(\sigma, \mu)$ , where the collection  $\sigma = (\sigma_\theta)_{\theta \in \Theta}$  is a tuple of experiments that we index using a finite set  $\Theta$  and  $\mu = (\mu_\theta)_{\theta \in \Theta}$  is any element of  $\Delta(\Theta)$ . The sender may choose any ambiguous experiment.<sup>3</sup> The crucial characteristic of ambiguous experiments is that ambiguity averse players behave as if they value some robustness with respect to perturbations of  $\mu$ .

In the stress-test setting, a specification of the exact model/test the bank must run and report the results of, would correspond to a statistical experiment (i.e., an unambiguous communication strategy on the part of the regulator). One channel through which ambiguity could be introduced into communication in this context is the use of contingent “bottom-up” tests – tests conducted by individual banks based on their own in-house models and data – as input to the stress tests. By making which model/test a bank is to run contingent on the range a parameter,  $\alpha \in [0, 1]$ , belongs to, where, for example,  $\alpha$  is something to be calculated based on data private to the bank (and not *directly* payoff-relevant for either the regulator or the investor), the regulator may cause the statistical experiment generating the announced result to vary with these ranges.<sup>4</sup> As is plausible for an  $\alpha$  for which the sender and receiver have little data, from their perspective the realization of  $\alpha$  is ambiguous and elements of finite partitions of  $[0, 1]$  are ambiguous events. Thus, if the regulator says that the bank should use one model/test if  $\alpha \in [0, 1/4)$ , another if  $\alpha \in [1/4, 3/4)$ , and a third if  $\alpha \in [3/4, 1]$ , this is an example of an ambiguous experiment. By varying the partition of  $[0, 1]$  used to define the contingencies under which the three models/tests will be run by the bank, the regulator may vary  $\mu$ .

An SEU player treats the ambiguous experiment  $(\sigma, \mu)$  as equivalent to the unambiguous experiment  $\sum_\theta \mu_\theta \sigma_\theta$ . If the receiver is SEU, the sender cannot do better than using the experiment  $\sigma^{BP}$  and thus ambiguity adds no value. We assume instead that the receiver is ambiguity averse and represent their preference with the smooth ambiguity model of [Klibanoff et al. \(2005\)](#). Specifically, let  $u_r(\sigma_\theta, \tau^*)$  be the receiver’s payoff when the (canonical) experiment is  $\sigma_\theta$  and the receiver is obedient.<sup>5</sup> The receiver values the ambiguous experiment as  $\phi_r^{-1}(\sum_\theta \mu_\theta \phi_r(u_r(\sigma_\theta, \tau^*)))$ , where

<sup>3</sup>Notice that ambiguous experiments are a generalization of experiments in the sense that any experiment  $\sigma$  can be viewed as an ambiguous experiment with a collection  $\sigma$  such that  $\sigma_\theta = \sigma$  for all  $\theta$  in the support of  $\mu$ .

<sup>4</sup>The use of bottom-up tests is common (see e.g., Table 1 in [Dent et al. \(2016\)](#)). Making the instructions for them contingent is something already done in practice. For instance, in recent EU stress tests (see [European Banking Authority \(2023\)](#), Section 2.4.4.): “Banks with *significant* foreign currency exposure are required to take into account the altered creditworthiness of their respective obligors, given the FX development under the baseline and adverse scenarios. In particular, banks are only required to evaluate this impact if the exposures of certain asset classes in foreign currencies are above certain thresholds.”

<sup>5</sup>Obedient in the sense of following the action recommendations. We denote the obedient strategy by  $\tau^*$ .

$\phi_r$  is some strictly increasing, concave and differentiable function.<sup>6</sup> The concavity of  $\phi_r$  captures ambiguity aversion. Greater concavity corresponds to more ambiguity aversion. At one extreme, when the receiver is infinitely ambiguity averse, we have an instance of the maxmin expected utility (MEU) model (Gilboa and Schmeidler, 1989). At the other, when  $\phi_r$  is affine, we have the SEU model (implying ambiguity neutrality).

As a preliminary result, we show (Lemma 1) that such an ambiguity-averse receiver faced with an ambiguous experiment  $(\sigma, \mu)$  is obedient if, and only if, they are obedient when facing the unambiguous experiment  $\sum_{\theta} \nu_{\theta} \sigma_{\theta}$  with  $\nu_{\theta} = \frac{\mu_{\theta} \phi'_r(u_r(\sigma_{\theta}, \tau^*))}{\sum_{\bar{\theta}} \mu_{\bar{\theta}} \phi'_r(u_r(\sigma_{\bar{\theta}}, \tau^*))}$ . We use this result throughout, and refer to  $\nu$  as the receiver's *effective measure* given  $(\sigma, \mu)$ . Assume  $\phi_r$  is strictly concave. Then  $u_r(\sigma_{\theta}, \tau^*) < u_r(\sigma_{\theta'}, \tau^*)$  implies  $\nu_{\theta}/\nu_{\theta'} > \mu_{\theta}/\mu_{\theta'}$ , that is, the effective measure assigns a *higher* (relative) probability than  $\mu$  to *lower* payoffs. This relative pessimism of the effective measure reflects the value that an ambiguity-averse receiver places on some robustness with respect to perturbations of  $\mu$ . The more ambiguity averse the receiver, the stronger the relative pessimism. Lemma 1 also makes clear that, in addition to depending on the receiver's ambiguity aversion,  $\nu_{\theta}$  is endogenous in the sense that it is a function of the profile  $(u_r(\sigma_{\theta}, \tau^*), \mu_{\theta})_{\theta \in \Theta}$ . Even local changes in the ambiguous experiment, say only changing  $\sigma_{\theta}$  to  $\sigma'_{\theta}$ , might impact *all*  $\nu_{\theta}$ . These endogenous pessimism properties stemming from ambiguity aversion distinguish our model from a model with exogenously fixed heterogeneous priors, e.g., Alonso and Câmara (2016), Laclau and Renou (2017) and Galperti (2019).

We now illustrate (see Figure 2) how ambiguous experiments can benefit an SEU sender in the example.<sup>7</sup> Consider an ambiguous experiment such that only two experiments  $\sigma_{\underline{\theta}}$  and  $\sigma_{\bar{\theta}}$  (see the figure) get positive  $\mu$ -weight. The experiment  $\sigma_{\underline{\theta}}$  is uninformative, while the experiment  $\sigma_{\bar{\theta}}$  is fully informative. In the stress-test context, think of  $\sigma_{\bar{\theta}}$  as a more comprehensive test than  $\sigma_{\underline{\theta}}$ . Observe that the interpretation of the message  $a_2$  is unambiguous: the receiver learns that the state is  $\omega_2$ . The interpretation of the message  $a_1$  is, however, ambiguous: either it means that the state is  $\omega_1$  (if  $\sigma_{\bar{\theta}}$  generated the message) or it is uninformative (if  $\sigma_{\underline{\theta}}$  generated the message). The associated payoff profiles are  $(u_s(\sigma_{\underline{\theta}}, \tau^*), u_r(\sigma_{\underline{\theta}}, \tau^*)) = (1/2, 1/2)$  and  $(u_s(\sigma_{\bar{\theta}}, \tau^*), u_r(\sigma_{\bar{\theta}}, \tau^*)) = (3/2, 3/2)$ . Thus, if  $\mu_{\bar{\theta}} > 3/4$ , an ambiguity-neutral sender's expected payoff,  $\mu_{\underline{\theta}} u_s(\sigma_{\underline{\theta}}, \tau^*) + \mu_{\bar{\theta}} u_s(\sigma_{\bar{\theta}}, \tau^*)$ , is strictly higher than the Bayesian persuasion payoff of  $5/4$ . We now argue that we can simultaneously choose  $\mu_{\bar{\theta}} > 3/4$  and guarantee obedience. First, observe that  $(1/4)\sigma_{\underline{\theta}} + (3/4)\sigma_{\bar{\theta}} = \sigma^{BP}$  – we call such a configuration a splitting of  $\sigma^{BP}$ . Since the receiver is obedient when facing  $\sigma^{BP}$ , by Lemma 1 the receiver is obedient when the effective weight  $\nu_{\bar{\theta}}$  equals  $3/4$ .

<sup>6</sup>We similarly model the sender's preferences, substituting  $u_s$  and  $\phi_s$ .

<sup>7</sup>Similar arguments remain valid as long as the sender is not too ambiguity averse. In particular, the sender continues to benefit even if they are as ambiguity averse as the receiver (and even a bit more so) assuming the sender is not infinitely ambiguity averse. This demonstrates that the essential source of the sender's benefit is *not* a less ambiguity averse sender insuring a more ambiguity averse receiver.

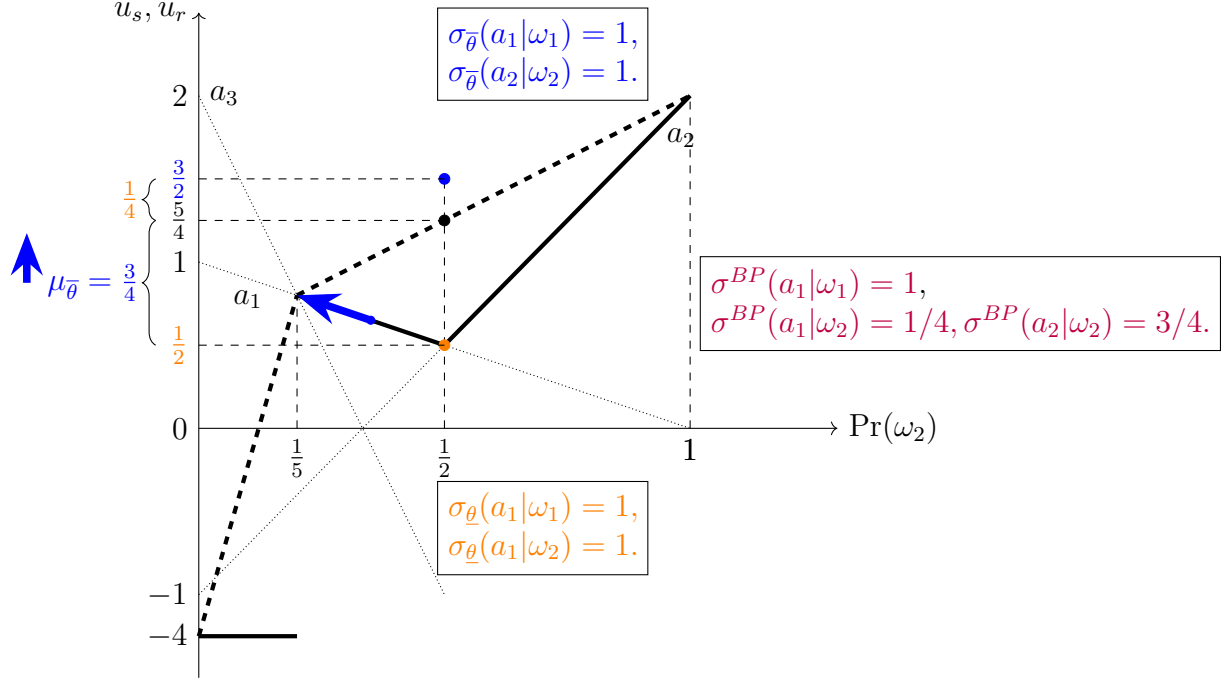


Figure 2: Construction of the ambiguous experiment

In fact, the receiver continues to be obedient for any effective weight weakly below  $3/4$ . Second, since  $1/2 = u_r(\sigma_{\underline{\theta}}, \tau^*) < u_r(\sigma_{\bar{\theta}}, \tau^*) = 3/2$ ,  $\nu_{\bar{\theta}}$  is strictly lower than  $\mu_{\bar{\theta}}$  (unless the receiver is ambiguity neutral) – as mentioned above, this is a consequence of ambiguity aversion. Therefore, since  $\nu_{\bar{\theta}} < 3/4$  when  $\mu_{\bar{\theta}} = 3/4$ , there is room to increase  $\mu_{\bar{\theta}}$  above  $3/4$  and maintain obedience until the point where  $\nu_{\bar{\theta}}$  equals  $3/4$ .<sup>8</sup> In the figure, the thick arrow moving along the sender's indirect utility curve indicates the movement of  $\nu_{\bar{\theta}}$  towards  $3/4$  from below as  $\mu_{\bar{\theta}}$  increases above  $3/4$  (along the thick arrow next to  $\mu_{\bar{\theta}}$ ). Thus, the ambiguous communication strategy allows the sender to place more weight on the better experiment  $\sigma_{\bar{\theta}}$  while maintaining obedience, than would be possible with unambiguous communication. This is how ambiguous communication provides benefits.<sup>9</sup>

An important observation is that the experiments  $\sigma_{\underline{\theta}}$  and  $\sigma_{\bar{\theta}}$  are Pareto-ranked – both players

<sup>8</sup>The effective weight  $\nu_{\bar{\theta}}$  is  $3/4$  when  $\mu_{\bar{\theta}} = \frac{3\phi'_r(1/2)}{3\phi'_r(1/2) + \phi'_r(3/2)}$ , with  $\phi'_r$  the derivative of  $\phi_r$ . Observe, moreover, that if  $3/4 < \mu_{\bar{\theta}} < \frac{3\phi'_r(1/2)}{3\phi'_r(1/2) + \phi'_r(3/2)}$  then the sender continues to benefit from the ambiguous communication even if the receiver slightly misperceives the ambiguous experiment and/or the sender slightly misperceives  $\phi'_r$ . Section 4.2 shows that this robustness holds quite generally.

<sup>9</sup>Though the particular formula for  $\nu_{\bar{\theta}}$  is special to the smooth ambiguity model, many models of ambiguity averse preferences (for example, everything in the very general class of Uncertainty Averse preferences, [Cerreiia-Vioglio et al. \(2011\)](#)) allow one to derive an effective measure that is similarly pessimistic and endogenous. The logic and explanation of how, when the receiver is more ambiguity averse than an SEU receiver with beliefs  $(\mu_{\theta})_{\theta}$ , ambiguous communication can allow the sender to improve beyond Bayesian persuasion in this example applies to any of these models.



prefer  $\sigma_{\bar{\theta}}$  assuming action recommendations are followed. If they were not Pareto-ranked, then ambiguity aversion would push the receiver’s effective measure in a direction that would *hurt* rather than help the sender – for example, if the receiver thought  $\sigma_{\underline{\theta}}$  were better this would cause  $\nu_{\bar{\theta}}$  to exceed  $\mu_{\bar{\theta}}$ , leading the receiver to defect to the speculative action  $a_3$  when the message is  $a_1$ . In fact, we prove that the existence of a splitting of some (unambiguous) obedient experiment into two Pareto-ranked experiments is necessary for ambiguous experiments to benefit the sender over Bayesian persuasion (Theorem 3). The ambiguity generated using Pareto-ranked experiments serves to beneficially misalign the (endogenous) effective beliefs of the sender and receiver. Any ambiguous experiment delivering the sender more than Bayesian persuasion must assign positive  $\mu_{\theta}$ -weight to some pair of experiments that are Pareto-ranked (Theorem 2). If, as here, an optimal Bayesian persuasion experiment can be split in this way, this is sufficient for an ambiguity-neutral sender (and the receiver too!) to benefit from ambiguous communication (Theorem 4 and Corollary 3). Furthermore, if an optimal Bayesian persuasion experiment violates a necessary condition for efficiency from [Arieli et al. \(2024\)](#), then, for generic payoffs, it can be split in this way (implied by Proposition 4).

We close this section with a brief discussion of a few closely related papers. A more extensive discussion can be found in Section 7.3. [Beauchêne et al. \(2019\)](#) (BLL henceforth) were first to study strategic use of ambiguous communication in persuasion (see also [Cheng \(2022\)](#)). The key difference in assumptions between BLL and our paper is how the receiver best responds given the sender’s ambiguous experiment. We assume the receiver chooses an ex-ante optimal message-contingent strategy (see Remark 1 for an equivalent interim implementation). BLL assume the receiver chooses, for each message, actions maximizing interim preferences formed using a belief updating rule that leads to dynamic inconsistency with their ex-ante preference. Thus, one contribution of our paper is establishing and analyzing benefits of ambiguous persuasion that do not stem from receiver’s behavior that is suboptimal with respect to their given ex-ante preferences (see our further discussion in Section 7.3, including the approach to consistency of [Pahlke \(2023\)](#)). The bulk of BLL’s analysis imposes the infinitely ambiguity-averse extreme for both the sender and receiver – a polar case of our model, though they show that their approach extends more broadly. [Cheng \(2025\)](#) shows that all benefits from ambiguous communication identified by BLL in the case of such a sender disappear if the receiver is assumed, as in our paper, to maximize their given ex-ante preference. In light of [Cheng \(2025\)](#)’s result, it is essential that we allow at least the sender to be less than infinitely ambiguity averse for benefits from ambiguous communications to possibly exist. Our analysis allows for varying degrees of ambiguity aversion for both the sender and the receiver.

The remainder of the paper is organized as follows. The next section presents the model and two key preliminary results – a revelation principle and an incentive-compatibility lemma. Main



results are in Sections 3 through 6. Section 7 contains further discussion. The main proofs are in the Appendix. Additional material and proofs of auxiliary results are in the Online Appendix (Cheng et al., 2025).

## 2 The Persuasion Problem with Ambiguous Communication

We consider a persuasion game between a sender and a receiver, where the sender can choose *ambiguous* experiments if they wish.

### 2.1 The Model

There is a finite set  $\Omega$  of payoff-relevant states  $\omega$ , with common prior probability distribution  $p \in \Delta(\Omega)$ . There is a finite set  $A$  of actions the receiver can choose from. If the receiver chooses  $a \in A$ , the payoff to the sender (resp., receiver) is  $u_s(a, \omega) \in \mathbb{R}$  (resp.,  $u_r(a, \omega) \in \mathbb{R}$ ), when the state is  $\omega$ . A *statistical experiment* is a finite set of messages  $M$  and a map  $\sigma$  from  $\Omega$  to  $\Delta(M)$ , and we write  $\sigma(m|\omega)$  for the probability of  $m$  given  $\omega$ .

We assume that the sender can condition their statistical experiment on the realization of a finite partition of a *source of ambiguity*. A source of ambiguity is a continuum,  $\mathcal{A}$ , of payoff-irrelevant ambiguous states,  $\alpha$ , along with a continuous  $\rho \in \Delta(\mathcal{A})$  that represents the common view of uncertainty over  $\mathcal{A}$  that an ambiguity-neutral (SEU) player would use to compute their expected utility.

Let  $B$  be any finite partition of  $\mathcal{A}$  and define  $\tilde{\mu} \in \Delta(B)$  by  $\tilde{\mu}(b) \equiv \int_{\alpha \in b} \rho(\alpha) d\alpha$  for each  $b \in B$ . By richness of  $\mathcal{A}$  and continuity of  $\rho$ , any finitely-supported distribution over experiments can be induced by the choice of some finite partition  $B$  together with some mapping from  $B$  to experiments. Hence, as justified by payoff-irrelevance of  $\mathcal{A}$ , we model an *ambiguous experiment* as a pair  $(\sigma, \mu)$ , where  $\sigma = (\sigma_\theta)_{\theta \in \Theta}$  is a tuple of experiments that we index using a finite set  $\Theta$  and  $\mu = (\mu_\theta)_{\theta \in \Theta}$  is any element of  $\Delta(\Theta)$ .<sup>10</sup> The sender may choose any ambiguous experiment – any finite length tuple of experiments together with any probability distribution over the experiments in that tuple.<sup>11</sup> Henceforth, whenever we use the term “experiment” without a modifier, it refers to a standard, unambiguous statistical experiment. Our model enlarges the sender’s strategy space relative to the standard Bayesian persuasion model in the sense that any experiment  $\sigma$  can be

<sup>10</sup>It is without loss to assume that all statistical experiments in the tuple share the same message space.

<sup>11</sup>The sender chooses and commits to  $(\sigma, \mu)$  before  $\alpha$  (which determines  $\theta$ ) and  $\omega$  are realized. Thus, just as in standard Bayesian persuasion where the sender chooses and commits to  $\sigma$  before  $\omega$  is realized, the sender’s choice of a communication strategy is influenced by their beliefs about how uncertainty may unfold. Since  $\mathcal{A}$  (and thus  $\Theta$ ) is viewed as ambiguous, any ambiguity aversion on the part of the sender may influence their choice of  $(\sigma, \mu)$ . See Section 7.1 for a discussion of how things would change under the alternative assumption that it is common knowledge that the sender privately learns  $\alpha$  before committing to an ambiguous experiment.

viewed as an ambiguous experiment with a collection  $\sigma$  such that  $\sigma_\theta = \sigma$  for all  $\theta \in \text{supp}(\mu)$  (where  $\text{supp}(\mu)$  denotes the support of  $\mu$ ). Of special interest in some of our later constructions are *binary ambiguous experiments*, those in which  $\sigma$  is a pair of experiments.

We analyze the receiver's behavior from the perspective of their ex-ante preferences, that is, we assume that the receiver observes the sender's choice of  $(\sigma, \mu)$  and then chooses a strategy  $\tau : M \rightarrow \Delta(A)$  to maximize the receiver's ex-ante preference.<sup>12</sup> We further assume that  $\tau$  is indeed carried out after the message is observed. This follows from either assuming that the receiver can commit to  $\tau$ , or that the receiver is dynamically consistent.<sup>13</sup> The main motivation for this ex-ante perspective is that we want to study whether the sender benefits from ambiguous communication even if the channel of dynamic *inconsistency* — the channel at work in nearly all previous literature on mechanism or information design with ambiguity — is shut down. We refer the interested reader to Section 7.3 for more discussion on this point.

We write  $u_i(\sigma, \tau)$  for the expected payoff of player  $i \in \{s, r\}$  when the experiment generating the message  $m$  is  $\sigma$  and the receiver's strategy is  $\tau$ , that is,

$$u_i(\sigma, \tau) = \sum_{\omega, m, a} p(\omega) \sigma(m|\omega) \tau(a|m) u_i(a, \omega). \quad (1)$$

Note that the payoff-irrelevance of  $\mathcal{A}$  implies that neither  $\alpha$  nor  $\theta$  appears in (1).

To isolate and clarify the role of intentional ambiguous communication, we work in a stylized environment where ambiguity is not payoff-relevant unless it becomes so by strategic choice of the sender to condition their communication on the realization of ambiguous events. Thus, while the payoff-irrelevant events generating  $\theta$  are viewed as ambiguous, the payoff-relevant events  $\omega$  and any randomization over messages induced by an experiment are viewed as unambiguous.<sup>14</sup> It follows that a message  $m$  is viewed as ambiguous by the sender and receiver only if the experiments the sender chooses to associate with distinct possible  $\theta$ 's generate  $m$  with different positive likelihoods. These different likelihoods may lead the expected payoff  $u_i(\sigma_\theta, \tau)$  to vary with  $\theta$  and thus itself be viewed as ambiguous.

How does such ambiguity enter the sender's and receiver's preferences? We assume, as in the smooth ambiguity model (Klibanoff et al., 2005), that player  $i$  evaluates the strategy profile  $((\sigma, \mu), \tau)$  as

$$U_i(\sigma, \mu, \tau) = \phi_i^{-1} \left( \sum_{\theta} \mu_{\theta} \phi_i(u_i(\sigma_{\theta}, \tau)) \right), \quad (2)$$

where  $\phi_i : \mathbb{R} \rightarrow \mathbb{R}$  is a weakly concave and strictly increasing and differentiable function. An

<sup>12</sup>For compactness, this notation suppresses the allowed dependence of  $\tau$  on  $(\sigma, \mu)$ .

<sup>13</sup>For a simple receiver's updating rule that guarantees dynamic consistency, see Remark 1 in Section 2.2.2.

<sup>14</sup>Section 7.2 discusses extensions to environments with pre-existing ambiguity about  $\omega$ .

affine  $\phi_i(\cdot)$  corresponds to ambiguity neutrality, in which case the preferences reduce to SEU with belief  $\mu$ . Greater concavity of  $\phi_i(\cdot)$  corresponds to greater ambiguity aversion. We do not consider ambiguity loving behavior, as this would build-in a direct preference benefit from ambiguous communication, while with ambiguity aversion, ambiguous communication can only be valuable if it has a strategic benefit.

When the expected payoff  $u_i(\sigma_\theta, \tau)$  varies with  $\theta$ , an ambiguity-averse player  $i$  responds to  $(\sigma, \mu)$  as if they are ambiguity neutral and using an *effective* measure over  $\Theta$  that is more pessimistic than  $\mu$ . As mentioned in the introduction (see footnote 9), while (2) delivers a particularly tractable formula for this effective measure (Section 2.2.2), the key qualitative properties of comparative pessimism and endogeneity would apply to the effective measure under many models of ambiguity averse preferences.

The following maxmin expected utility (MEU) objective can be viewed as an appropriate limit of (2) as ambiguity aversion tends to infinity (Klibanoff et al., 2005, Proposition 3):

$$U_i^{MEU}(\sigma, \mu, \tau) = \min_{\theta \in \text{supp}(\mu)} u_i(\sigma_\theta, \tau).$$

Such an MEU receiver together with an ambiguity-neutral sender is considered in Section SA.1 of the Online Appendix (Cheng et al., 2025).<sup>15</sup>

Writing  $BR(\sigma, \mu)$  for the set of best replies of the receiver (i.e., the maximizers of  $U_r(\sigma, \mu, \tau)$  with respect to  $\tau$ ), the sender's problem is:

$$(\mathcal{P}) = \begin{cases} \max_{(\sigma, \mu, \tau)} U_s(\sigma, \mu, \tau), \\ \text{subject to } \tau \in BR(\sigma, \mu). \end{cases}$$

Observe that the sender's Bayesian persuasion problem (Kamenica and Gentzkow, 2011) corresponds to the special case of our model where the sender is restricted to choosing an experiment:

$$(\mathcal{P}^{BP}) = \begin{cases} \max_{(\sigma, \tau)} u_s(\sigma, \tau), \\ \text{subject to } \tau \in br(\sigma), \end{cases}$$

where  $br(\sigma)$  denotes the set of best replies to  $\sigma$ , i.e., the maximizers of  $u_r(\sigma, \tau)$  with respect to  $\tau$ . Let  $u_s^{BP}$  denote the value of  $(\mathcal{P}^{BP})$ , i.e., the sender's payoff at a solution to  $(\mathcal{P}^{BP})$ .

Our analysis will focus on optimal persuasion with ambiguous communication (the solution to  $(\mathcal{P})$ ) and its properties, as well as when and how ambiguous communication may benefit the

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<sup>15</sup>Key differences from our main analysis are that (1) the sender never needs more than binary ambiguous experiments to approach their optimal payoff, and (2) the sender can approach their optimal payoff without providing the receiver any benefits from communication, and this may lead the receiver's payoff to drop discontinuously when passing to the MEU limit.

sender compared to the standard, unambiguous case of Bayesian persuasion.

**Definition 1.** *Ambiguous communication benefits the sender if the value of  $(\mathcal{P})$  is strictly higher than  $u_s^{BP}$ .*

We next present two preliminary results – a revelation principle and a characterization of incentive compatibility for ambiguous experiments – that play a central role in our analysis.

## 2.2 A Revelation Principle and Incentive Compatibility

### 2.2.1 A Revelation Principle

**Definition 2.** *An ambiguous experiment  $(\sigma, \mu)$  is **canonical** if  $M = A$ .*

We write  $\tau^* : A \rightarrow \Delta(A)$  for the receiver’s obedient strategy, that is,  $\tau^*(a|a) = 1$  for all  $a$ . We will refer to any canonical ambiguous experiment that induces such obedience as itself obedient.

**Definition 3.** *A canonical ambiguous experiment  $(\sigma, \mu)$  is **obedient** if  $\tau^* \in BR(\sigma, \mu)$ .*

We start with a preliminary observation: a revelation principle holds – for payoff purposes, it is without loss of generality to restrict attention to canonical and obedient ambiguous experiments.

**Proposition 1.** *For any  $((\sigma, \mu), \tau)$  such that  $\tau \in BR(\sigma, \mu)$ , there exists a canonical and obedient ambiguous experiment  $(\sigma^*, \mu)$  such that  $u_i(\sigma_\theta, \tau) = u_i(\sigma_\theta^*, \tau^*)$  for all  $i \in \{s, r\}$  and  $\theta$ .*

It is well-known that such a revelation principle holds in the persuasion game setting without ambiguity. However, one might have thought of at least two reasons why the same might not be true in our environment. First, an ambiguity averse receiver might strictly prefer a mixed strategy to any pure strategy for hedging reasons in the face of ambiguity. How can the receiver’s desire to mix be reconciled with the revelation principle, which states that it is without loss of generality to have the receiver play the pure strategy  $\tau^*$ ? The answer is that any mixing the receiver might desire to do can always be emulated through the use of experiments that mix over action recommendations. It is the standard Bayesian persuasion assumption of sender’s commitment that guarantees that this emulation is always possible. Second, dynamic inconsistency, generated by ambiguity aversion together with assumptions on updating, is the main channel leading to the failure of such a revelation principle in existing literature. As previously mentioned, we shut down this channel by modeling the receiver as choosing a strategy  $\tau$  to maximize  $U_r(\sigma, \mu, \tau)$ , their ex-ante payoff from  $((\sigma, \mu), \tau)$ , which imposes dynamic consistency on the receiver.

From here on, we restrict attention to canonical experiments, and represent incentive compatibility via obedience. Given the prominent role obedient experiments play, understanding when obedience holds is important. We next present a characterization of such incentive compatibility for ambiguous experiments.

### 2.2.2 Incentive Compatibility and Effective Measure

We present a central result linking the obedience of an *ambiguous* experiment to the obedience of an *unambiguous* experiment that is derived from the ambiguous experiment. We repeatedly use this result throughout the paper. To state the result, we need the following definition:

**Definition 4.** Given an ambiguous experiment  $(\sigma, \mu)$ , the receiver's **effective measure**  $em^{(\sigma, \mu)} \in \Delta(\Theta)$  is given by:

$$em_{\theta}^{(\sigma, \mu)} := \frac{\mu_{\theta} \phi_r'(u_r(\sigma_{\theta}, \tau^*))}{\sum_{\tilde{\theta}} \mu_{\tilde{\theta}} \phi_r'(u_r(\sigma_{\tilde{\theta}}, \tau^*))}, \text{ for all } \theta \in \Theta. \quad (3)$$

The effective measure  $em^{(\sigma, \mu)}$  is a probability measure with the same support as  $\mu$ . It is equal to  $\mu$  when the receiver is ambiguity neutral (i.e.,  $\phi_r$  is affine), and is more pessimistic than  $\mu$  for an ambiguity averse receiver (i.e.,  $\phi_r$  concave). Pessimism here means shifting weight toward  $\theta$  yielding lower expected receiver's payoffs, i.e., if  $u_r(\sigma_{\theta}, \tau^*) < u_r(\sigma_{\theta'}, \tau^*)$ , then  $em_{\theta}^{(\sigma, \mu)} / em_{\theta'}^{(\sigma, \mu)} > \mu_{\theta} / \mu_{\theta'}$ . Notice also that the effective measure of a given  $\theta$  depends on the specification of the ambiguous experiment for all  $\theta \in \text{supp}(\mu)$ .

The next result states that  $\tau^*$  is the receiver's best response to the ambiguous experiment  $(\sigma, \mu)$  if, and only if, it is a best response to the experiment,  $\sigma^*$ , defined below as the convex combination of the experiments in the collection  $\sigma$  with weights given by the receiver's effective measure.

**Lemma 1.** The ambiguous experiment  $(\sigma, \mu)$  is obedient if, and only if, the (unambiguous) experiment  $\sigma^*$  is obedient, where

$$\sigma^* = \sum_{\theta} em_{\theta}^{(\sigma, \mu)} \sigma_{\theta}.$$

Lemma 1 follows from the first-order conditions of the receiver's maximization problem  $\max_{\tau} U_r(\sigma, \mu, \tau)$ , evaluated at  $\tau^*$ . Some intuition is that obedience will differ from the best response to the experiment  $\sum_{\theta} \mu_{\theta} \sigma_{\theta}$  in that it will be better hedged against uncertainty about the weights on the experiments. In our introductory example, for instance,

$$\begin{aligned} 2 &= u_r(\bar{\sigma}, br(\mu\bar{\sigma} + (1 - \mu)\underline{\sigma})) > u_r(\bar{\sigma}, \tau^*) = 3/2 \\ &> u_r(\underline{\sigma}, \tau^*) = 1/2 \\ &> u_r(\underline{\sigma}, br(\mu\bar{\sigma} + (1 - \mu)\underline{\sigma})) = -1, \end{aligned}$$

showing that the obedience strategy  $\tau^*$  is hedged against the uncertainty about the weight  $\mu$  more than the strategy  $br(\mu\bar{\sigma} + (1 - \mu)\underline{\sigma})$ . Thus the relative pessimism of the effective measure reflects the fact that an ambiguity averse receiver values such hedging.

Lemma 1 gives rise to the following interpretation of the receiver's effective measure: It is an “ambiguity-neutral measure supporting obedience” in the sense that *if* the receiver were ambiguity neutral, the ambiguous experiment  $(\sigma, em^{(\sigma, \mu)})$  would be obedient.

**Remark 1.** These properties of the effective measure also give rise to an updating implementation of the receiver’s ex-ante optimality – a receiver who updates after observing a message using Bayes’ rule with the effective measure in place of  $\mu$  and adopts this update as their effective posterior will be dynamically consistent.

Finally, for later reference, observe that, fixing  $\sigma$ , we can invert (3) to express  $\mu$  as a function of the effective measure it generates:

$$\mu_\theta = \frac{em_\theta^{(\sigma, \mu)} / \phi'_r(u_r(\sigma_\theta, \tau^*))}{\sum_{\tilde{\theta}} em_{\tilde{\theta}}^{(\sigma, \mu)} / \phi'_r(u_r(\sigma_{\tilde{\theta}}, \tau^*))}. \quad (4)$$

### 3 Properties of Optimal Persuasion with Ambiguous Communication

Two experiments are *Pareto-ranked* if the sender and receiver agree on their strict ranking under the assumption of obedience. As we shall see, Pareto-ranking and splittings into Pareto-ranked experiments play a key role in optimal persuasion and, more generally, in the sender benefiting from ambiguous communication.

**Definition 5.** Two experiments  $\bar{\sigma}$  and  $\underline{\sigma}$  are **weakly Pareto-ranked** if either the two inequalities

$$u_s(\bar{\sigma}, \tau^*) \geq u_s(\underline{\sigma}, \tau^*) \text{ and } u_r(\bar{\sigma}, \tau^*) \geq u_r(\underline{\sigma}, \tau^*), \quad (5)$$

hold or both reversed inequalities hold. They are **Pareto-ranked** if the same holds true with strict inequalities.

A **Pareto-ranked splitting** of the experiment  $\sigma$  is a triple  $(\bar{\sigma}, \underline{\sigma}, \lambda)$  such that (i)  $\lambda \bar{\sigma} + (1 - \lambda) \underline{\sigma} = \sigma$ , (ii)  $\lambda \in (0, 1)$ , and (iii) (5) holds with strict inequalities, i.e.,  $\bar{\sigma}$  and  $\underline{\sigma}$  are Pareto-ranked.

Our next result shows that these concepts are fundamental in describing properties of optimal ambiguous communication. Part (i) of the result says that if two experiments in the support of  $\mu$  bracket the sender’s payoff from an optimal ambiguous experiment, they must be weakly Pareto-ranked. The proof shows that if they were not, then improvement could be achieved by merging the two experiments. Part (ii) shows there must not exist any opportunities to introduce additional ambiguity through Pareto-ranked splittings that bracket the sender’s payoff from the ambiguous experiment. The proof shows that any such splittings would be beneficial for the sender. Unlike Part (i), Part (ii) requires the sender to be ambiguity neutral. However, Theorem A.2 in the Appendix shows that a similar result holds whenever the sender is not too ambiguity averse compared to the receiver.

**Theorem 1.** Suppose  $(\sigma, \mu)$  is optimal (i.e., is a solution to  $(\mathcal{P})$ ), and that  $\phi_r$  is strictly concave.<sup>16</sup> Then

(i) for all  $\theta, \theta' \in \text{supp}(\mu)$  such that  $u_s(\sigma_\theta, \tau^*) \geq U_s(\sigma, \mu, \tau^*) \geq u_s(\sigma_{\theta'}, \tau^*)$  with at least one inequality strict,  $\sigma_\theta$  and  $\sigma_{\theta'}$  are weakly Pareto-ranked,

and,

(ii) if  $\phi_s$  is affine, then for all  $\theta \in \text{supp}(\mu)$ , there does not exist a Pareto-ranked splitting of  $\sigma_\theta$ ,  $(\bar{\sigma}, \underline{\sigma}, \lambda)$ , such that  $u_s(\bar{\sigma}, \tau^*) \geq U_s(\sigma, \mu, \tau^*) \geq u_s(\underline{\sigma}, \tau^*)$ .

To gain intuition for part (i), first observe that if such  $\sigma_\theta$  and  $\sigma_{\theta'}$  are not weakly Pareto-ranked, then the receiver must get a strictly higher expected payoff from  $\sigma_{\theta'}$  than from  $\sigma_\theta$ , while the reverse is true for the sender. Ambiguity aversion then implies that the receiver's effective measure places more weight on  $\sigma_\theta$  relative to  $\sigma_{\theta'}$  than the ambiguity neutral weights do, i.e.,  $em_\theta^{(\sigma, \mu)} / em_{\theta'}^{(\sigma, \mu)} > \mu_\theta / \mu_{\theta'}$ . If  $\sigma_\theta$  and  $\sigma_{\theta'}$  are the only two experiments in the support of  $\mu$ , the sender can merge them into the (unambiguous) experiment  $em_\theta^{(\sigma, \mu)} \sigma_\theta + em_{\theta'}^{(\sigma, \mu)} \sigma_{\theta'}$ . By construction, the receiver would continue to be obedient, and the sender would strictly benefit from this merging – a profitable deviation. When  $\sigma_\theta$  and  $\sigma_{\theta'}$  are not the only two experiments in the support of  $\mu$ , however, this is not the complete story as this merging may also impact the weighting of the merged experiment relative to the other experiments. Part of the additional insight of the proof is that when  $u_s(\sigma_\theta, \tau^*) \geq U_s(\sigma, \mu, \tau^*) \geq u_s(\sigma_{\theta'}, \tau^*)$  holds, this impact is at least weakly beneficial to the sender. The intuition for part (ii) is similar.

In Theorem 1, the conditions refer to pairs of experiments for which the sender's payoffs bracket  $U_s(\sigma, \mu, \tau^*)$ . Intuition for why similar conclusions may not apply when the pairs involved in the Pareto-ranking or the Pareto-ranked splitting lie on the same side of  $U_s(\sigma, \mu, \tau^*)$  is related to how the receiver's ambiguity aversion, as reflected in properties of  $\phi_r$ , connects  $\mu$  with the effective measure  $em^{(\sigma, \mu)}$  via (3). In particular, when there are more than two experiments in  $\sigma$ , splitting or merging experiments on the same side of  $U_s(\sigma, \mu, \tau^*)$  may shift their combined weights in the effective measure relative to the other experiments in a manner unfavorable to the sender. In the Appendix, we show that concavity (resp. convexity) of  $1/\phi_r'$  is sufficient to extend the conclusions to pairs of experiments on a particular side of  $U_s(\sigma, \mu, \tau^*)$ , and assuming linearity of  $1/\phi_r'$  leads to the following simpler necessary conditions for optimal persuasion:

**Proposition 2.** Suppose  $(\sigma, \mu)$  is a solution to  $(\mathcal{P})$ , and

$$\phi_r(x) = c \ln(ax + b) + d \quad (6)$$

<sup>16</sup>As the proof in the Appendix makes clear, the only role of strict concavity of  $\phi_r$  is to simplify the statement of the theorem. Without it, one needs to add conditions checking if  $\phi_r'(u_r(\sigma_\theta, \tau^*)) \neq \phi_r'(u_r(\sigma_{\theta'}, \tau^*))$  to each part of the theorem.



for some  $a, b, c, d \in \mathbb{R}$  where  $a, c > 0$  and  $ax + b > 0$  for all  $x \in [\min_{a,\omega} u_r(a, \omega), \max_{a,\omega} u_r(a, \omega)]$ . Then, all experiments are weakly Pareto-ranked, that is, for all  $\theta, \theta' \in \text{supp}(\mu)$ ,  $\sigma_\theta$  and  $\sigma_{\theta'}$  are weakly Pareto-ranked.

If, in addition, the sender is ambiguity neutral, no Pareto-ranked splitting of  $\sigma_\theta$  exists for any  $\theta \in \text{supp}(\mu)$ .

Note that (6) may be interpreted as constant relative ambiguity aversion (see [Klibanoff et al. \(2005\)](#)). The result that all experiments used must be weakly Pareto-ranked is reminiscent of a key Pareto-ranking result ([Rayo and Segal, 2010](#), p. 959, Lemma 2) in an entirely different persuasion setting (one in which ambiguity plays no role). The more general results of our Theorem 1 have no obvious analogue in the setting of [Rayo and Segal \(2010\)](#).

We now solve our introductory example for a  $\phi_r$  satisfying (6):

**Example 1** (Introductory Example Continued). Suppose  $\phi_r(x) = \ln(x + 5)$  and  $\phi_s(x) = x$ . Then a sender's optimal persuasion strategy is the  $((\sigma_{\bar{\theta}}, \sigma_{\underline{\theta}}), \mu)$  described in Figure 2 with  $\mu_{\bar{\theta}} = 39/50$ . The payoffs from this optimal persuasion are as follows:

$$U_s((\sigma_{\bar{\theta}}, \sigma_{\underline{\theta}}), \mu, \tau^*) = 39/50 \times 3/2 + 11/50 \times 1/2 = 1.28,$$

$$U_r((\sigma_{\bar{\theta}}, \sigma_{\underline{\theta}}), \mu, \tau^*) = e^{(39/50 \ln(13/2) + 11/50 \ln(11/2))} - 5 \approx 1.265.$$

Thus both the sender and receiver do better than the payoff of  $5/4$  they would each obtain under Bayesian persuasion.

So far, the analysis was devoted to properties of optimal communication strategies when ambiguous experiments are allowed. However, it does not directly tell us whether the sender strictly benefits from introducing ambiguity into their communication. We now turn to this issue, which we view as a primary focus of the paper.

## 4 When Does Ambiguous Communication Benefit The Sender?

### 4.1 Necessary Conditions for Ambiguity to Benefit The Sender

We show that Pareto-ranked experiments continue to be key in determining whether ambiguous communication is better for the sender than unambiguous communication. The following theorem shows having Pareto-ranked experiments in the collection (in particular, better and worse ones having sender's expected payoffs bracketing  $u_s^{BP}$ ) is necessary for an ambiguous experiment to benefit the sender.

**Theorem 2.** *If an obedient ambiguous experiment  $(\sigma, \mu)$  benefits the sender, then there exist  $\theta, \theta' \in \text{supp}(\mu)$  such that  $\sigma_\theta$  and  $\sigma_{\theta'}$  are Pareto-ranked, with  $u_s(\sigma_\theta, \tau^*) > u_s^{BP} \geq u_s(\sigma_{\theta'}, \tau^*)$ .*

Comparing with part (i) of Theorem 1, we see that while optimal persuasion requires *weak* Pareto-ranking of experiments that bracket the sender's payoff from that ambiguous experiment, Theorem 2 says that any improvement over Bayesian persuasion requires some Pareto-ranked experiments (and thus strictly ranked) bracketing  $u_s^{BP}$  for the sender.

We next present two equivalent sets of necessary conditions for ambiguity to benefit the sender, and show that these conditions imply that ambiguous communication can never benefit the sender when the receiver has only two available actions – a common assumption in many examples and applications in the literature. Whereas Theorem 2 described a necessary property of any sender's strategy that improves on Bayesian persuasion, these next conditions relate the possibility of ambiguity benefiting the sender in a given persuasion game to the existence of Pareto-ranked experiments with certain properties.

**Theorem 3.** *Ambiguous communication benefits the sender only if  $\phi_r$  is not affine and there exists a Pareto-ranked splitting,  $(\bar{\sigma}, \underline{\sigma}, \lambda)$ , of an obedient experiment  $\hat{\sigma}$  such that  $u_s(\bar{\sigma}, \tau^*) > u_s^{BP}$ .*

**Example 2** (Introductory Example Continued). *For the collection  $\sigma = (\sigma_{\bar{\theta}}, \sigma_{\underline{\theta}})$  constructed in Figure 2 of the introductory example,  $(\sigma_{\bar{\theta}}, \sigma_{\underline{\theta}}, \frac{3}{4})$  is a Pareto-ranked splitting of  $\sigma^{BP}$  and  $u_s(\sigma_{\bar{\theta}}, \tau^*) > u_s^{BP}$ . Thus, for this example, the existence required in Theorem 3 is satisfied for  $\hat{\sigma} = \sigma^{BP}$ .*

**Remark 2** (Not necessary for  $\hat{\sigma}$  to be an optimal Bayesian persuasion experiment). The reader might wonder if a stronger version of the theorem that requires  $\hat{\sigma}$  to be an optimal Bayesian persuasion experiment holds. This is false. There are examples in which the sender benefits from ambiguous communication even though no Pareto-ranked splitting of any optimal Bayesian persuasion experiment exists (as is true, for instance, whenever all such experiments are efficient). In such cases, it is splittings of some other obedient experiment that generate the gains over Bayesian persuasion for the sender.

The conditions in Theorem 3 are deceptively powerful: From these conditions alone, strict benefit from ambiguity can be ruled out for a simple yet important class of problems – those in which the receiver has a binary action space.

**Corollary 1.** *If the receiver has only two actions, the sender cannot benefit from ambiguous communication.*

An important step in the proofs of Theorem 3 and Corollary 1 is showing that, when  $\phi_r$  is not affine, the conditions in Theorem 3 are equivalent to the existence of Pareto-ranked experiments,  $\sigma$  and  $\sigma^*$  such that: (i)  $\text{supp } \sigma(\cdot|\omega) = \text{supp } \sigma^*(\cdot|\omega)$  for all  $\omega$ , (ii)  $u_s(\sigma, \tau^*) > u_s^{BP}$ , and (iii)  $\tau^* \in$

$br(\sigma^*) \setminus br(\sigma)$ . The argument that this formulation of the conditions are necessary is constructive, and the effective measure plays a key role. Suppose that there exists a solution  $(\sigma^*, \mu^*)$  to the sender's program  $(\mathcal{P})$  that benefits the sender. Construct  $\sigma$  and  $\sigma^*$  by letting  $\sigma = \sum_{\theta} \mu_{\theta}^* \sigma_{\theta}^*$  and  $\sigma^* = \sum_{\theta} em_{\theta}^{(\sigma^*, \mu^*)} \sigma_{\theta}^*$ .

The intuition for Corollary 1 is as follows. From (iii), above, we have that part of a necessary condition for ambiguity helping the sender is the existence of an experiment  $\sigma$  that strictly improves the receiver's expected payoff compared to some other experiment  $\sigma^*$ , with the added property that obedience of  $\sigma$  is not optimal, i.e.,  $\tau^* \notin br(\sigma)$ . Intuitively, such an improvement is possible only when  $\sigma$  is more informative for the receiver and the benefit of this extra information outweighs the cost of not best responding. When there are only two actions, taking advantage of extra information requires best responding. To see this, note that not best responding implies either taking the same action always (and thus ignoring any information) or always doing the opposite of what is optimal for the receiver (which hurts more when there is more information). In contrast, when there are three or more actions, it becomes possible to have some beneficial responsiveness to information without going all the way to best responding. As we saw in the introductory example, this indeed can leave scope for possible improvements.

## 4.2 Robust Benefits

So far, we have assumed that if the sender designs the ambiguous experiment  $(\sigma, \mu)$ , the receiver perceives it correctly. More realistically, the receiver might have a somewhat different perception of the experiment than the one the sender intends to convey. After all, conveying the exact specifications of an experiment is a complex task, let alone of an ambiguous one. Yet, we show that if the sender benefits from ambiguous communication, they continue to benefit even if the receiver somewhat misperceives the intended experiment.

**Proposition 3.** *Suppose ambiguous communication benefits the sender and that the set of obedient experiments has a non-empty interior. Then, there exists a non-empty open set of obedient ambiguous experiments that benefit the sender.*

A sketch of the argument (full details are in Section SA.4 of the Online Appendix (Cheng et al., 2025)) is to show the persuasion problem is sufficiently continuous to guarantee the existence of an open set of obedient ambiguous experiments that benefit the sender. In fact, this continues to be true even under small perturbations of  $\phi_r$ , so that the existence of benefits does not rest on exact knowledge of the receiver's ambiguity aversion.

**Corollary 2.** *Suppose ambiguous communication benefits the sender and that the set of obedient experiments has a non-empty interior. Then, there exists an ambiguous experiment that benefits the sender, and continues to do so under small perturbations of  $\phi_r$ .*

## 5 Benefits from Binary Ambiguous Communication

This section restricts attention to binary ambiguous experiments. This restriction is not without loss of generality because there are examples in which the sender benefitting from ambiguous communication requires ambiguous experiments with  $\sigma$  containing more than two distinct experiments (see Section SA.2 in the Online Appendix (Cheng et al., 2025)). Nonetheless, this restriction allows us to derive sufficient conditions for the sender to benefit from ambiguous communication and how these conditions vary with the extent of the sender's and/or receiver's ambiguity aversion. It also allows us to see such ambiguous communication may also improve the receiver's payoff.

If a binary ambiguous experiment benefits the sender compared to Bayesian persuasion, it follows from Theorem 2 that the experiments must be Pareto-ranked. We therefore focus on Pareto-ranked binary ambiguous experiments in what follows.

The next theorem, Theorem 4, provides necessary and sufficient conditions for a binary ambiguous experiment based on a Pareto-ranked splitting of any obedient experiment  $\sigma^*$  to strictly improve the sender's payoff compared to  $\sigma^*$ . Additionally, it provides necessary and sufficient conditions for such an experiment to strictly improve the receiver's payoff compared to  $\sigma^*$ . We later apply the theorem to the case in which  $u_s(\sigma^*, \tau^*) = u_s^{BP}$ , thereby obtaining sufficient conditions for the sender to benefit from ambiguous communication (see Corollary 3). Proposition 4 provides conditions on the primitives sufficient for existence of a Pareto-ranked splitting of a given experiment and relates them to conditions on inefficiency of Bayesian persuasion.

The theorem uses the following notion of *probability premium*.

**Definition 6.** Given  $\phi$ ,  $u$ , and experiments  $\bar{\sigma}$  and  $\underline{\sigma}$  such that  $u(\bar{\sigma}, \tau^*) > u(\underline{\sigma}, \tau^*)$ , the  $((\bar{\sigma}, \underline{\sigma}), \lambda)$ -*probability premium* required to compensate for replacing the unambiguous experiment  $\sigma^* := \lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma}$  by the ambiguous experiment  $((\bar{\sigma}, \underline{\sigma}), \lambda)$ , assuming obedience, is:

$$\rho^{\phi, u}((\bar{\sigma}, \underline{\sigma}), \lambda) := \frac{\phi(u(\sigma^*, \tau^*)) - \lambda\phi(u(\bar{\sigma}, \tau^*)) - (1 - \lambda)\phi(u(\underline{\sigma}, \tau^*))}{\phi(u(\bar{\sigma}, \tau^*)) - \phi(u(\underline{\sigma}, \tau^*))}.$$

The probability premium  $\rho^{\phi, u}((\bar{\sigma}, \underline{\sigma}), \lambda)$  is exactly the  $\phi$ -payoff difference between  $\sigma^*$  and the ambiguous experiment  $((\bar{\sigma}, \underline{\sigma}), \lambda)$ , normalized to lie in  $[0, 1]$ . This premium is non-negative under ambiguity aversion, and is zero under ambiguity neutrality. Similar notions of probability premium in the context of risk go back to at least Pratt (1964) (see Eeckhoudt and Laeven (2015) for a graphical representation of Pratt's concept).

Thus, if we let

$$\mu_{\bar{\sigma}} = \lambda + \rho^{\phi, u}((\bar{\sigma}, \underline{\sigma}), \lambda) \in [0, 1],$$

be the probability of  $\bar{\sigma}$ , then

$$\begin{aligned} U((\bar{\sigma}, \underline{\sigma}), \mu, \tau^*) &= \phi^{-1}\left((\lambda + \rho^{\phi, u}((\bar{\sigma}, \underline{\sigma}), \lambda))\phi(u(\bar{\sigma}, \tau^*)) + (1 - \lambda - \rho^{\phi, u}((\bar{\sigma}, \underline{\sigma}), \lambda))\phi(u(\underline{\sigma}, \tau^*))\right) \\ &= \phi^{-1}\left(\phi(u(\sigma^*, \tau^*))\right) = u(\sigma^*, \tau^*), \end{aligned}$$

meaning that the premium  $\rho^{\phi, u}((\bar{\sigma}, \underline{\sigma}), \lambda)$  is exactly the increase in  $\mu_{\bar{\sigma}}$  above  $\lambda$  needed to make the player indifferent between the ambiguous experiment  $((\bar{\sigma}, \underline{\sigma}), \mu)$  and  $\sigma^*$ . Thus, assuming obedience, this premium is the smallest increase in the  $\mu$ -probability of the higher payoff experiment required to compensate for exposure to the ambiguous experiment:

**Lemma 2.** *Let  $\bar{\sigma}$  and  $\underline{\sigma}$  be experiments such that  $u_i(\bar{\sigma}, \tau^*) > u_i(\underline{\sigma}, \tau^*)$ . For all  $\mu_{\bar{\sigma}}, \lambda \in [0, 1]$ ,  $U_i((\bar{\sigma}, \underline{\sigma}), \mu, \tau^*) > u_i(\lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma}, \tau^*)$  if, and only if, player  $i$ 's  $((\bar{\sigma}, \underline{\sigma}), \lambda)$ -probability premium is strictly less than  $\mu_{\bar{\sigma}} - \lambda$ .*

As a consequence, we have the following result:

**Theorem 4.** *Let  $\sigma^*$  be an obedient experiment. Suppose that  $(\bar{\sigma}, \underline{\sigma}, \lambda)$  is a Pareto-ranked splitting of  $\sigma^*$  satisfying  $u_s(\bar{\sigma}, \tau^*) > u_s(\underline{\sigma}, \tau^*)$ . The binary ambiguous experiment  $(\sigma, \mu)$ , with  $\sigma = (\bar{\sigma}, \underline{\sigma})$  and*

$$\mu_{\bar{\sigma}} = \frac{\lambda\phi'_r(u_r(\underline{\sigma}, \tau^*))}{\lambda\phi'_r(u_r(\underline{\sigma}, \tau^*)) + (1 - \lambda)\phi'_r(u_r(\bar{\sigma}, \tau^*))}, \quad (7)$$

*satisfies the following properties:*

- (i)  $(\sigma, \mu)$  is obedient,
- (ii)  $U_r(\sigma, \mu, \tau^*) > u_r(\sigma^*, \tau^*)$  if, and only if,  $\mu_{\bar{\sigma}} > \lambda$ ,
- (iii)  $U_s(\sigma, \mu, \tau^*) > u_s(\sigma^*, \tau^*)$  if, and only if, the sender's  $((\bar{\sigma}, \underline{\sigma}), \lambda)$ -probability premium is strictly less than  $\mu_{\bar{\sigma}} - \lambda$ .

*Furthermore, the sender's  $((\bar{\sigma}, \underline{\sigma}), \lambda)$ -probability premium is increasing in the sender's ambiguity aversion, and  $\mu_{\bar{\sigma}}$  is increasing in the receiver's ambiguity aversion.*

That a  $\mu$  satisfying (7) ensures that the obedience of  $\sigma^*$  extends to the binary ambiguous experiment  $(\sigma, \mu)$  as in (i) is a straightforward consequence of Lemma 1. The necessary and sufficient conditions in (ii) for the receiver to be better off when the sender communicates ambiguously using  $(\sigma, \mu)$  rather than unambiguously using  $\sigma^*$  require some elaboration. First, the condition  $\mu_{\bar{\sigma}} > \lambda$  is equivalent to  $\phi'_r(u_r(\underline{\sigma}, \tau^*)) > \phi'_r(u_r(\bar{\sigma}, \tau^*))$ , i.e., the receiver is, within this range of payoffs, not everywhere ambiguity neutral. In particular, this condition is always satisfied if  $\phi_r$  is strictly concave. Second, the condition  $\mu_{\bar{\sigma}} > \lambda$  can be shown to be equivalent to the receiver's  $((\bar{\sigma}, \underline{\sigma}), \lambda)$ -probability premium being strictly less than  $\mu_{\bar{\sigma}} - \lambda$ , which, by Lemma 2, characterizes when the

receiver is better off under  $(\sigma, \mu)$  than under  $\sigma^*$ . The necessary and sufficient conditions in (iii) for  $(\sigma, \mu)$  to be better for the sender than  $\sigma^*$  follow directly from Lemma 2. For an ambiguity neutral sender, the probability premium is zero, and thus the condition in (iii) reduces to  $\mu_{\bar{\sigma}} > \lambda$ , as in (ii). Thus, for an ambiguity neutral sender facing a strictly ambiguity averse receiver, the ambiguity introduced in  $(\sigma, \mu)$  improves on  $\sigma^*$  for both sender and receiver.

The source of the economic gain from ambiguous communication, for both sender and receiver, is the greater use, as measured by  $\mu_{\bar{\sigma}} - \lambda$ , of the Pareto-better experiment  $\bar{\sigma}$ . This gain has to be netted-off against the probability premium, which encapsulates the cost due to the player's own ambiguity aversion of the exposure to ambiguity from the ambiguous experiment. The fact that  $\mu_{\bar{\sigma}}$  is constructed to respect obedience taking into account the receiver's ambiguity aversion but not the sender's, is what explains why the condition for this net gain to be positive can be simplified for the receiver, but not the sender. Though our result (and proof) is global in the sense that they apply to any Pareto-ranked splitting, intuition based on a local, envelope-theorem type argument may be helpful.<sup>17</sup> The idea is that the effects on the receiver of exposure to a small amount of ambiguity around  $\sigma^*$  are second-order because they are partially mitigated by the endogeneity of the receiver's best response which, because of ambiguity aversion, is partly designed to hedge against ambiguity and thus reduce its negative effect. In contrast, the positive effects of increasing  $\mu_{\bar{\sigma}}$  are first-order. See Section SA.3 of the Online Appendix (Cheng et al., 2025) for a formal argument along these lines.

The comparative static about  $\mu_{\bar{\sigma}}$  in the final section of the theorem, when combined with (ii) and (iii), shows that the payoff difference between the ambiguous experiment  $(\sigma, \mu)$  and the unambiguous  $\sigma^*$  satisfies single-crossing with respect to the receiver's ambiguity aversion for both the sender and receiver. Similarly, the comparative static in the sender's probability premium, together with (iii), shows that the negative of this payoff difference for the sender satisfies single-crossing with respect to the sender's ambiguity aversion.

Starting from any given obedient experiment, Theorem 4 provides necessary and sufficient conditions for binary ambiguous communication to strictly improve the sender's payoff, and thus sufficient conditions for *some* ambiguous communication to do so. Thus, if we apply Theorem 4 to the case where  $\sigma^*$  is an optimal Bayesian persuasion, we obtain sufficient conditions for ambiguity to benefit the sender:

**Corollary 3.** *Let  $\sigma^{BP}$  be an obedient experiment such that  $u_s(\sigma^{BP}, \tau^*) = u_s^{BP}$ . If there exists a Pareto-ranked splitting of  $\sigma^{BP}$ ,  $(\bar{\sigma}, \underline{\sigma}, \lambda)$ , for which  $\rho^{\phi_s, u_s}((\bar{\sigma}, \underline{\sigma}), \lambda) < \mu_{\bar{\sigma}} - \lambda$ , with  $\mu_{\bar{\sigma}}$  given by equation (7), then ambiguous communication benefits the sender.*

Therefore, whenever a Pareto-ranked splitting of an optimal Bayesian persuasion experiment ex-

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<sup>17</sup>We thank Dilip Abreu for suggesting this line of reasoning.

ists, an ambiguity-neutral sender benefits from ambiguous communication as long as the receiver is not completely ambiguity neutral over the payoff range of the splitting.<sup>18</sup>

Theorem 4 and Corollary 3 require the existence of a Pareto-ranked splitting. This existence is not guaranteed. For instance, if the obedient experiment  $\sigma^*$  induces an efficient payoff profile, no Pareto-ranked splitting of it exists. The next result provides sufficient conditions on the primitives for the existence of a Pareto-ranked splitting of  $\sigma^*$ .

**Proposition 4.** *Given any experiment  $\sigma^*$ , fix, for each  $\omega \in \Omega$ ,  $a_\omega \in \text{supp}(\sigma^*(\cdot|\omega))$  and consider the following set of vectors,*

$$\left\{ \begin{bmatrix} p(\omega)(u_s(a, \omega) - u_s(a_\omega, \omega)) \\ p(\omega)(u_r(a, \omega) - u_r(a_\omega, \omega)) \end{bmatrix} : a \in \text{supp}(\sigma^*(\cdot|\omega)), \omega \in \Omega \right\}. \quad (8)$$

*If this set spans  $\mathbb{R}^2$ , then there exists a Pareto-ranked splitting of  $\sigma^*$ .*

Arieli et al. (2024) argue that Bayesian persuasion solutions are typically inefficient<sup>19</sup> and provide a necessary condition,  $\sum_{\omega \in \Omega} |\text{supp}(\sigma(\cdot|\omega))| \leq |\Omega| + 1$ , for their efficiency. If a Bayesian persuasion solution violates this condition, the spanning condition for (8) holds for a generic specification of the payoffs,  $u_i(a, \omega)$ , implying, by Corollary 3, that both the receiver and an ambiguity-neutral sender benefit from ambiguous communication.

**Remark 3** (Non-necessity). That neither the spanning condition nor violation of the Arieli et al. (2024) condition are necessary for the existence of a Pareto-ranked splitting can be seen from our introductory example. For  $\sigma^* = \sigma^{BP}$ , the example satisfies neither condition but there are, as depicted in Figure 2, Pareto-ranked splittings of  $\sigma^{BP}$ .

While the optimal persuasion does depend on  $\phi_r$  and  $\phi_s$ , i.e., the ambiguity attitudes, we next show that the possibility of strict improvement for the sender from using binary ambiguous experiments is robust in several respects. First, the same ambiguous experiment remains beneficial to any less ambiguity averse sender. Second, it is robust to the sender underestimating the extent of ambiguity aversion of the receiver. In other words, if an improvement is possible when facing a given receiver, it is also possible when facing a more ambiguity-averse receiver. While we show that the same collection  $\sigma$  can be used to generate the improvement for all more ambiguity-averse receivers, in general, the  $\mu$  guaranteeing improvement may need to change. Part (iii) of the result shows that adding the requirement that  $\underline{\sigma}$  is obedient allows a stronger robustness: the same  $\mu$  that generates an improvement for the sender when facing a receiver with  $\phi_r$  also generates an improvement when facing any more ambiguity averse receiver (more concave  $\phi_r$ ).

<sup>18</sup>Recall from Remark 2 that existence of a Pareto-ranked splitting of  $\sigma^{BP}$  is not necessary to benefit.

<sup>19</sup>Though Ichihashi (2019) proves they are always efficient when the receiver has only two actions.



**Theorem 5.** Suppose there exist a  $\sigma = (\bar{\sigma}, \underline{\sigma})$  and a non-degenerate  $\mu$  such that  $(\sigma, \mu)$  is obedient and benefits the sender (compared to  $u_s^{BP}$ ). Then:

- (i)  $(\sigma, \mu)$  also benefits all less ambiguity averse senders, and
- (ii) for any more ambiguity averse receiver, there exists some  $\tilde{\mu}$  such that  $(\sigma, \tilde{\mu})$  benefits the sender (and all less ambiguity averse senders), and
- (iii) if  $\underline{\sigma}$  is obedient, then  $\tilde{\mu}$  in (ii) can be set equal to  $\mu$ .

## 6 A Concavification-like Characterization

In this section, we present a concavification-like characterization of optimal persuasion with ambiguous communication as well as a characterization of when the sender can strictly benefit from communicating ambiguously rather than unambiguously. We stress that the concavification-like characterization is not immediate from the one for Bayesian persuasion of [Kamenica and Gentzkow \(2011\)](#). A key complication is the non-separability across recommendations in determining obedience (coming from the appearance of the  $u_r(\sigma_\theta, \tau^*)$  terms in the effective measure formula (3)), which is a consequence of ambiguity aversion.

### 6.1 A Concavification-like Characterization of Optimal Persuasion with Ambiguous Communication

Let  $\Sigma$  denote the set of all experiments and  $\Sigma^* \subseteq \Sigma$  the set of obedient experiments (i.e.,  $\Sigma^* := \{\sigma \in \Sigma : \tau^* \in br(\sigma)\}$ ). Notice that both  $\Sigma$  and  $\Sigma^*$  are non-empty convex sets and can be embedded into an  $|\Omega| \times (|A| - 1)$ -dimensional Euclidean space since an experiment specifies, for each state  $\omega \in \Omega$ , a probability distribution over actions in  $A$ .

For each scalar  $u \in \mathbb{R}$ , define the function  $\Phi_u : \Sigma \rightarrow \mathbb{R}$  by

$$\Phi_u(\sigma) := \frac{\phi_s(u_s(\sigma, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\sigma, \tau^*))},$$

and consider the following maximization problem:

$$(\Phi^*(u)) := \begin{cases} \max_{(\lambda_\theta, \sigma_\theta)_{\theta \in \Theta}} \sum_{\theta \in \Theta} \lambda_\theta \Phi_u(\sigma_\theta), \\ \text{subject to: } \sum_{\theta \in \Theta} \lambda_\theta \sigma_\theta \in \Sigma^*, \sum_{\theta \in \Theta} \lambda_\theta = 1, \lambda_\theta \geq 0, \sigma_\theta \in \Sigma, \forall \theta \in \Theta. \end{cases}$$

Theorem 6 states that the value of the optimal ambiguous persuasion program ( $\mathcal{P}$ ) is the unique utility level  $u$  such that the value of the program  $(\Phi^*(u))$  is equal to zero. An optimal ambiguous

persuasion strategy can be directly constructed from a solution to  $(\Phi^*(u))$ , and there always exists such an optimal strategy that makes use of no more than  $|\Omega| \times (|A| - 1) + 1$  experiments.

**Theorem 6.** *The value of  $(\mathcal{P})$  is  $u$  if, and only if, the value of  $(\Phi^*(u))$  is 0. Moreover, there exists a solution  $(\sigma^*, \mu^*)$  to  $(\mathcal{P})$  such that  $|\text{supp}(\mu^*)| \leq |\Omega| \times (|A| - 1) + 1$ .*

To understand the relationship between the programs  $(\mathcal{P})$  and  $(\Phi^*(u))$ , we first note that the definition of  $U_s$ , the fact that  $\phi_s^{-1}$  is strictly increasing, and the characterization of obedience in Lemma 1, implies that the value of  $(\mathcal{P})$  is  $u$  if, and only if, the value of the program

$$(\hat{\mathcal{P}}) = \begin{cases} \max_{(\mu_\theta, \sigma_\theta)_{\theta \in \Theta}} \sum_{\theta \in \Theta} \mu_\theta \phi_s(u_s(\sigma_\theta, \tau^*)), \\ \text{subject to: } \sum_{\theta \in \Theta} em_\theta^{(\sigma, \mu)} \sigma_\theta \in \Sigma^*, \sigma_\theta \in \Sigma, \forall \theta \in \Theta, \end{cases}$$

is  $\phi_s(u)$ . Next, we can do a change of variables to maximize over the choice of effective measures and experiments. Formally, if we write  $\lambda_\theta$  for  $em_\theta^{(\sigma, \mu)}$ , we can use (4) to substitute for  $\mu_\theta$  in terms of  $\lambda_\theta$  to yield:

$$(\hat{\mathcal{P}}) = \begin{cases} \max_{(\lambda_\theta, \sigma_\theta)_{\theta \in \Theta}} \left( \sum_{\tilde{\theta} \in \Theta} \frac{\lambda_{\tilde{\theta}}}{\phi'_r(u_r(\sigma_{\tilde{\theta}}, \tau^*))} \right)^{-1} \sum_{\theta \in \Theta} \frac{\lambda_\theta}{\phi'_r(u_r(\sigma_\theta, \tau^*))} \phi_s(u_s(\sigma_\theta, \tau^*)), \\ \text{subject to: } \sum_{\theta \in \Theta} \lambda_\theta \sigma_\theta \in \Sigma^*, \sum_{\theta} \lambda_\theta = 1, \lambda_\theta \geq 0, \sigma_\theta \in \Sigma, \forall \theta \in \Theta. \end{cases}$$

Finally, observe that the normalization factor  $\left( \sum_{\tilde{\theta} \in \Theta} \frac{\lambda_{\tilde{\theta}}}{\phi'_r(u_r(\sigma_{\tilde{\theta}}, \tau^*))} \right)^{-1}$  makes the objective function highly non-linear in the maximizers  $(\lambda_\theta, \sigma_\theta)_{\theta \in \Theta}$ . This is the motivation for subtracting off  $\phi_s(u)$ . Indeed, if the value of  $(\hat{\mathcal{P}})$  is  $\phi_s(u)$ , then

$$\sum_{\theta \in \Theta} \lambda_\theta \frac{\phi_s(u_s(\sigma_\theta, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\sigma_\theta, \tau^*))} = 0.$$

Conversely, if the value of  $(\Phi^*(u))$  is zero, then the value of  $(\hat{\mathcal{P}})$  is  $\phi_s(u)$ . In effect, this reformulation discards the messy (but strictly positive) normalization term without changing the solution.

The constraint  $\sum_{\theta \in \Theta} \lambda_\theta \sigma_\theta \in \Sigma^*$  makes clear that any solution is a splitting of an obedient experiment, where the  $(\lambda_\theta)_{\theta \in \Theta}$  are the splitting weights. Importantly,  $(\Phi^*(u))$  is linear in these splitting weights.

An implication of this linearity and Theorem 6 is to provide a concavification-like characterization (Aumann and Maschler, 1966, 1995) of the value of optimal persuasion with ambiguous communication. Notice that concavification can be used to compute the value of program  $(\Phi^*(u))$ : For each  $u \in \mathbb{R}$ , the program  $(\Phi^*(u))$  maximizes over convex combinations of points on the graph of  $\Phi_u$ , exactly the type of program that concavification characterizes. Specifically, for each  $u \in \mathbb{R}$ ,

let  $\text{cav}\Phi_u : \Sigma \rightarrow \mathbb{R}$  denote the concavification of  $\Phi_u$ , that is,

$$\text{cav}\Phi_u(\sigma) = \begin{cases} \max_{(\lambda_\theta, \sigma_\theta)_{\theta \in \Theta}} \sum_{\theta \in \Theta} \lambda_\theta \Phi_u(\sigma_\theta), \\ \text{subject to: } \sum_{\theta \in \Theta} \lambda_\theta \sigma_\theta = \sigma, \sum_{\theta} \lambda_\theta = 1, \lambda_\theta \geq 0, \sigma_\theta \in \Sigma, \forall \theta \in \Theta, \end{cases}$$

and the maximum over  $\sigma \in \Sigma^*$  of  $\text{cav}\Phi_u(\sigma)$  is the value of  $(\Phi^*(u))$ . Observe that any such maximum is achieved by a splitting of some obedient experiment, with the splitting weights given by the effective measure. The following immediate corollary of Theorem 6 thus provides a concavification-like characterization of the value of  $(\mathcal{P})$ .

**Corollary 4.** *The value of  $(\mathcal{P})$  is  $u$  if, and only if,  $\max_{\sigma \in \Sigma^*} \text{cav}\Phi_u(\sigma) = 0$ .*

Algorithmically, we can start with  $u_0 = u_s^{BP}$ , the payoff the sender obtains at a solution to  $(\mathcal{P}^{BP})$ , which is a lower bound on what the sender can achieve with ambiguous communication. If  $\max_{\sigma \in \Sigma^*} \text{cav}\Phi_{u_0}(\sigma) = 0$ , then we are done – the sender’s best payoff is  $u_s^{BP}$ . If  $\max_{\sigma \in \Sigma^*} \text{cav}\Phi_{u_0}(\sigma) > 0$ , we can increase  $u_0$  to  $u_1 = \max_{a, \omega} u_s(a, \omega)$  and check again. If the solution is zero, we are done. If it is strictly negative, we can then consider the mid-point  $u_2 = (1/2)u_0 + (1/2)u_1$ . If  $\max_{\sigma \in \Sigma^*} \text{cav}\Phi_{u_2}(\sigma) > 0$  (resp.,  $< 0$ ), we can then consider the midpoint  $u_3 = (1/2)u_2 + (1/2)u_1$  (resp.,  $u_3 = (1/2)u_0 + (1/2)u_2$ ) and repeat the maximization problem, and so on.

We now relate this concavification-like result with the concavification characterization of Bayesian persuasion in [Kamenica and Gentzkow \(2011\)](#) and its extension to allow for exogenously heterogeneous priors in [Alonso and Câmara \(2016\)](#). Both of these characterizations are formulated in terms of splittings of priors, rather than, as in our characterization, splittings on the higher-dimensional space of experiments. Suppose we try to write a program in which the sender maximizes with respect to splittings of the prior. Consider the simplest case of an ambiguity neutral sender, i.e.,  $\phi_s$  linear. Any ambiguous experiment  $(\mu_\theta, \sigma_\theta)_{\theta \in \Theta}$  induces a distribution over the receiver’s *effective* posteriors, that is, the posteriors that the “effective” experiment  $\sum_{\theta} \lambda_\theta \sigma_\theta$  induces, where  $\lambda_\theta = em_\theta^{(\sigma, \mu)}$ , the effective measure. Thus, the splitting the “effective” experiment  $\sum_{\theta} \lambda_\theta \sigma_\theta$  induces may differ from the splitting the experiment  $\sum_{\theta} \mu_\theta \sigma_\theta$  induces. The latter is the one the ambiguity neutral sender uses to evaluate their payoff. To be amenable to a concavification approach on this space, the sender’s objective function would therefore need to be, as in [Alonso and Câmara \(2016\)](#), an increasing transformation of a function that is linear in the distribution over the receiver’s effective posteriors. However, since the relationship between the  $em_\theta^{(\sigma, \mu)}$  and  $(\mu_\theta, \sigma_\theta)_{\theta \in \Theta}$  is highly non-linear, the desired linearity is impossible. Economically, this non-linearity has its source in the fact that ambiguity aversion causes the effective measure to be proportional to the product (and thus, essentially, the covariance) of the ambiguity neutral probability  $\mu_\theta$  and the marginal utility  $\phi'_r(u_r(\sigma_\theta, \tau^*))$  and the latter is non-separable across action

recommendations. This explains how the non-separability across action recommendations in determining obedience is what prevents adopting the strategies of [Kamenica and Gentzkow \(2011\)](#) and [Alonso and Câmara \(2016\)](#) to establish our characterization.

## 6.2 When is Ambiguous Communication Better than Unambiguous?

A characterization of when ambiguous communication benefits the sender can be derived from our characterization of optimal persuasion (Theorem 6 and Corollary 4). More specifically, Lemma A.2 in the proof of Theorem 6 shows that ambiguous communication gives the sender a strictly higher payoff than  $u$  if, and only if, the value of the program  $(\Phi^*(u))$  is strictly positive. By letting  $u = u_s^{BP}$ , we obtain the following.

**Corollary 5.** *Ambiguous communication benefits the sender if, and only if, the value of  $(\Phi^*(u_s^{BP}))$  is strictly positive, or, equivalently,  $\max_{\sigma \in \Sigma^*} \text{cav} \Phi_{u_s^{BP}}(\sigma) > 0$ .*

## 7 Further Discussion

### 7.1 What if the sender knows the resolution of ambiguity in advance?

We have assumed that the sender commits to an ambiguous experiment  $(\sigma, \mu)$  without knowing in advance how this ambiguity will resolve. Here we consider the alternative assumption that the sender privately observes the resolution of the source of ambiguity,  $\alpha \in \mathcal{A}$ , before committing to  $(\sigma, \mu)$  (and this is common knowledge). In this case, the “type  $\alpha$ ” sender privately knows that the experiment generating the message will be  $\sigma_\theta$  for the  $\theta$  corresponding to the partition element from  $\mathcal{A}$  containing  $\alpha$ . The receiver remains uncertain. We now argue that, in any perfect Bayesian equilibria of this modified game, the sender does *not* benefit from ambiguous communication.<sup>20</sup> First observe that the payoff of any privately informed sender must be at least  $u_s^{BP}$  in any equilibrium. To see this, note that a sender can always offer an *unambiguous* optimal Bayesian persuasion experiment and, thereby, guarantee a payoff of  $u_s^{BP}$ .

One subtlety is that observing the sender’s choice of  $(\sigma, \mu)$  may lead the receiver to update their beliefs about the underlying state  $\alpha$  (and thus  $\theta$ ) because senders informed of different  $\alpha$ ’s might make different choices in equilibrium. Letting  $\beta^\sigma \in \Delta(\Theta)$  denote this possibly updated belief over  $\Theta$ , rather than reacting to the ambiguous experiment  $(\sigma, \mu)$ , the receiver views the ambiguous

<sup>20</sup>A (pure) strategy for the sender maps from  $\mathcal{A}$  to ambiguous experiments, whereas a (pure) strategy for the receiver is a mapping from ambiguous experiments and messages to actions. Upon observing the sender’s choice of an ambiguous experiment, the receiver may revise their beliefs about the sender’s private information, as in cheap-talk games. We assume that whenever the receiver faces an *unambiguous* experiment, the receiver’s action is optimal given that experiment and the realized message, regardless of whether it is on or off-path.

experiment as  $(\sigma, \beta^\sigma)$ . Thus, the receiver best responds to  $(\sigma, \beta^\sigma)$ , which is equivalent to best responding to an unambiguous experiment that is a particular convex combination of the  $\sigma_\theta$  for  $\theta \in \text{supp}(\beta^\sigma)$ . Denote this unambiguous experiment by  $\sum_\theta \zeta_\theta^{(\sigma, \beta^\sigma)} \sigma_\theta$ . For all senders informed of  $\hat{\alpha}$  corresponding to some  $\hat{\theta} \in \Theta$  committing to  $(\sigma, \mu)$ , the sender's expected payoff is

$$u_s \left( \sigma_{\hat{\theta}}, \text{br} \left( \sum_\theta \zeta_\theta^{(\sigma, \beta^\sigma)} \sigma_\theta \right) \right) \geq u_s^{BP}, \quad (9)$$

where the inequality follows from the preceding observation that  $u_s^{BP}$  is a lower bound on the informed sender's equilibrium payoff.

Since  $\text{supp}(\zeta^{(\sigma, \beta^\sigma)}) \subseteq \text{supp}(\beta^\sigma)$ , we have

$$u_s^{BP} \geq u_s \left( \sum_\theta \zeta_\theta^{(\sigma, \beta^\sigma)} \sigma_\theta, \text{br} \left( \sum_\theta \zeta_\theta^{(\sigma, \beta^\sigma)} \sigma_\theta \right) \right) = \sum_{\hat{\theta}} \zeta_{\hat{\theta}}^{(\sigma, \beta^\sigma)} u_s \left( \sigma_{\hat{\theta}}, \text{br} \left( \sum_\theta \zeta_\theta^{(\sigma, \beta^\sigma)} \sigma_\theta \right) \right) \geq u_s^{BP},$$

where the first inequality follows since  $u_s^{BP}$  is the sender's best expected payoff from any unambiguous experiment to which the receiver best responds, the equality from linearity of  $u_s$  in its first argument, and the second inequality from (9). Therefore, to benefit from ambiguous communication, the sender needs to use a source of ambiguity about which they do not have private information at the time of committing to the ambiguous experiment.

## 7.2 Robustness to Pre-Existing Ambiguity about Payoff-Relevant States

To isolate the role of ambiguous communication, we have assumed throughout that there is no pre-existing ambiguity about the payoff-relevant states. A full analysis of persuasion with ambiguous communication given arbitrary pre-existing ambiguity about payoff-relevant states is beyond the scope of this paper and is an interesting topic for future research. Nonetheless, many of the results and concepts emphasized in this paper remain relevant when there is pre-existing ambiguity. Here we establish (Theorem 7) the continued role of Pareto-ranked splittings of obedient experiments in generating benefits from ambiguous communication even under pre-existing ambiguity. Additionally, the conditions under which a sender benefits from ambiguous communication are robust to the introduction of a small amount of pre-existing ambiguity (Theorem 8).

In the context of our model, pre-existing ambiguity over the payoff-relevant state space  $\Omega$  can be represented by a common subjective distribution over priors  $\eta \in \Delta(\Delta(\Omega))$ . Denote by  $u_i(p, \sigma, \tau^*)$  the obvious extension of  $u_i(\sigma, \tau^*)$  to allow for different priors  $p \in \text{supp}(\eta)$ . Theorem 7 establishes the benefit of Pareto-ranked splittings of obedient experiments where both obedience and Pareto ranking are satisfied for each  $p \in \text{supp}(\eta)$  rather than simply the unique  $p$  as in the rest

of the paper.

**Theorem 7.** *Let  $\sigma^*$  be obedient under all  $p \in \text{supp}(\eta)$ . Suppose that  $\phi_r$  is strictly concave and  $(\bar{\sigma}, \underline{\sigma}, \lambda)$  is a Pareto-ranked splitting of  $\sigma^*$  satisfying  $u_s(p, \bar{\sigma}, \tau^*) > u_s(p, \underline{\sigma}, \tau^*)$  for all  $p \in \text{supp}(\eta)$  and  $\underline{\sigma}$  is obedient under all  $p \in \text{supp}(\eta)$ . Then the binary ambiguous experiment  $(\sigma, \mu)$  with  $\sigma = (\bar{\sigma}, \underline{\sigma})$  and  $\mu_{\bar{\sigma}}$  given by*

$$\mu_{\bar{\sigma}} = \min_{p \in \text{supp}(\eta)} \frac{\lambda \phi'_r(u_r(p, \underline{\sigma}, \tau^*))}{\lambda \phi'_r(u_r(p, \underline{\sigma}, \tau^*)) + (1 - \lambda) \phi'_r(u_r(p, \bar{\sigma}, \tau^*))} > \lambda, \quad (10)$$

*is obedient and an ambiguity neutral sender does strictly better using  $(\sigma, \mu)$  than using  $\sigma^*$ .*

**Theorem 8.** *Suppose an ambiguity neutral sender benefits from ambiguous communication when the prior is  $p \in \Delta(\Omega)$  and that the set of distributions over states and actions for which obedience is a best response for the receiver has a non-empty interior. Then, the sender continues to benefit from ambiguous communication under any small enough pre-existing ambiguity,  $\eta$ , around  $p$ . Specifically, there exists  $\delta > 0$  such that the sender benefits from ambiguous communication whenever  $\mathbb{E}[\eta] = p$  and  $\text{supp}(\eta) \subseteq \{q \in \Delta(\Omega) : \|q - p\| < \delta\}$ .*

For both theorems, similar results continue to hold as long as the sender is not too ambiguity averse.

### 7.3 Related Literature

In addition to the papers cited in the introduction, the following are also at the intersection of Bayesian persuasion (BP) and ambiguity. [Kosterina \(2022\)](#) studies BP when an MEU sender is ambiguous about the receiver's prior, while in [Dworczak and Pavan \(2022\)](#) an MEU sender (who also has a preference for selecting among MEU-optimal strategies those that perform best under a baseline conjecture) is ambiguous about the exogenous information a receiver might learn.<sup>21</sup> [Nikzad \(2021\)](#) studies BP when the receiver is MEU and has ambiguity about the prior over states. [Hedlund et al. \(2020\)](#) studies BP in problems with two states and two actions, when the receiver has  $\alpha$ -MEU preferences ([Ghirardato et al., 2004](#)) and considers an interval of priors and the sender has state-independent preferences over the action taken by the agent and is ambiguity neutral. In all of these papers, the sender is limited to standard, unambiguous experiments, and thus any ambiguity is exogenous. This stands in contrast to the endogeneity of ambiguity in our setting, where it becomes payoff-relevant only through the intentional communication choices of the sender.

[Kellner and Le Quement \(2018\)](#) study cheap talk communication assuming that the receiver has MEU preferences and the sender can choose to communicate ambiguously. The key difference

<sup>21</sup>[Dworczak and Pavan \(2022\)](#)'s model is not restricted to single-receiver persuasion settings.

between cheap talk and persuasion is the sender’s inability to commit to a communication strategy. Their receiver uses the same dynamically inconsistent update rule as in BLL. They find that both sender and receiver may benefit from the sender choosing to communicate ambiguously. [Kellner and Le Quement \(2017\)](#) studies cheap talk communication with purely exogenous ambiguity.

Papers studying mechanism design with ambiguity include [Bose and Renou \(2014\)](#), [Wolitzky \(2016\)](#), [Di Tillio et al. \(2017\)](#), [Guo \(2019\)](#) and [Tang and Zhang \(2021\)](#), among others. All but [Wolitzky \(2016\)](#) consider ambiguity that arises intentionally through design of the mechanism. [Dütting et al. \(2024\)](#) allow a principal to offer ambiguous contracts to an MEU agent and show how the principal may benefit and that optimal contracts have a simple form. All gains from ambiguous contracting disappear in their model if the agent can hedge against ambiguity by randomizing.

We conclude by returning to the discussion of BLL begun in the introduction. Broadly speaking, the gains we identify work through key properties, such as Pareto-ranking of (at least some) experiments in the collection  $\sigma$  chosen by the sender. Such properties contrast sharply with the “synonym” constructions emphasized in BLL that lead to collections in which each experiment yields the same expected payoff to the sender. BLL and our approach also lead to different outcomes. For example, our Corollary 1 shows that ambiguous communication never benefits the sender when the receiver has only two actions. In contrast, BLL find gains from ambiguous communication in such cases, including their main example. Conversely, there are examples in which there is no benefit for the sender according to BLL’s approach (even when extended to include sender preferences less extremely ambiguity averse than  $U_s^{MEU}$ ), in which the sender benefits from ambiguous communication in our approach.

As previously mentioned, the benefits from ambiguous communication in BLL involve in an essential way the receiver behaving suboptimally with respect to their ex-ante preferences as specified by BLL. [Pahlke \(2023\)](#) uses constructions based on rectangularity ([Epstein and Schneider, 2003](#)) to construct alternative ex-ante MEU preferences (different from BLL and from  $U_r^{MEU}$ ) that are consistent with the receiver’s interim behavior in BLL. When there are gains in BLL from ambiguous communication, some of the measures in Pahlke’s construction must reflect correlation between  $\Omega$  and which experiment from the ambiguous collection generates the messages. This is the manifestation of the dynamic inconsistency in BLL within the dynamically consistent reformulation of [Pahlke \(2023\)](#).<sup>22</sup>

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<sup>22</sup>For discussion and approaches to dynamic consistency issues in decision-making under ambiguity more broadly see e.g., [Hanany and Klibanoff \(2007, 2009\)](#) and [Siniscalchi \(2011\)](#). See [Hanany et al. \(2020\)](#) on dynamic games of incomplete information with ambiguity averse players.



# Appendices

## A Proofs of Main Results<sup>23</sup>

### A.1 Proofs for Section 2

*Proof of Proposition 1.* Fix  $(\sigma, \mu)$  and  $\tau \in BR(\sigma, \mu)$ . Construct a canonical ambiguous experiment  $(\sigma^*, \mu)$  as follows: For each  $\theta$ , define  $\sigma_\theta^*(a|\omega) = \sum_m \tau(a|m)\sigma_\theta(m|\omega)$ , for all  $(a, \omega)$ , and let  $\sigma^* = (\sigma_\theta^*)_\theta$ . For  $i \in \{s, r\}$ ,  $u_i(\sigma_\theta, \tau) = \sum_{\omega, a} p(\omega)u_i(a, \omega) \sum_m \tau(a|m)\sigma_\theta(m|\omega) = u_i(\sigma_\theta^*, \tau^*)$ . Therefore  $U_i(\sigma^*, \mu, \tau^*) = U_i(\sigma, \mu, \tau)$ , for  $i \in \{s, r\}$ .

Suppose  $(\sigma^*, \mu)$  is not obedient. Then there exists  $\delta : A \rightarrow \Delta(A)$  such that  $U_r(\sigma^*, \mu, \tau^*) < U_r(\sigma^*, \mu, \delta)$ . Then, define  $\tau'(a|m) = \sum_{a'} \delta(a|a')\tau(a'|m)$  for all  $(a, m)$ , and note that for all  $\theta$ ,

$$\begin{aligned} u_r(\sigma_\theta, \tau') &= \sum_{\omega, a} p(\omega)u_r(a, \omega) \sum_{a'} \delta(a|a') \sum_m \tau(a'|m)\sigma_\theta(m|\omega) \\ &= \sum_{\omega, a, a'} p(\omega)u_r(a, \omega)\delta(a|a')\sigma_\theta^*(a'|\omega) = u_r(\sigma_\theta^*, \delta). \end{aligned}$$

Thus,  $U_r(\sigma, \mu, \tau') = U_r(\sigma^*, \mu, \delta) > U_r(\sigma^*, \mu, \tau^*) = U_r(\sigma, \mu, \tau)$ , so  $\tau \notin BR(\sigma, \mu)$ .  $\square$

*Proof of Lemma 1. IF.* Let  $(\sigma, \mu)$  be an ambiguous experiment. We show that if  $\tau^* \in br(\sigma^*)$ , where  $\sigma^* = \sum_\theta em_\theta^{(\sigma, \mu)}\sigma_\theta$ , then  $\tau^* \in BR(\sigma, \mu)$ . Since  $\tau^* \in br(\sigma^*)$ , for all  $b, a \in A$ ,

$$\sum_\omega p(\omega)\sigma^*(a|\omega)u_r(a, \omega) \geq \sum_\omega p(\omega)\sigma^*(a|\omega)u_r(b, \omega). \quad (\text{A.1})$$

For any strategy  $\tau$ , there exists  $\delta \in \mathbb{R}^{|A| \times |A|}$  such that  $\tau = \tau^* + \delta$ , where  $\delta$  satisfies the following:

$$\forall a \neq b \in A, \quad \delta(b|a) \geq 0, \quad \delta(a|a) \leq 0, \quad \text{and} \quad \sum_{\tilde{a} \in A} \delta(\tilde{a}|a) = 0 \quad (\text{A.2})$$

The concavity of  $\phi_r$  implies that  $\phi_r(U_r(\sigma, \mu, \tau))$  is concave in  $\tau$ . Hence, for all  $\delta$ ,

$$\phi_r(U_r(\sigma, \mu, \tau^* + \delta)) \leq \phi_r(U_r(\sigma, \mu, \tau^*)) + \sum_{b, a \in A} \left. \frac{\partial \phi_r(U_r(\sigma, \mu, \tau))}{\partial \tau(b|a)} \right|_{\tau=\tau^*} \delta(b|a).$$

<sup>23</sup>Proofs for auxiliary results may be found in Section SA.4 of the Online Appendix (Cheng et al., 2025).

For  $\tau^*$  to be a solution to the receiver's program, it suffices that for all  $\delta$  satisfying (A.2),

$$\sum_{b,a \in A} \frac{\partial \phi_r(U_r(\sigma, \mu, \tau))}{\partial \tau(b|a)} \Big|_{\tau=\tau^*} \delta(b|a) \leq 0. \quad (\text{A.3})$$

To show (A.3) holds, note that  $\frac{\partial \phi_r(U_r(\sigma, \mu, \tau))}{\partial \tau(b|a)} \Big|_{\tau=\tau^*} = \sum_{\tilde{\theta}} \mu_{\tilde{\theta}} \phi'_r(u_r(\sigma_{\tilde{\theta}}, \tau^*)) \sum_{\omega} p(\omega) \sigma^*(a|\omega) u_r(b, \omega)$ . Then by (A.2),  $-\delta(a|a) = \sum_{b \neq a} \delta(b|a)$ , and we have:

$$\begin{aligned} \frac{\sum_{b,a \in A} \frac{\partial \phi_r(U_r(\sigma, \mu, \tau))}{\partial \tau(b|a)} \Big|_{\tau=\tau^*} \delta(b|a)}{\sum_{\tilde{\theta}} \mu_{\tilde{\theta}} \phi'_r(u_r(\sigma_{\tilde{\theta}}, \tau^*))} &= \sum_{b,a \in A} \sum_{\omega} p(\omega) \sigma^*(a|\omega) u_r(b, \omega) \delta(b|a) \\ &= \sum_{a \in A} \sum_{b \neq a} \delta(b|a) \left( \sum_{\omega} p(\omega) \sigma^*(a|\omega) u_r(b, \omega) - \sum_{\omega} p(\omega) \sigma^*(a|\omega) u_r(a, \omega) \right) \leq 0, \end{aligned}$$

where the last inequality follows from  $\delta(b|a) \geq 0$  for all  $b \neq a$  and (A.1). This implies (A.3) as  $\sum_{\tilde{\theta}} \mu_{\tilde{\theta}} \phi'_r(u_r(\sigma_{\tilde{\theta}}, \tau^*)) > 0$ . Therefore, we have shown that  $\tau^* \in BR(\sigma, \mu)$ .

**ONLY IF.** The proof is nearly identical to the if direction and left to the reader.  $\square$

## A.2 Proofs for Sections 3 and 6

The proofs in Section 3 make use of our concavification-like characterization of optimal persuasion (Theorem 6) from Section 6. We therefore establish Theorem 6 first.

*Proof of Theorem 6.* First, we show that there is a unique  $u$  that solves  $\Phi^*(u) = 0$ .

**Lemma A.1.**  $\Phi^*(u)$  satisfies single-crossing, i.e., for any  $u > u'$ , if  $\Phi^*(u) \geq 0$  then  $\Phi^*(u') > 0$ . Thus, there exists a unique  $u \in \mathbb{R}$  such that  $\Phi^*(u) = 0$ .

*Proof of Lemma A.1.*  $\Phi^*(u) \geq 0$  and  $\phi_s, \phi_r$  strictly increasing implies the existence of  $(\lambda_{\theta}, \sigma_{\theta})_{\theta \in \Theta}$  such that  $\sum_{\theta \in \Theta} \lambda_{\theta} \frac{\phi_s(u_s(\sigma_{\theta}, \tau^*))}{\phi'_r(u_r(\sigma_{\theta}, \tau^*))} \geq \sum_{\theta \in \Theta} \lambda_{\theta} \frac{\phi_s(u)}{\phi'_r(u_r(\sigma_{\theta}, \tau^*))} > \sum_{\theta \in \Theta} \lambda_{\theta} \frac{\phi_s(u')}{\phi'_r(u_r(\sigma_{\theta}, \tau^*))}$ , which implies  $\Phi^*(u') > 0$  for  $u' < u$ . Therefore, there is at most one solution to  $\Phi^*(u) = 0$ . Since  $\phi_s$  is continuous,  $\Phi_u$  is continuous in  $(\sigma, u)$  and, thus, by Berge's Maximum Theorem,  $\Phi^*(u)$  is continuous. Since  $\Phi^*(u) < 0$  for  $u > \max\{u_s(a, \omega)\}$  and  $\Phi^*(u) > 0$  for  $u < \min\{u_s(a, \omega)\}$ , there exists  $u$  such that  $\Phi^*(u) = 0$ .  $\square$

By Proposition 1, we can rewrite  $(\mathcal{P})$  as

$$(\mathcal{P}) = \begin{cases} \max_{(\sigma, \mu)} U_s(\sigma, \mu, \tau^*), \\ \text{subject to } \tau^* \in BR(\sigma, \mu). \end{cases}$$

We then show the conclusion using the following lemma.

**Lemma A.2.** *For each  $u \in \mathbb{R}$ ,  $(\mathcal{P}) > u$  if, and only if,  $\Phi^*(u) > 0$ .*

*Proof of Lemma A.2. IF.* Suppose there exists a solution  $(\lambda_\theta, \sigma_\theta)_{\theta \in \Theta}$  such that  $\Phi^*(u) > 0$ . Let  $\sigma = (\sigma_\theta)_{\theta \in \Theta}$  and  $\mu$  be defined by  $\mu_\theta := \frac{\lambda_\theta / \phi'_r(u_r(\sigma_\theta, \tau^*))}{\sum_j \lambda_j / \phi'_r(u_r(\sigma_j, \tau^*))}$  for each  $\theta \in \Theta$ . By construction,  $(\sigma, \mu)$  satisfies  $em_\theta^{(\sigma, \mu)} = \lambda_\theta$  and  $\sum_{\theta \in \Theta} \lambda_\theta \sigma_\theta \in \Sigma^*$ . Lemma 1 implies  $\tau^* \in BR(\sigma, \mu)$ . Thus,  $(\mathcal{P}) > u$  since  $U_s(\sigma, \mu, \tau^*) > u$  as

$$\phi_s(U_s(\sigma, \mu, \tau^*)) - \phi_s(u) = \frac{1}{\sum_{j \in I} \lambda_j / \phi'_r(u_r(\sigma_j, \tau^*))} \sum_{\theta \in \Theta} \lambda_\theta \frac{\phi_s(u_s(\sigma_\theta, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\sigma_\theta, \tau^*))} > 0,$$

**ONLY IF.**  $(\mathcal{P}) > u$  implies there exists an obedient  $(\sigma, \mu)$  such that  $U_s(\sigma, \mu, \tau^*) > u$ . Let  $\lambda_\theta = em_\theta^{(\sigma, \mu)}$ . Lemma 1 implies that  $\sum_\theta \lambda_\theta \sigma_\theta \in \Sigma^*$ . Thus,  $\Phi^*(u) > 0$  as  $\sum_\theta \lambda_\theta \Phi_u(\sigma_\theta) = (\phi_s(U_s(\sigma, \mu, \tau^*)) - \phi_s(u)) \sum_\theta \lambda_\theta / \phi'_r(u_r(\sigma_\theta, \tau^*)) > 0$  since  $U_s(\sigma, \mu, \tau^*) > u$ .  $\square$

We now complete the proof by showing that  $(\mathcal{P}) = u$  if, and only if,  $\Phi^*(u) = 0$ . Suppose  $(\mathcal{P}) = u$ . Then for all  $u' < u$ ,  $(\mathcal{P}) > u'$  and thus  $\Phi^*(u') > 0$  by Lemma A.2. By Lemma A.1, there exists a unique  $\hat{u}$  such that  $\Phi^*(\hat{u}) = 0$ . If  $\hat{u} > u$ , then  $\Phi^*(u) > 0$  by Lemma A.1, and thus  $(\mathcal{P}) > u$  by Lemma A.2, a contradiction. Thus,  $\Phi^*(u) = 0$ .

For the other direction, suppose  $\Phi^*(u) = 0$ . If  $(\mathcal{P}) > u$ , then Lemma A.2 implies  $\Phi^*(u) > 0$ , a contradiction. If  $(\mathcal{P}) < u$ , then there exists  $u' < u$  such that  $(\mathcal{P}) = u'$ . Then by the previous direction,  $\Phi^*(u') = 0$ , contradicting Lemma A.1. Thus,  $(\mathcal{P}) = u$ .

Recall that  $\Sigma$  is a convex subset of  $\mathbb{R}^{|\Omega| \times (|A|-1)}$ . As a result, the graph of  $\Phi_u$  is a subset of  $\mathbb{R}^{(|\Omega| \times (|A|-1) + 1)}$ . Suppose that  $(\mathcal{P}) = u$ . Then  $\Phi^*(u) = 0$ . Let  $(\lambda_\theta, \sigma_\theta)_{\theta \in \Theta}$  be such that  $\sum_{\theta \in \Theta} \lambda_\theta \Phi_u(\sigma_\theta) = 0$  and  $\sum_{\theta \in \Theta} \lambda_\theta \sigma_\theta = \sigma^* \in \Sigma^*$ . Thus,  $(\sigma^*, 0)$  is on the boundary of, and thus an element of a supporting hyperplane of, the convex hull of the graph of  $\Phi_u$ . The intersection of this hyperplane and the set forms a face and thus has dimension at most  $|\Omega| \times (|A| - 1)$ . Extreme points of the face are also extreme points of the convex hull of the graph of  $\Phi_u$ . Any such extreme point has the form  $(\sigma, \Phi_u(\sigma))$  for some  $\sigma \in \Sigma$ . By Caratheodory's theorem applied to the face,  $(\sigma^*, 0)$  can be written as a convex combination of at most  $(|\Omega| \times (|A| - 1) + 1)$  such extreme points. Denote the coefficients in the convex combination and the experiments corresponding to these extreme points by  $(\hat{\lambda}_\theta, \hat{\sigma}_\theta)_{\theta \in \hat{\Theta}}$  with  $|\hat{\Theta}| \leq (|\Omega| \times (|A| - 1) + 1)$ . Thus, there exists a solution to  $(\mathcal{P})$  with  $\sigma^* = (\hat{\sigma}_\theta)_{\theta \in \hat{\Theta}}$  and  $\mu^*$  with  $\text{supp}(\mu^*) \subseteq \hat{\Theta}$  so that  $|\text{supp}(\mu^*)| \leq (|\Omega| \times (|A| - 1) + 1)$ .  $\square$

*Proof of Theorem 1.* Suppose  $(\sigma, \mu)$  is obedient and  $\phi_r$  is strictly concave. Let  $\hat{u} := U_s(\sigma, \mu, \tau^*)$  and  $\lambda_\theta = em_\theta^{(\sigma, \mu)}$  for all  $\theta$ . Then  $\sum_i \lambda_\theta \frac{\phi_s(u_s(\sigma_\theta, \tau^*)) - \phi_s(\hat{u})}{\phi'_r(u_r(\sigma_\theta, \tau^*))} = 0$ . If there exists  $(\hat{\lambda}_\theta, \hat{\sigma}_\theta)$  such that  $\sum_\theta \hat{\lambda}_\theta \frac{\phi_s(u_s(\hat{\sigma}_\theta, \tau^*)) - \phi_s(\hat{u})}{\phi'_r(u_r(\hat{\sigma}_\theta, \tau^*))} > 0$ , then the ambiguous experiment  $(\hat{\sigma}, \hat{\mu})$  with  $em_\theta^{(\hat{\sigma}, \hat{\mu})} := \hat{\lambda}_\theta$  will

strictly improve upon  $(\sigma, \mu)$ . The existence of such  $(\hat{\lambda}_\theta, \hat{\sigma}_\theta)$  under the conditions in (i) and (ii) of Theorem 1 follows from (i) and (ii) of the following Lemma (and its swapped version), respectively. To state the lemma, we need the following definition.

**Definition A.1.** For any  $u \in \mathbb{R}$ , let  $\Sigma_+(u) = \{\sigma \in \Sigma : u_s(\sigma, \tau^*) > u\}$  and  $\Sigma_-(u) = \{\sigma \in \Sigma : u_s(\sigma, \tau^*) \leq u\}$ .

**Lemma A.3.** Let  $(\lambda_\theta, \sigma_\theta)_{\theta \in \Theta}$  be a solution to  $(\Phi^*(u))$ . Then the following hold:

- (i) For all  $\theta$  and  $\theta'$  such that  $(\sigma_\theta, \sigma_{\theta'}) \in \Sigma_+(u) \times \Sigma_-(u)$ , if  $\phi'_r(u_r(\sigma_\theta, \tau^*)) \neq \phi'_r(u_r(\sigma_{\theta'}, \tau^*))$ , then they are Pareto-ranked,
- (ii) If  $\phi_s$  is affine, then for all  $\sigma_\theta$ , there does not exist a Pareto-ranked splitting,  $(\bar{\sigma}, \underline{\sigma}, \lambda)$  with  $(\bar{\sigma}, \underline{\sigma}) \in \Sigma_+(u) \times \Sigma_-(u)$  and  $\phi'_r(u_r(\bar{\sigma}, \tau^*)) < \phi'_r(u_r(\underline{\sigma}, \tau^*))$ .

A swapped version of Lemma A.3 with  $\Sigma_+(u)$  and  $\Sigma_-(u)$  defined by swapping the strict and weak inequalities in Definition A.1 also holds, and the proof is identical.

*Proof of Lemma A.3.* Fix  $u \in \mathbb{R}$ . Let  $(\sigma_\theta, \lambda_\theta)_{\theta \in \Theta}$  be feasible for the maximization problem  $\Phi^*(u)$ .

To prove (i), suppose that there exists a pair  $(\sigma_\theta, \sigma_{\theta'})$  with  $\lambda_\theta > 0$  and  $\lambda_{\theta'} > 0$  and such that there exists a  $\lambda \in (0, 1)$  for which,  $\Phi_u(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}) > \lambda\Phi_u(\sigma_\theta) + (1-\lambda)\Phi_u(\sigma_{\theta'})$ . Then,  $(\sigma_\theta, \lambda_\theta)_{\theta \in \Theta}$  cannot be a solution to the maximization problem  $\Phi^*(u)$ . This can be seen from the following construction of a strict improvement: If  $\frac{\lambda_\theta}{\lambda} \leq \frac{\lambda_{\theta'}}{1-\lambda}$ , then replacing  $\sigma_\theta$  by the merged experiment  $\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}$  and replacing  $\lambda_\theta$  by  $\hat{\lambda}_\theta = \frac{\lambda_\theta}{\lambda}$  and  $\lambda_{\theta'}$  by  $\hat{\lambda}_{\theta'} = \lambda_{\theta'} - (1-\lambda)\frac{\lambda_\theta}{\lambda}$  yield such an improvement. If instead  $\frac{\lambda_\theta}{\lambda} > \frac{\lambda_{\theta'}}{1-\lambda}$ , then replacing  $\sigma_{\theta'}$  by the merged experiment  $\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}$  and replacing  $\lambda_{\theta'}$  by  $\hat{\lambda}_{\theta'} = \frac{\lambda_{\theta'}}{1-\lambda}$  and  $\lambda_\theta$  by  $\hat{\lambda}_\theta = \lambda_\theta - \lambda\frac{\lambda_{\theta'}}{1-\lambda}$  is such an improvement.

Towards a contradiction, suppose in the solution there exists  $(\sigma_\theta, \sigma_{\theta'}) \in \Sigma_+(u) \times \Sigma_-(u)$  with  $\phi'_r(u_r(\sigma_\theta, \tau^*)) \neq \phi'_r(u_r(\sigma_{\theta'}, \tau^*))$  and they are not Pareto-ranked, i.e.,  $u_s(\sigma_\theta, \tau^*) > u \geq u_s(\sigma_{\theta'}, \tau^*)$ , and  $u_r(\sigma_\theta, \tau^*) < u_r(\sigma_{\theta'}, \tau^*)$ . Then there exists  $\lambda \in (0, 1)$  such that  $\phi'_r(u_r(\sigma_\theta, \tau^*)) > \phi'_r(u_r(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}, \tau^*))$  (by differentiability of  $\phi_r$ ), and  $\Phi_u(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}) > \lambda\Phi_u(\sigma_\theta) + (1-\lambda)\Phi_u(\sigma_{\theta'})$ . To see the last point, notice that

$$\begin{aligned}
\Phi_u(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}) &= \frac{\phi_s(u_s(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}, \tau^*))} \\
&\geq \frac{\lambda[\phi_s(u_s(\sigma_\theta, \tau^*)) - \phi_s(u)] + (1-\lambda)[\phi_s(u_s(\sigma_{\theta'}, \tau^*)) - \phi_s(u)]}{\phi'_r(u_r(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}, \tau^*))} \\
&= \frac{\lambda\phi'_r(u_r(\sigma_\theta, \tau^*))}{\phi'_r(u_r(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}, \tau^*))} \Phi_u(\sigma_\theta) + \frac{(1-\lambda)\phi'_r(u_r(\sigma_{\theta'}, \tau^*))}{\phi'_r(u_r(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}, \tau^*))} \Phi_u(\sigma_{\theta'}) \\
&> \lambda\Phi_u(\sigma_\theta) + (1-\lambda)\Phi_u(\sigma_{\theta'}),
\end{aligned}$$

where the first inequality follows from concavity of  $\phi_s$ , and the second inequality follows from concavity of  $\phi_r$  and  $\Phi_u(\sigma_\theta) > 0$  and  $\Phi_u(\sigma_{\theta'}) \leq 0$ , and  $\phi'_r(u_r(\sigma_\theta, \tau^*)) > \phi'_r(u_r(\lambda\sigma_\theta + (1 - \lambda)\sigma_{\theta'}, \tau^*))$ .

To prove (ii), if there exist  $\sigma_\theta$  satisfying  $\lambda_\theta > 0$  and two experiments  $\sigma$  and  $\sigma'$  such that  $\sigma_\theta = \lambda\sigma + (1 - \lambda)\sigma'$  for some  $\lambda \in (0, 1)$  and  $\Phi_u(\lambda\sigma + (1 - \lambda)\sigma') < \lambda\Phi_u(\sigma) + (1 - \lambda)\Phi_u(\sigma')$ , then  $(\sigma_\theta, \lambda_\theta)_{\theta \in \Theta}$  cannot be a solution to  $(\Phi^*(u))$ . This follows by noting that splitting  $\sigma_\theta$  into  $\sigma$  with probability  $\lambda\lambda_\theta$  and  $\sigma'$  with probability  $(1 - \lambda)\lambda_\theta$  induces a strict improvement.

Towards a contradiction, suppose there exists such a Pareto-ranked splitting  $(\bar{\sigma}, \underline{\sigma}, \lambda)$  with  $\lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma} \in \Sigma_+(u)$  (the other case is symmetric), then we have

$$\begin{aligned}
& \lambda\Phi_u(\bar{\sigma}) + (1 - \lambda)\Phi_u(\underline{\sigma}) - \Phi_u(\lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma}) \\
&= \lambda \left( \frac{\phi_s(u_s(\bar{\sigma}, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\bar{\sigma}, \tau^*))} - \frac{\phi_s(u_s(\lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma}, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma}, \tau^*))} \right) \\
& \quad + (1 - \lambda) \left( \frac{\phi_s(u_s(\underline{\sigma}, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\underline{\sigma}, \tau^*))} - \frac{\phi_s(u_s(\lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma}, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma}, \tau^*))} \right) \\
&\geq \frac{\lambda}{\phi'_r(u_r(\bar{\sigma}, \tau^*))} (\phi_s(u_s(\bar{\sigma}, \tau^*)) - \phi_s(u_s(\lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma}, \tau^*))) \\
& \quad + \frac{1 - \lambda}{\phi'_r(u_r(\lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma}, \tau^*))} (\phi_s(u_s(\underline{\sigma}, \tau^*)) - \phi_s(u_s(\lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma}, \tau^*))) \\
&\geq \frac{\lambda}{\phi'_r(u_r(\bar{\sigma}, \tau^*))} \phi'_s(u_s(\bar{\sigma}, \tau^*)) (u_s(\bar{\sigma}, \tau^*) - \lambda u_s(\bar{\sigma}, \tau^*) - (1 - \lambda)u_s(\underline{\sigma}, \tau^*)) \\
& \quad + \frac{1 - \lambda}{\phi'_r(u_r(\lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma}, \tau^*))} \phi'_s(u_s(\underline{\sigma}, \tau^*)) (u_s(\underline{\sigma}, \tau^*) - \lambda u_s(\bar{\sigma}, \tau^*) - (1 - \lambda)u_s(\underline{\sigma}, \tau^*)) \\
&= \lambda(1 - \lambda)(u_s(\bar{\sigma}, \tau^*) - u_s(\underline{\sigma}, \tau^*)) \left( \frac{\phi'_s(u_s(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\bar{\sigma}, \tau^*))} - \frac{\phi'_s(u_s(\underline{\sigma}, \tau^*))}{\phi'_r(u_r(\lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma}, \tau^*))} \right) > 0,
\end{aligned}$$

where the first inequality follows from  $u_s(\lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma}) > u \geq u_s(\underline{\sigma}, \tau^*)$ ,  $\phi'_r(u_r(\bar{\sigma}, \tau^*)) \leq \phi'_r(u_r(\lambda\bar{\sigma} + (1 - \lambda)\underline{\sigma}, \tau^*)) \leq \phi'_r(u_r(\underline{\sigma}, \tau^*))$ , the second inequality follows from concavity of  $\phi_s$ , and the last inequality follows from linearity of  $\phi_s$ .  $\square$

**Remark 4.** From the proof of Lemma A.3 (ii), observe that linearity of  $\phi_s$  can be relaxed to  $\frac{\phi'_s(u_s(\bar{\sigma}, \tau^*))}{\phi'_s(u_s(\underline{\sigma}, \tau^*))} > \frac{\phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\sigma_\theta, \tau^*))}$ , if  $\sigma_\theta \in \Sigma_+(u)$ , and  $\frac{\phi'_s(u_s(\bar{\sigma}, \tau^*))}{\phi'_s(u_s(\underline{\sigma}, \tau^*))} > \frac{\phi'_r(u_r(\sigma_\theta, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*))}$ , if  $\sigma_\theta \in \Sigma_-(u)$ .  $\square$

*Proof of Proposition 2.* Proposition 2 is implied by the following two auxiliary theorems that extend Lemma A.3. The proofs of these theorems are similar to that of Lemma A.3 and may be found in the Online Appendix (Cheng et al., 2025).

**Theorem A.1.** Let  $(\lambda_\theta, \sigma_\theta)_{\theta \in \Theta}$  be a solution to  $(\Phi^*(u))$ . Then:

- (i) If  $1/\phi'_r$  is concave, then for all  $\theta, \theta'$  such that  $(\sigma_\theta, \sigma_{\theta'}) \in \Sigma_+(u) \times \Sigma_+(u)$ , if  $u_s(\sigma_\theta, \tau^*) \neq u_s(\sigma_{\theta'}, \tau^*)$  and  $\phi'_r(u_r(\sigma_\theta, \tau^*)) \neq \phi'_r(u_r(\sigma_{\theta'}, \tau^*))$ , they are Pareto-ranked.
- (ii) If  $1/\phi'_r$  is convex, then for all  $\theta, \theta'$  such that  $(\sigma_\theta, \sigma_{\theta'}) \in \Sigma_-(u) \times \Sigma_-(u)$ , if  $u_s(\sigma_\theta, \tau^*) \neq u_s(\sigma_{\theta'}, \tau^*)$  and  $\phi'_r(u_r(\sigma_\theta, \tau^*)) \neq \phi'_r(u_r(\sigma_{\theta'}, \tau^*))$ , they are Pareto-ranked.

**Theorem A.2.** Let  $(\lambda_\theta, \sigma_\theta)_{\theta \in \Theta}$  be a solution to  $(\Phi^*(u))$ . Then:

- (i) If  $1/\phi'_r$  is concave, then for all  $\sigma_\theta$ , there does not exist a Pareto-ranked splitting,  $(\bar{\sigma}, \underline{\sigma}, \lambda)$  with  $(\bar{\sigma}, \underline{\sigma}) \in \Sigma_-(u) \times \Sigma_-(u)$ ,  $\phi'_r(u_r(\bar{\sigma}, \tau^*)) < \phi'_r(u_r(\underline{\sigma}, \tau^*))$ , and  $\frac{\phi'_s(u_s(\bar{\sigma}, \tau^*))}{\phi'_s(u_s(\underline{\sigma}, \tau^*))} > \frac{\phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*))}$ .
- (ii) If  $1/\phi'_r$  is convex, then for all  $\sigma_\theta$ , there does not exist a Pareto-ranked splitting,  $(\bar{\sigma}, \underline{\sigma}, \lambda)$  with  $(\bar{\sigma}, \underline{\sigma}) \in \Sigma_+(u) \times \Sigma_+(u)$ ,  $\phi'_r(u_r(\bar{\sigma}, \tau^*)) < \phi'_r(u_r(\underline{\sigma}, \tau^*))$ , and  $\frac{\phi'_s(u_s(\bar{\sigma}, \tau^*))}{\phi'_s(u_s(\underline{\sigma}, \tau^*))} > \frac{\phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*))}$ .

To prove Proposition 2, observe that when both  $\phi_s$  and  $1/\phi'_r$  are linear all the conditions in Theorem A.1 and Theorem A.2 (and their swapped versions) are satisfied.  $\square$

### A.3 Proofs for Section 4

*Proof of Theorem 2.* Suppose that  $U_s(\sigma, \mu, \tau^*) > u_s^{BP}$ . Observe that  $\Sigma_+(u_s^{BP})$  and  $\Sigma_-(u_s^{BP})$  have non-empty intersections with the support of  $\mu$  since, if  $\Sigma_+(u_s^{BP})$  did not, then  $(\sigma, \mu)$  could not benefit the sender, while if  $\Sigma_-(u_s^{BP})$  did not, then  $\tau^* \notin BR(\sigma, \mu)$ , contradicting obedience. Thus, there exists  $\theta, \theta' \in \text{supp}(\mu)$  such that  $\sigma_\theta \in \Sigma_+(u_s^{BP})$  and  $\sigma_{\theta'} \in \Sigma_-(u_s^{BP})$ . Define  $\sigma^* := \sum_{\hat{\theta}} em_{\hat{\theta}}^{(\sigma, \mu)} \sigma_{\hat{\theta}}$ . Then,

$$\sigma^* = \sum_{\hat{\theta}: \sigma_{\hat{\theta}} \in \Sigma_+(u_s^{BP})} \frac{\mu_{\hat{\theta}} \phi'_r(u_r(\sigma_{\hat{\theta}}, \tau^*))}{\sum_{\tilde{\theta}} \mu_{\tilde{\theta}} \phi'_r(u_r(\sigma_{\tilde{\theta}}, \tau^*))} \sigma_{\hat{\theta}} + \sum_{\hat{\theta}: \sigma_{\hat{\theta}} \in \Sigma_-(u_s^{BP})} \frac{\mu_{\hat{\theta}} \phi'_r(u_r(\sigma_{\hat{\theta}}, \tau^*))}{\sum_{\tilde{\theta}} \mu_{\tilde{\theta}} \phi'_r(u_r(\sigma_{\tilde{\theta}}, \tau^*))} \sigma_{\hat{\theta}}.$$

By Lemma 1, we have  $\sigma^*$  is obedient. By definition of  $u_s^{BP}$ , this implies

$$\phi_s(u_s(\sigma^*, \tau^*)) \leq \phi_s(u_s^{BP}). \quad (\text{A.4})$$

Suppose that for all  $\theta, \theta' \in \text{supp}(\mu)$  such that  $\sigma_\theta \in \Sigma_+(u_s^{BP})$  and  $\sigma_{\theta'} \in \Sigma_-(u_s^{BP})$ ,  $\sigma_\theta$  and  $\sigma_{\theta'}$  are not Pareto-ranked. This is equivalent to

$$u_r(\sigma_\theta, \tau^*) \leq u_r(\sigma_{\theta'}, \tau^*). \quad (\text{A.5})$$

The remainder of the proof shows that this contradicts (A.4). From (A.5) and concavity of  $\phi_r$ ,  $\phi'_r(u_r(\sigma_\theta, \tau^*)) \geq \phi'_r(u_r(\sigma_{\theta'}, \tau^*))$ . Observe that,

$$\begin{aligned}
& (\phi_s(u_s(\sigma^*, \tau^*)) - \phi_s(u_s^{BP})) \sum_{\hat{\theta}} \mu_{\hat{\theta}} \phi'_r(u_r(\sigma_{\hat{\theta}}, \tau^*)) \\
& \geq \left( \sum_{\hat{\theta}: \sigma_{\hat{\theta}} \in \Sigma_+(u_s^{BP})} \frac{\mu_{\hat{\theta}} \phi'_r(u_r(\sigma_{\hat{\theta}}, \tau^*))}{\sum_{\tilde{\theta}} \mu_{\tilde{\theta}} \phi'_r(u_r(\sigma_{\tilde{\theta}}, \tau^*))} (\phi_s(u_s(\sigma_{\hat{\theta}})) - \phi_s(u_s^{BP})) \right. \\
& \quad \left. + \sum_{\hat{\theta}: \sigma_{\hat{\theta}} \in \Sigma_-(u_s^{BP})} \frac{\mu_{\hat{\theta}} \phi'_r(u_r(\sigma_{\hat{\theta}}, \tau^*))}{\sum_{\tilde{\theta}} \mu_{\tilde{\theta}} \phi'_r(u_r(\sigma_{\tilde{\theta}}, \tau^*))} (\phi_s(u_s(\sigma_{\hat{\theta}})) - \phi_s(u_s^{BP})) \right) \sum_{\tilde{\theta}} \mu_{\tilde{\theta}} \phi'_r(u_r(\sigma_{\tilde{\theta}}, \tau^*)) \\
& = \sum_{\hat{\theta}: \sigma_{\hat{\theta}} \in \Sigma_+(u_s^{BP})} \mu_{\hat{\theta}} \phi'_r(u_r(\sigma_{\hat{\theta}}, \tau^*)) (\phi_s(u_s(\sigma_{\hat{\theta}}, \tau^*)) - \phi_s(u_s^{BP})) \\
& \quad + \sum_{\hat{\theta}: \sigma_{\hat{\theta}} \in \Sigma_-(u_s^{BP})} \mu_{\hat{\theta}} \phi'_r(u_r(\sigma_{\hat{\theta}}, \tau^*)) (\phi_s(u_s(\sigma_{\hat{\theta}}, \tau^*)) - \phi_s(u_s^{BP})) \\
& \geq \phi'_r \left( \max_{\hat{\theta} \in \Sigma_+(u_s^{BP})} u_r(\sigma_{\hat{\theta}}, \tau^*) \right) \sum_{\hat{\theta}: \sigma_{\hat{\theta}} \in \Sigma_+(u_s^{BP})} \mu_{\hat{\theta}} (\phi_s(u_s(\sigma_{\hat{\theta}}, \tau^*)) - \phi_s(u_s^{BP})) \\
& \quad + \phi'_r \left( \min_{\hat{\theta} \in \Sigma_-(u_s^{BP})} u_r(\sigma_{\hat{\theta}}, \tau^*) \right) \sum_{\hat{\theta}: \sigma_{\hat{\theta}} \in \Sigma_-(u_s^{BP})} \mu_{\hat{\theta}} (\phi_s(u_s(\sigma_{\hat{\theta}}, \tau^*)) - \phi_s(u_s^{BP})) \\
& \geq \phi'_r \left( \min_{\hat{\theta} \in \Sigma_-(u_s^{BP})} u_r(\sigma_{\hat{\theta}}, \tau^*) \right) \left[ \left( \sum_{\hat{\theta}} \mu_{\hat{\theta}} \phi_s(u_s(\sigma_{\hat{\theta}}, \tau^*)) \right) - \phi_s(u_s^{BP}) \right] > 0,
\end{aligned}$$

implying  $\phi_s(u_s(\sigma^*, \tau^*)) > \phi_s(u_s^{BP})$ , contradicting (A.4). The first inequality follows from substituting for  $\sigma^*$  and concavity of  $\phi_s$ , the second from the definitions of  $\Sigma_+(u_s^{BP})$  and  $\Sigma_-(u_s^{BP})$  and concavity of  $\phi_r$ , the third since (A.5) implies  $\max_{\hat{\theta} \in \Sigma_+(u_s^{BP})} u_r(\sigma_{\hat{\theta}}, \tau^*) \leq \min_{\hat{\theta} \in \Sigma_-(u_s^{BP})} u_r(\sigma_{\hat{\theta}}, \tau^*)$ , and the final one since  $(\sigma, \mu)$  benefits the sender.  $\square$

*Proof of Theorem 3.* We prove the theorem by first showing the following lemma and then establish that the conditions in Theorem 3 are equivalent to the conditions in the lemma.

**Lemma A.4.** *Ambiguous communication benefits the sender only if  $\phi_r$  is not affine and there exists Pareto-ranked experiments,  $\sigma$  and  $\sigma^*$  such that: (i)  $\text{supp } \sigma(\cdot|\omega) = \text{supp } \sigma^*(\cdot|\omega)$  for all  $\omega$ , (ii)  $u_s(\sigma, \tau^*) > u_s^{BP}$ , and (iii)  $\tau^* \in \text{br}(\sigma^*) \setminus \text{br}(\sigma)$ .*

*Proof of Lemma A.4.* Suppose that there exists a solution  $(\sigma^*, \mu^*, \tau^*)$  to the maximization problem  $(\mathcal{P})$  that benefits the sender. Let  $\sigma := \sum_{\theta} \mu_{\theta}^* \sigma_{\theta}^*$  and  $\sigma^* := \sum_{\theta} \text{em}_{\theta}^{(\sigma^*, \mu^*)} \sigma_{\theta}^*$ . These experiments satisfy the conditions in Lemma A.4. Since  $\text{em}^{(\sigma^*, \mu^*)}$  and  $\mu^*$  have the same support on  $\Theta$ ,  $\text{supp } \sigma(\cdot|\omega) = \text{supp } \sigma^*(\cdot|\omega)$  for all  $\omega$ . From Lemma 1, since  $\tau^* \in \text{BR}(\sigma^*, \mu^*)$ ,



$\tau^* \in br(\sigma^*)$ . Since the sender benefits from ambiguous communication, we have that  $u_s^{BP} < \phi_s^{-1}(\sum_{\theta} \mu_{\theta}^* \phi_s(u_s(\sigma_{\theta}^*, \tau^*))) \leq \sum_{\theta} \mu_{\theta}^* u_s(\sigma_{\theta}^*, \tau^*) = u_s(\sigma, \tau^*)$ . This further implies that  $\tau^* \notin br(\sigma)$ , and thus  $em(\sigma^*, \mu^*) \neq \mu^*$ , implying that  $\phi_r$  is not affine. Since  $\tau^* \in br(\sigma^*)$ , we have that  $u_s^{BP} \geq u_s(\sigma^*, \tau^*)$  and, thus,  $u_s(\sigma, \tau^*) > u_s^{BP} \geq u_s(\sigma^*, \tau^*)$ .

Since  $em(\sigma^*, \mu^*) \neq \mu^*$ , there must exist a pair  $(\theta, \theta')$  in the support of  $\mu^*$  such that  $u_r(\sigma_{\theta}, \tau^*) > u_r(\sigma_{\theta'}, \tau^*)$  and  $\phi_r'(u_r(\sigma_{\theta}, \tau^*)) < \phi_r'(u_r(\sigma_{\theta'}, \tau^*))$ . We next use the following lemma and concavity of  $\phi_r$  to show  $u_r(\sigma, \tau^*) > u_r(\sigma^*, \tau^*)$ .

**Lemma A.5.** *Fix any two monotonic sequences  $x_1 \geq x_2 \geq \dots \geq x_n$ ,  $0 < y_1 \leq y_2 \leq \dots \leq y_n$ , and a probability  $\mu \in \Delta(\{1, 2, \dots, n\})$ . Assume that there exist indices  $i^* < j^*$  such that  $\mu_{i^*} > 0$ ,  $\mu_{j^*} > 0$ ,  $x_{i^*} > x_{j^*}$  and  $y_{i^*} < y_{j^*}$ . Then  $\sum_{i=1}^n x_i \frac{\mu_i y_i}{\sum_{j=1}^n \mu_j y_j} < \sum_{i=1}^n x_i \mu_i$ .*

(Lemma A.5 seems like it should be a known result, but we could not locate a reference and so include a proof in the Online Appendix (Cheng et al., 2025) for completeness.)

To prove  $u_r(\sigma, \tau^*) > u_r(\sigma^*, \tau^*)$ , we apply the lemma to the decreasing rearrangement of the sequence  $(u_r(\sigma_{\theta}^*, \tau^*))_{\theta}$  (the  $x_i$ 's) and the increasing rearrangement of  $(\phi_r'(u_r(\sigma_{\theta}^*, \tau^*)))_{\theta}$  (the  $y_i$ 's). Since  $\phi_r$  is strictly increasing and concave, we have that  $\phi_r'(u_r(\sigma_{\theta}^*, \tau^*)) > 0$ , and  $u_r(\sigma_{\theta}^*, \tau^*) \geq u_r(\sigma_{\theta'}, \tau^*)$  implies that  $\phi_r'(u_r(\sigma_{\theta}^*, \tau^*)) \leq \phi_r'(u_r(\sigma_{\theta'}, \tau^*))$ . There exists  $i^* < j^*$  such that  $\mu_{i^*} > 0$ ,  $\mu_{j^*} > 0$ ,  $x_{i^*} > x_{j^*}$  and  $y_{i^*} < y_{j^*}$  since there exists a pair  $(\theta, \theta')$  in the support of  $\mu$  such that  $u_r(\sigma_{\theta}, \tau^*) > u_r(\sigma_{\theta'}, \tau^*)$  and  $\phi_r'(u_r(\sigma_{\theta}, \tau^*)) < \phi_r'(u_r(\sigma_{\theta'}, \tau^*))$ . Applying the lemma and the definition of  $\sigma$  and  $\sigma^*$  yields  $u_r(\sigma, \tau^*) > u_r(\sigma^*, \tau^*)$ . This establishes that the conditions in Lemma A.4 are necessary for ambiguous communication to benefit the sender.  $\square$

To establish that these conditions imply the conditions in Theorem 3, we rely on the following sufficient condition for the existence of a Pareto-ranked splitting of an experiment  $\sigma$ :

**Lemma A.6.** *Let  $\sigma$  be an experiment. If there exists  $\hat{\sigma}$  such that  $u_s(\hat{\sigma}, \tau^*) > u_s(\sigma, \tau^*)$ ,  $u_r(\hat{\sigma}, \tau^*) > u_r(\sigma, \tau^*)$  and for all  $\omega \in \Omega$ ,  $\text{supp}(\hat{\sigma}(\cdot|\omega)) \subseteq \text{supp}(\sigma(\cdot|\omega))$ , then there exists a Pareto-ranked splitting of  $\sigma$ ,  $(\bar{\sigma}, \underline{\sigma}, \lambda)$  with  $\bar{\sigma} = \hat{\sigma}$ .*

(A proof of Lemma A.6 is in the Online Appendix (Cheng et al., 2025).)

Observe that the conditions in Lemma A.4 imply the conditions in Lemma A.6. Therefore a Pareto-ranked splitting of  $\sigma^*$ ,  $(\bar{\sigma}, \underline{\sigma}, \lambda)$  exists with  $\bar{\sigma} = \sigma$ . Then by condition (ii),  $u_s(\bar{\sigma}, \tau^*) > u_s^{BP}$ . This shows that the conditions in Lemma A.4 imply those in Theorem 3.

Finally, we show the conditions in Theorem 3 imply those in Lemma A.4. Suppose there exists an obedient experiment  $\hat{\sigma}$  satisfying the conditions in Theorem 3. Since  $u_s(\bar{\sigma}, \tau^*) > u_s^{BP}$ , there exists a  $\underline{\gamma} \in (\lambda, 1)$  such that for all  $\gamma \in (\underline{\gamma}, 1)$ ,  $\gamma u_s(\bar{\sigma}, \tau^*) + (1 - \gamma) u_s(\underline{\sigma}, \tau^*) > u_s^{BP}$ , and thus  $\tau^* \notin br(\gamma \bar{\sigma} + (1 - \gamma) \underline{\sigma})$ . Fix any such  $\gamma$  and let  $\sigma = \gamma \bar{\sigma} + (1 - \gamma) \underline{\sigma}$ . Let  $\sigma^* = \hat{\sigma}$ . It follows that  $\tau^* \in br(\sigma^*)$  since  $\hat{\sigma}$  is obedient, and  $u_s^{BP} \geq u_s(\sigma^*, \tau^*)$ , hence  $\gamma > \lambda$ . Since both  $\sigma$  and  $\sigma^*$

are strict mixtures of  $\bar{\sigma}$  and  $\underline{\sigma}$ , they satisfy the common support condition (i). By the definition of Pareto-ranked splitting and  $\gamma > \lambda$ ,  $u_r(\sigma, \tau^*) > u_r(\sigma^*, \tau^*)$ .  $\square$

*Proof of Corollary 1.* Suppose that there exist experiments,  $\sigma$  and  $\sigma^*$ , satisfying the conditions in Lemma A.4. Since condition (iii) implies  $\tau^* \notin br(\sigma)$ , either

$$\sum_{\omega} u_r(a_1, \omega) \sigma(a_1|\omega) p(\omega) < \sum_{\omega} u_r(a_2, \omega) \sigma(a_1|\omega) p(\omega), \text{ or} \quad (\text{A.6})$$

$$\sum_{\omega} u_r(a_2, \omega) \sigma(a_2|\omega) p(\omega) < \sum_{\omega} u_r(a_1, \omega) \sigma(a_2|\omega) p(\omega). \quad (\text{A.7})$$

Assume (A.6). From (iii),  $\tau^* \in br(\sigma^*)$ , implying  $\sum_{\omega} u_r(a_1, \omega) \sigma^*(a_1|\omega) p(\omega) \geq \sum_{\omega} u_r(a_2, \omega) \sigma^*(a_1|\omega) p(\omega)$ . It follows that:  $\sum_{\omega} u_r(a_1, \omega) [\sigma(a_1|\omega) - \sigma^*(a_1|\omega)] p(\omega) < \sum_{\omega} u_r(a_2, \omega) [\sigma(a_1|\omega) - \sigma^*(a_1|\omega)] p(\omega) = \sum_{\omega} u_r(a_2, \omega) [\sigma^*(a_2|\omega) - \sigma(a_2|\omega)] p(\omega)$ . Therefore,  $\sum_{\omega} u_r(a_2, \omega) [\sigma^*(a_2|\omega) - \sigma(a_2|\omega)] p(\omega) + \sum_{\omega} u_r(a_1, \omega) [\sigma^*(a_1|\omega) - \sigma(a_1|\omega)] p(\omega) > 0$ , contradicting (ii). Assuming (A.7) is analogous.  $\square$

## A.4 Proofs for Section 5

*Proof of Lemma 2.*

$$\begin{aligned} & \frac{\phi_i(u_i(\lambda \bar{\sigma} + (1 - \lambda) \underline{\sigma}, \tau^*)) - (\lambda \phi_i(u_i(\bar{\sigma}, \tau^*)) + (1 - \lambda) \phi_i(u_i(\underline{\sigma}, \tau^*)))}{\phi_i(u_i(\bar{\sigma}, \tau^*)) - \phi_i(u_i(\underline{\sigma}, \tau^*))} < \mu - \lambda \\ \Leftrightarrow & \phi_i(u_i(\lambda \bar{\sigma} + (1 - \lambda) \underline{\sigma}, \tau^*)) < \mu \phi_i(u_i(\bar{\sigma}, \tau^*)) + (1 - \mu) \phi_i(u_i(\underline{\sigma}, \tau^*)) \end{aligned}$$

Thus, equivalently,  $u_i(\lambda \bar{\sigma} + (1 - \lambda) \underline{\sigma}, \tau^*) < U_i(\sigma, \mu, \tau^*)$ .  $\square$

*Proof of Theorem 4.* Fix an obedient  $\sigma^*$  and let  $(\bar{\sigma}, \underline{\sigma}, \lambda)$  be a Pareto-ranked splitting of  $\sigma^*$  satisfying  $u_i(\bar{\sigma}, \tau^*) > u_i(\underline{\sigma}, \tau^*)$  for  $i \in \{s, r\}$ .

Observe that  $U_i(\sigma, \mu, \tau^*) = \phi_i^{-1}(\mu \phi_i(u_i(\bar{\sigma}, \tau^*)) + (1 - \mu) \phi_i(u_i(\underline{\sigma}, \tau^*)))$ , and  $u_i(\sigma^*, \tau^*) = \lambda u_i(\bar{\sigma}, \tau^*) + (1 - \lambda) u_i(\underline{\sigma}, \tau^*)$ . (7) implies that  $em_{\theta_1}^{(\sigma, \mu)} = \lambda$ , and  $em_{\theta_2}^{(\sigma, \mu)} = 1 - \lambda$ . Lemma 1 then implies that  $(\sigma, \mu)$  is obedient since  $\sigma^*$  is. This proves part (i).

By Lemma 2,  $U_r(\sigma, \mu, \tau^*) > u_r(\sigma^*, \tau^*)$  if and only if the receiver's  $(\{\bar{\sigma}, \underline{\sigma}\}, \lambda)$ -probability premium is strictly less than  $\mu - \lambda$ . The latter is equivalent to:

$$\begin{aligned} & \frac{\phi_r(u_r(\sigma^*, \tau^*)) - \lambda \phi_r(u_r(\bar{\sigma}, \tau^*)) - (1 - \lambda) \phi_r(u_r(\underline{\sigma}, \tau^*))}{\phi_r(u_r(\bar{\sigma}, \tau^*)) - \phi_r(u_r(\underline{\sigma}, \tau^*))} + \lambda < \frac{\lambda \phi_r'(u_r(\underline{\sigma}, \tau^*))}{\lambda \phi_r'(u_r(\underline{\sigma}, \tau^*)) + (1 - \lambda) \phi_r'(u_r(\bar{\sigma}, \tau^*))} \\ \Leftrightarrow & \frac{\phi_r(u_r(\sigma^*, \tau^*)) - \phi_r(u_r(\underline{\sigma}, \tau^*))}{\phi_r(u_r(\sigma^*, \tau^*)) - \phi_r(u_r(\underline{\sigma}, \tau^*)) + \phi_r(u_r(\bar{\sigma}, \tau^*)) - \phi_r(u_r(\sigma^*, \tau^*))} \end{aligned}$$

$$\begin{aligned}
&< \frac{\phi'_r(u_r(\underline{\sigma}, \tau^*))(u_r(\sigma^*, \tau^*) - u_r(\underline{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*))(u_r(\sigma^*, \tau^*) - u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))(u_r(\bar{\sigma}, \tau^*) - u_r(\sigma^*, \tau^*))} \\
&\Leftrightarrow \frac{1}{1 + \frac{\phi_r(u_r(\bar{\sigma}, \tau^*)) - \phi_r(u_r(\sigma^*, \tau^*))}{\phi_r(u_r(\sigma^*, \tau^*)) - \phi_r(u_r(\underline{\sigma}, \tau^*))}} < \frac{1}{1 + \frac{\phi'_r(u_r(\bar{\sigma}, \tau^*))(u_r(\bar{\sigma}, \tau^*) - u_r(\sigma^*, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*))(u_r(\sigma^*, \tau^*) - u_r(\underline{\sigma}, \tau^*))}} \\
&\Leftrightarrow \phi'_r(u_r(\bar{\sigma}, \tau^*)) < \phi'_r(u_r(\underline{\sigma}, \tau^*)) \Leftrightarrow \lambda < \frac{\lambda \phi'_r(u_r(\underline{\sigma}, \tau^*))}{\lambda \phi'_r(u_r(\underline{\sigma}, \tau^*)) + (1 - \lambda) \phi'_r(u_r(\bar{\sigma}, \tau^*))} = \mu.
\end{aligned}$$

where the first equivalence uses the fact that  $\lambda = \frac{u_r(\sigma^*, \tau^*) - u_r(\underline{\sigma}, \tau^*)}{u_r(\bar{\sigma}, \tau^*) - u_r(\underline{\sigma}, \tau^*)}$ , the second is algebra, the third follows from concavity and differentiability of  $\phi_r$ , and the fourth from the strict positivity of  $\phi'_r$ .<sup>24</sup>

This proves part (ii). Part (iii) follows directly from Lemma 2.

Within the smooth ambiguity model, an increase in ambiguity aversion corresponds to  $\phi$  becoming more concave (Klibanoff et al., 2005). Given differentiability,  $\tilde{\phi}$  more concave than  $\phi$  means that  $\tilde{\phi} := \varphi \circ \phi$  for some strictly increasing, concave, and differentiable  $\varphi$ . To see that the sender's probability premium increases in the sender's ambiguity aversion, observe that  $\rho^{\tilde{\phi}, u_s}((\bar{\sigma}, \underline{\sigma}), \lambda) + \lambda$  is equal to

$$\begin{aligned}
&\frac{\varphi(\phi(u(\sigma^*, \tau^*))) - \varphi(\phi(u(\underline{\sigma}, \tau^*)))}{\varphi(\phi(u(\bar{\sigma}, \tau^*))) - \varphi(\phi(u(\sigma^*, \tau^*))) + \varphi(\phi(u(\sigma^*, \tau^*))) - \varphi(\phi(u(\underline{\sigma}, \tau^*)))} \\
&\geq \frac{\varphi'(\phi(u(\sigma^*, \tau^*))) (\phi(u(\sigma^*, \tau^*)) - \phi(u(\underline{\sigma}, \tau^*)))}{\varphi'(\phi(u(\sigma^*, \tau^*))) (\phi(u(\bar{\sigma}, \tau^*)) - \phi(u(\sigma^*, \tau^*))) + \varphi'(\phi(u(\sigma^*, \tau^*))) (\phi(u(\sigma^*, \tau^*)) - \phi(u(\underline{\sigma}, \tau^*)))} \\
&= \frac{\phi(u(\sigma^*, \tau^*)) - \phi(u(\underline{\sigma}, \tau^*))}{\phi(u(\bar{\sigma}, \tau^*)) - \phi(u(\underline{\sigma}, \tau^*))} = \rho^{\phi, u_s}(\{\bar{\sigma}, \underline{\sigma}\}, \lambda) + \lambda,
\end{aligned}$$

where the inequality follows from the concavity of  $\varphi$ . The inequality is strict if and only if  $\varphi'(\phi(u(\bar{\sigma}, \tau^*))) < \varphi'(\phi(u(\underline{\sigma}, \tau^*)))$ , as the strict inequality on  $\varphi'$  implies that either  $\varphi'(\phi(u(\bar{\sigma}, \tau^*))) < \varphi'(\phi(u(\sigma^*, \tau^*)))$  or  $\varphi'(\phi(u(\sigma^*, \tau^*))) < \varphi'(\phi(u(\underline{\sigma}, \tau^*)))$  or both.

It remains to show that the r.h.s. of (7) is increasing in the receiver's ambiguity aversion. This follows since it is increasing in  $\frac{\phi'_r(u_r(\underline{\sigma}, \tau^*))}{\phi'_r(u_r(\bar{\sigma}, \tau^*))}$ , and

$$\frac{\tilde{\phi}'(u_r(\underline{\sigma}, \tau^*))}{\tilde{\phi}'(u_r(\bar{\sigma}, \tau^*))} = \frac{\varphi'(\phi(u(\underline{\sigma}, \tau^*))) \phi'(u_r(\underline{\sigma}, \tau^*))}{\varphi'(\phi(u(\bar{\sigma}, \tau^*))) \phi'(u_r(\bar{\sigma}, \tau^*))} \geq \frac{\phi'(u_r(\underline{\sigma}, \tau^*))}{\phi'(u_r(\bar{\sigma}, \tau^*))},$$

by concavity of  $\varphi$ , and strict if and only if  $\varphi'(\phi(u(\bar{\sigma}, \tau^*))) < \varphi'(\phi(u(\underline{\sigma}, \tau^*)))$ .  $\square$

*Proof of Proposition 4.* Let  $\Sigma_{\sigma^*} \subseteq \Sigma$  denote the set of experiments that, for all  $\omega \in \Omega$ , have the same support as  $\sigma^*$ . For each  $\omega \in \Omega$ , fix any  $a_\omega \in \text{supp}(\sigma^*(\cdot|\omega))$ . For any  $\sigma \in \Sigma_{\sigma^*}$ , by substituting

<sup>24</sup>If  $\phi_r$  were concave, but not differentiable at  $u_r(\sigma^*, \tau^*)$ , the third equivalence would fail in one direction since we could then have a linear piece from  $u_r(\underline{\sigma}, \tau^*)$  to  $u_r(\sigma^*, \tau^*)$  with slope  $\phi'_r(u_r(\underline{\sigma}, \tau^*))$  and another one from  $u_r(\sigma^*, \tau^*)$  to  $u_r(\bar{\sigma}, \tau^*)$  with slope  $\phi'_r(u_r(\bar{\sigma}, \tau^*))$ .

$\sigma(a_\omega|\omega) = 1 - \sum_{a \neq a_\omega} \sigma(a|\omega)$ , we have, for  $i \in \{s, r\}$ ,

$$u_i(\sigma, \tau^*) = \sum_{\omega, a} \sigma(a|\omega) u_i(a, \omega) = \sum_{\omega \in \Omega_{\sigma^*}, a} p(\omega) \sigma(a|\omega) (u_i(a, \omega) - u_i(a_\omega, \omega)) + \sum_{\omega} p(\omega) u_i(a_\omega, \omega),$$

where  $\Omega_{\sigma^*} \subseteq \Omega$  denotes the set of  $\omega$  such that  $|\text{supp}(\sigma^*(\cdot|\omega))| > 1$ .

Given any  $\sigma \in \Sigma_{\sigma^*}$ , we use  $\tilde{\sigma} \in \mathbb{R}^{\sum_{\omega \in \Omega_{\sigma^*}} |\text{supp}(\sigma^*(\cdot|\omega)) - 1|} =: \mathbb{R}^{\sigma^*}$  to denote the vector of those components of  $\sigma(a|\omega)$  with  $\omega \in \Omega_{\sigma^*}$  and  $a \in \text{supp}(\sigma^*(\cdot|\omega)) \setminus \{a_\omega\}$ . Thus, we can write

$$\begin{bmatrix} u_s(\sigma, \tau^*) \\ u_r(\sigma, \tau^*) \end{bmatrix} = \begin{bmatrix} p(\omega)(u_s(a, \omega) - u_s(a_\omega, \omega)) & \cdots \\ p(\omega)(u_r(a, \omega) - u_r(a_\omega, \omega)) & \cdots \end{bmatrix} \tilde{\sigma} + \begin{bmatrix} \sum_{\omega} p(\omega) u_s(a_\omega, \omega) \\ \sum_{\omega} p(\omega) u_r(a_\omega, \omega) \end{bmatrix}$$

Notice that any non-zero vectors in (8) are exactly the non-zero columns of the first matrix on the right hand side above. When the former set spans  $\mathbb{R}^2$ , the latter matrix has full rank and thus the linear mapping from  $\mathbb{R}^{\sigma^*}$  to  $\mathbb{R}^2$  defined from the right hand side above is surjective. Since  $\{\tilde{\sigma} : \sigma \in \Sigma_{\sigma^*}\} \ni \tilde{\sigma}^*$  is open in  $\mathbb{R}^{\sigma^*}$ , by the open mapping theorem,  $\{(u_s(\sigma, \tau^*), u_r(\sigma, \tau^*)) : \sigma \in \Sigma_{\sigma^*}\}$  is open. Thus, there exists  $\hat{\sigma} \in \Sigma_{\sigma^*}$  such that  $u_i(\hat{\sigma}, \tau^*) > u_i(\sigma^*, \tau^*)$  for  $i \in \{s, r\}$ . By Lemma A.6, there exists a Pareto-ranked splitting of  $\sigma^*$  with  $\bar{\sigma} = \hat{\sigma}$ .  $\square$

*Proof of Theorem 5.* That  $(\sigma, \mu)$  benefits the sender means that

$$\phi_s^{-1}(\mu \phi_s(u_s(\bar{\sigma}, \tau^*)) + (1 - \mu) \phi_s(u_s(\underline{\sigma}, \tau^*))) > u_s^{BP}. \quad (\text{A.8})$$

Any less ambiguity averse sender will have  $\tilde{\phi}_s$  weakly less concave than  $\phi_s$ , weakly increasing the left-hand side of (A.8), while leaving the right-hand side unchanged. This proves (i).

Let  $\sigma^* = em_{\theta_1}^{(\sigma, \mu)} \bar{\sigma} + (1 - em_{\theta_1}^{(\sigma, \mu)}) \underline{\sigma}$ . By Lemma 1,  $\tau^* \in br(\sigma^*)$ . In light of (A.8) and Theorem 2,  $\bar{\sigma}$  and  $\underline{\sigma}$  must be Pareto-ranked. Without loss of generality, assume that  $\bar{\sigma}$  is the better one. A weakly more ambiguity averse receiver will have a  $\tilde{\phi}_r = \varphi \circ \phi_r$  for some increasing, differentiable and concave  $\varphi$ , resulting in an effective measure  $\tilde{em}^{(\sigma, \mu)}$  such that  $\tilde{em}_{\theta_1}^{(\sigma, \mu)} \leq em_{\theta_1}^{(\sigma, \mu)}$ .

If, as in (iii),  $\tau^* \in br(\underline{\sigma})$ , then since  $\tau^* \in br(\sigma^*) = br(em_{\theta_1}^{(\sigma, \mu)} \bar{\sigma} + (1 - em_{\theta_1}^{(\sigma, \mu)}) \underline{\sigma})$ ,  $\tau^* \in br(\tilde{em}_{\theta_1}^{(\sigma, \mu)} \bar{\sigma} + (1 - \tilde{em}_{\theta_1}^{(\sigma, \mu)}) \underline{\sigma})$ . By Lemma 1, this implies  $\tau^* \in BR(\sigma, \mu)$  for a receiver with any such  $\tilde{\phi}_r$ , proving that  $\sigma$  together with  $\mu$  continues to benefit the sender (and, by the same argument as for (i), any less ambiguity averse senders as well). This proves (iii).

Finally, if  $\tau^* \notin br(\underline{\sigma})$ , define  $\tilde{\mu}$  by

$$\tilde{\mu}(\theta_1) = \frac{em_{\theta_1}^{(\sigma, \mu)} \tilde{\phi}_r'(u_r(\underline{\sigma}, \tau^*))}{em_{\theta_1}^{(\sigma, \mu)} \tilde{\phi}_r'(u_r(\underline{\sigma}, \tau^*)) + (1 - em_{\theta_1}^{(\sigma, \mu)}) \tilde{\phi}_r'(u_r(\bar{\sigma}, \tau^*))}, \text{ and } \tilde{\mu}_{\theta_2} = 1 - \tilde{\mu}_{\theta_1},$$

so that  $e\tilde{m}^{(\sigma, \tilde{\mu})} = em^{(\sigma, \mu)}$ . Lemma 1 and  $\tau^* \in br(\sigma^*)$  then implies  $\tau^* \in BR(\sigma, \tilde{\mu})$  for a receiver with  $\tilde{\phi}_r$ . Since  $(u_s(\bar{\sigma}, \tau^*) > u_s(\underline{\sigma}, \tau^*))$ , to show that  $(\sigma, \tilde{\mu})$  benefits the sender (and all less ambiguity averse senders) it suffices to show that  $\tilde{\mu}_{\theta_1} \geq \mu$ . Indeed,

$$\begin{aligned} \tilde{\mu}_{\theta_1} &= \frac{em_{\theta_1}^{(\sigma, \mu)} \phi'_r(u_r(\underline{\sigma}, \tau^*))}{em_{\theta_1}^{(\sigma, \mu)} \phi'_r(u_r(\underline{\sigma}, \tau^*)) + (1 - em_{\theta_1}^{(\sigma, \mu)}) \frac{\varphi'(\phi_r(u_r(\bar{\sigma}, \tau^*)))}{\varphi'(\phi_r(u_r(\underline{\sigma}, \tau^*)))} \phi'_r(u_r(\bar{\sigma}, \tau^*))} \\ &\geq \frac{em_{\theta_1}^{(\sigma, \mu)} \phi'_r(u_r(\underline{\sigma}, \tau^*))}{em_{\theta_1}^{(\sigma, \mu)} \phi'_r(u_r(\underline{\sigma}, \tau^*)) + (1 - em_{\theta_1}^{(\sigma, \mu)}) \phi'_r(u_r(\bar{\sigma}, \tau^*))} = \mu, \end{aligned}$$

where the inequality follows from  $\varphi'(\phi_r(u_r(\underline{\sigma}, \tau^*))) \geq \varphi'(\phi_r(u_r(\bar{\sigma}, \tau^*)))$ .  $\square$

## A.5 Proofs for Section 7

Before proving Theorems 7 and 8, we first characterize the receiver's obedience condition under pre-existing ambiguity about the payoff-relevant states. For any joint distribution  $\pi \in \Delta(\Omega \times A)$ ,  $\tau^*$  is the receiver's best response if

$$\tau^* \in \arg \max_{\tau} \sum_{\omega, a, a'} \pi(\omega, a) \tau(a'|a) u_r(a', \omega).$$

Let  $\Pi^* \subseteq \Delta(\Omega \times A)$  denote the set of all such distributions. Note that  $\Pi^*$  is convex and closed. Thus, an experiment  $\sigma$  is obedient under prior  $p$  if the induced joint distribution  $\pi^{(p, \sigma)} = p \times \sigma \in \Pi^*$ . An argument similar to that in the proof of Lemma 1 can be used to show that when there is pre-existing ambiguity  $\eta$  and the sender chooses an ambiguous experiment  $(\sigma, \mu)$ ,  $\tau^*$  is a best response of the receiver if and only if

$$\pi^{(\eta, (\sigma, \mu))} := \sum_{p, \theta} \frac{\eta_p \mu_\theta \phi'_r(u_r(p, \sigma_\theta, \tau^*))}{\sum_{\tilde{p}, \tilde{\theta}} \eta_{\tilde{p}} \mu_{\tilde{\theta}} \phi'_r(u_r(\tilde{p}, \sigma_{\tilde{\theta}}, \tau^*))} p \times \sigma \in \Pi^*,$$

where we explicitly include  $p$  as an argument of the receiver's expected utility  $u_r(p, \sigma_\theta, \tau^*)$ . In this case, we say the ambiguous experiment  $(\sigma, \mu)$  is obedient under  $\eta$ .

*Proof of Theorem 7.* Because  $\phi_r$  is strictly concave, and  $(\bar{\sigma}, \underline{\sigma}, \lambda)$  is a Pareto-ranked splitting of  $\sigma^*$  for all  $p \in \text{supp}(\eta)$ , it holds that  $\mu_{\bar{\theta}} > \lambda$ . Thus, if the binary ambiguous experiment  $(\sigma, \mu)$  is obedient under  $\eta$ , then an ambiguity neutral sender does strictly better using it than using  $\sigma^*$ .

We next establish its obedience. For each  $p \in \text{supp}(\eta)$ , let  $em_{\bar{\sigma}}^{(p, (\sigma, \mu))}$  denote the effective measure on  $\bar{\sigma}$  under prior  $p$ . With  $\mu$  given by (10), it implies that for the minimizing  $p^*$ ,  $em_{\bar{\sigma}}^{(p^*, (\sigma, \mu))} = \lambda$ ; and for any other  $p \in \text{supp}(\eta)$ ,  $em_{\bar{\sigma}}^{(p, (\sigma, \mu))} < \lambda$ . It further implies that, for each  $p \in \text{supp}(\eta)$ , the experiment  $em_{\bar{\sigma}}^{(p, (\sigma, \mu))} \bar{\sigma} + (1 - em_{\bar{\sigma}}^{(p, (\sigma, \mu))}) \underline{\sigma}$  is a convex combination of  $\sigma^*$  and  $\underline{\sigma}$ ,

and thus, by the obedience of  $\sigma^*$  and  $\underline{\sigma}$  under  $p$ , is also obedient under  $p$ . The obedience of  $(\sigma, \mu)$  under  $\eta$  is implied by the following lemma:

**Lemma A.7.** *If an ambiguous experiment  $(\sigma, \mu)$  is obedient under  $p$  for every  $p \in \text{supp}(\eta)$ , then it is obedient under  $\eta$ .*

*Proof of Lemma A.7.* Given  $p \in \Delta(\Omega)$  and  $(\sigma, \mu)$ , let

$$\sigma_p^{(\sigma, \mu)} := \sum_{\theta} \frac{\mu_{\theta} \phi'_r(u_r(p, \sigma_{\theta}, \tau^*))}{\sum_{\tilde{\theta}} \mu_{\tilde{\theta}} \phi'_r(u_r(\tilde{p}, \sigma_{\tilde{\theta}}, \tau^*))} \sigma_{\theta}.$$

If  $(\sigma, \mu)$  is obedient under prior  $p$ , then it holds that  $\pi^{(p, (\sigma, \mu))} = p \times \sigma_p^{(\sigma, \mu)} \in \Pi^*$ . Consider an ambiguous experiment  $(\sigma, \mu)$  under  $\eta$ , the induced joint distribution satisfies

$$\begin{aligned} \pi^{(\eta, (\sigma, \mu))} &= \sum_{p, \theta} \frac{\eta_p \mu_{\theta} \phi'_r(u_r(p, \sigma_{\theta}, \tau^*))}{\sum_{\tilde{p}, \tilde{\theta}} \eta_{\tilde{p}} \mu_{\tilde{\theta}} \phi'_r(u_r(\tilde{p}, \sigma_{\tilde{\theta}}, \tau^*))} p \times \sigma_{\theta} \\ &= \sum_p \frac{\eta_p \sum_{\tilde{\theta}} \mu_{\tilde{\theta}} \phi'_r(u_r(\tilde{p}, \sigma_{\tilde{\theta}}, \tau^*))}{\sum_{\tilde{p}, \tilde{\theta}} \eta_{\tilde{p}} \mu_{\tilde{\theta}} \phi'_r(u_r(\tilde{p}, \sigma_{\tilde{\theta}}, \tau^*))} p \times \sum_{\theta} \frac{\mu_{\theta} \phi'_r(u_r(p, \sigma_{\theta}, \tau^*))}{\sum_{\tilde{\theta}} \mu_{\tilde{\theta}} \phi'_r(u_r(\tilde{p}, \sigma_{\tilde{\theta}}, \tau^*))} \sigma_{\theta} \\ &= \sum_p \frac{\eta_p \sum_{\tilde{\theta}} \mu_{\tilde{\theta}} \phi'_r(u_r(\tilde{p}, \sigma_{\tilde{\theta}}, \tau^*))}{\sum_{\tilde{p}, \tilde{\theta}} \eta_{\tilde{p}} \mu_{\tilde{\theta}} \phi'_r(u_r(\tilde{p}, \sigma_{\tilde{\theta}}, \tau^*))} \pi^{(p, (\sigma, \mu))}. \end{aligned}$$

In other words,  $\pi^{(\eta, (\sigma, \mu))}$  is a convex combination of  $\pi^{(p, (\sigma, \mu))}$  for  $p \in \text{supp}(\eta)$ . Thus, if  $(\sigma, \mu)$  is obedient under  $p$  for every  $p \in \text{supp}(\eta)$ , by the convexity of  $\Pi^*$ , it is obedient as well.  $\square$

$\square$

*Proof of Theorem 8.* By the supposition, there exists an ambiguous experiment  $(\sigma, \mu)$  such that (we explicitly write the dependence on the prior  $p$ ):

$$U_s(p, \sigma, \mu, \tau^*) > u_s^{BP}(p).$$

Let  $\sigma^*$  denote the effective experiment induced by  $(\sigma, \mu)$  and let  $\pi^* = p \times \sigma^*$ . By continuity and the fact that  $\text{int}(\Pi^*) \neq \emptyset$ , we can perturb  $\sigma$  and  $\mu$  slightly to  $\tilde{\sigma}$  and  $\tilde{\mu}$  such that

$$U_s(p, \tilde{\sigma}, \tilde{\mu}, \tau^*) > u_s^{BP}(p),$$

and the joint distribution  $\pi^{(p, (\tilde{\sigma}, \tilde{\mu}))} \in \text{int}(\Pi^*)$ .

Notice for any  $\eta$  that reduces to  $p$ , because the sender is ambiguity neutral, the sender's payoff remains the same under  $\eta$  as under  $p$  provided the receiver is obedient. Because  $\pi^{(p, (\tilde{\sigma}, \tilde{\mu}))} \in \text{int}(\Pi^*)$ , there exists  $\delta > 0$  such that for all  $q \in \Delta(\Omega)$  with  $\|q - p\| < \delta$ , the distribution  $\pi^{(q, (\tilde{\sigma}, \tilde{\mu}))}$  remains in

the interior of  $\Pi^*$ . Thus, by Lemma A.7, the ambiguous experiment  $(\tilde{\sigma}, \tilde{\mu})$  will continue to benefit the sender under any  $\eta$  that reduces to  $p$  and has support within this  $\delta$ -neighborhood of  $p$ .  $\square$

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# Online Supplemental Appendix to “Persuasion with Ambiguous Communication”

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## SA.1 A Polar Case: A Maxmin Receiver and Ambiguity Neutral Sender

This section analyzes the case of an ambiguity-neutral sender and an infinitely ambiguity averse receiver, represented by the maxmin preferences  $U_r^{MEU}$ . As the comparative statics statements in Theorem 4 suggest, this is the most favorable case for the sender to benefit from ambiguous communication.

More precisely, we show that in this case (a) binary ambiguous experiments are sufficient to exhaust all gains from persuasion, and (b) the sender can attain a payoff arbitrarily close to their best feasible payoff subject to the receiver getting at least the payoff they would obtain if no information were disclosed. A conclusion we draw is that assuming an infinitely ambiguity averse receiver is very powerful and, in our view, unrealistically so, further motivating the analysis in the rest of the paper which allows for more moderate levels of aversion.

Before turning to the analysis, we remark that the opposite cases, of either an infinitely ambiguity averse sender with payoffs  $U_s^{MEU}$  or an ambiguity neutral receiver, preclude any benefit from ambiguous persuasion. The latter case follows from Theorem 3, while Cheng (2025) shows that in the former case the sender never benefits from ambiguous communication.

The following lemma relates obedience for an ambiguous experiment to obedience for an experiment. It is thus the analogue of Lemma 1 for a receiver with preferences  $U_r^{MEU}$ :

**Lemma SA.1.1.**  *$(\sigma, \mu)$  is obedient if, and only if, the experiment  $\sigma^*$  is obedient, where*

$$\sigma^* := \frac{\sum_{\substack{\theta \in \arg \min_{\theta \in \text{supp}(\mu)} u_r(\sigma_\theta, \tau^*)}} \mu_\theta \sigma_\theta}{\sum_{\substack{\theta \in \arg \min_{\theta \in \text{supp}(\mu)} u_r(\sigma_\theta, \tau^*)}} \mu_\theta}.$$

*Proof of Lemma SA.1.1.* Fix  $(\sigma, \mu)$ . Obedience requires that  $\min_{\theta \in \Theta} u_r(\sigma_\theta, \tau^*) \geq \min_{\theta \in \Theta} u_r(\sigma_\theta, \tau)$ .

From the minmax theorem, this is equivalent to the existence of  $\mu^* \in \Delta(\Theta)$  such that  $\sum_{\theta \in \Theta} \mu_\theta u_r(\sigma_\theta, \tau^*) \geq \sum_{\theta \in \Theta} \mu_\theta^* u_r(\sigma_\theta, \tau)$ , for all  $(\mu, \tau)$ . The result follows since  $\mu^*$  is a minimizer of  $\sum_{\theta \in \Theta} \mu_\theta u_r(\sigma_\theta, \tau^*)$ .  $\square$

Observe that when the argmin in Lemma SA.1.1 is a singleton,  $\sigma^*$  equals the receiver's payoff-minimizing experiment from  $\sigma$ . More generally, it is a convex combination of the possibly multiple minimizing experiments in  $\sigma$  with relative weights inherited from  $\mu$ . Thus the analogue of the effective measure here may have a smaller support than  $\mu$  (something that never happens for a smooth ambiguity receiver). Lemma SA.1.1 says that *only* those payoff-minimizing experiments affect obedience of  $(\sigma, \mu)$ . Thus, the sender is free to include in  $\sigma$  and arbitrarily weight *any other experiments* as long as they don't disrupt the receiver's minimum.

Since the receiver can always ignore any recommendations made, they can guarantee themselves the payoff

$$\underline{u}_r^* := \max_{a \in A} \sum_{\omega} p(\omega) u_r(a, \omega),$$

which is the payoff they would obtain if no information were disclosed. The consequence of the great flexibility available to the sender given Lemma SA.1.1 is the next theorem, which states that the sender's optimal payoff approaches their highest feasible payoff subject to the receiver getting at least  $\underline{u}_r^*$ .<sup>25</sup> The corresponding communication strategy uses a binary ambiguous experiment with the  $\mu$ -weight on the better experiment approaching 1, and the worse experiment an obedient one holding the receiver to  $\underline{u}_r^*$ .

**Theorem SA.1.1.** *Suppose there exists  $\hat{\sigma}$  such that  $u_r(\hat{\sigma}, \tau^*) > \underline{u}_r^*$ . The value of the following program is the supremum of the payoff that an ambiguity neutral sender can obtain when the receiver has maxmin preferences  $U_r^{MEU}$ :*

$$\begin{aligned} & \max_{\sigma} u_s(\sigma, \tau^*), \\ & \text{s.t. } u_r(\sigma, \tau^*) \geq \underline{u}_r^*. \end{aligned}$$

*Proof of Theorem SA.1.1.* Let  $\bar{\sigma}$  attain the value of the program in the theorem for the sender and  $\underline{\sigma}$  be an obedient experiment with  $u_r(\underline{\sigma}, \tau^*) = \underline{u}_r^*$ . Assume  $u_r(\bar{\sigma}, \tau^*) > u_r(\underline{\sigma}, \tau^*)$ . If the sender chooses  $((\bar{\sigma}, \underline{\sigma}), (\mu, 1 - \mu))$ , the receiver is obedient since the worst payoff is  $u_r(\underline{\sigma}, \tau^*)$ . As  $\mu$  approaches 1, the sender's payoff approaches  $u_s(\bar{\sigma}, \tau^*)$ . The sender cannot do better than this, since the receiver's payoff is at least  $\underline{u}_r^*$ . If  $u_r(\bar{\sigma}, \tau^*) = u_r(\underline{\sigma}, \tau^*)$ , mix  $\bar{\sigma}$  with  $\hat{\sigma}$ . For any  $\varepsilon > 0$ ,  $u_r((1 - \varepsilon)\bar{\sigma} + \varepsilon\hat{\sigma}, \tau^*) > u_r(\underline{\sigma}, \tau^*)$ . As  $\varepsilon$  approaches 0 and  $\mu$  approaches 1, the payoff for the sender approaches  $u_s(\bar{\sigma}, \tau^*)$ .  $\square$

**Remark 5** (MEU receiver's payoff). Theorem SA.1.1 does not imply that an MEU receiver is held to (or even close to)  $\underline{u}_r^*$  by all sender-optimal strategies. As receiver ambiguity aversion

<sup>25</sup>This "efficiency subject to an outside option" approach to identifying the sender's optimal payoff is reminiscent of one developed in Smolin (2021) that can be applied to any Bayesian persuasion problem where the receiver has only two actions. In our case, it is the extremity of the receiver's MEU preference that creates the almost complete separation between the experiment providing the sender's payoff and the one providing the receiver's payoff.

passes to the MEU limit, while the value of the sender's program is continuous in this limit, the payoff to the receiver may drop discontinuously. In our introductory example, for instance, as  $\phi_r$  becomes more and more concave, the receiver's payoff under optimal ambiguous communication approaches their payoff under  $\sigma_{\bar{\theta}}$  of  $3/2$ , while in the MEU limit, their payoff is no higher than the Bayesian persuasion payoff of  $5/4$ .

There is a sense in which Theorem SA.1.1 could be argued to overstate what the sender can achieve. For MEU, the “effective” experiment  $\sigma^*$  could have a smaller support than  $(\sigma, \mu)$ . Lemma SA.1.1 treats action recommendations that could occur under  $(\sigma, \mu)$  but not under  $\sigma^*$  as zero probability events. However, observing such action recommendations would reveal to the receiver that  $\theta \notin \arg \min_{\theta \in \text{supp}(\mu)} u_r(\sigma_\theta, \tau^*)$ . In this case, the receiver may no longer be indifferent between obeying or not. Therefore, Theorem SA.1.1 could be seen as forcing the receiver to be obedient in such situations.

This issue can be addressed by strengthening obedience to further require that  $\sigma^*$  always has the same support as  $(\sigma, \mu)$  (which was always true for smooth ambiguity receivers). This strengthening does not substantially change the conclusions of Theorem SA.1.1, as it only replaces the program in Theorem SA.1.1 by:

$$\begin{aligned} & \sup_{\sigma} u_s(\sigma, \tau^*), \\ & \text{s.t. } u_r(\sigma, \tau^*) > \underline{u}_r^* \text{ and } \text{supp}(\sigma) \subseteq A_0, \end{aligned}$$

where  $A_0$  is the set of all actions which can be best responses for the receiver to some probability distribution over the states in the support of the prior  $p$ . The corresponding communication strategies would involve two experiments with  $\mu$ -weight on the better one approaching 1 as before, but with the worse experiment now adjusted to have full support on  $A_0$  by mixing it with an arbitrarily small amount of an obedient experiment with full support on  $A_0$  that yields the receiver more than  $\underline{u}_r^*$  (such an experiment exists under the assumptions of Theorem SA.1.1).

Formally, a stronger notion of obedience that does not allow positive  $\mu$  weight on experiments that recommend actions outside the support of the effective measure weighted experiment of a maxmin receiver is the following:

**Lemma SA.1.2.**  *$(\sigma, \mu)$  is (strongly) obedient if, and only if, the experiment  $\sigma^*$  is obedient, where*

$$\sigma^* := \frac{\sum_{\substack{\theta \in \arg \min_{\theta \in \text{supp}(\mu)} u_r(\sigma_\theta, \tau^*)}} \mu_\theta \sigma_\theta}{\sum_{\substack{\theta \in \arg \min_{\theta \in \text{supp}(\mu)} u_r(\sigma_\theta, \tau^*)}} \mu_\theta},$$

and  $\text{supp}(\sigma_\theta) \subseteq \text{supp}(\sigma^*)$  for all  $\theta \in \text{supp}(\mu)$ .

Define  $\underline{u}_r^*$  as before

$$\underline{u}_r^* := \max_{a \in A} \sum_{\omega} p(\omega) u_r(a, \omega).$$

Further define  $A_0$  as the set of all actions which can be best responses for the receiver:

$$A_0 := \{a \in A : \exists q \in \Delta(\Omega) \text{ s.t. } a \in \arg \max_{a' \in A} \sum_{\omega} q(\omega) u_r(a', \omega)\}.$$

The following is the version of Theorem SA.1.1 using the stronger obedience notion:

**Theorem SA.1.2.** *Suppose there exists  $\hat{\sigma}$  such that  $u_r(\hat{\sigma}, \tau^*) > \underline{u}_r^*$ . The value of the following program is the supremum of the payoff that an ambiguity neutral sender can obtain when the receiver has maxmin preferences  $U_r^{\text{MEU}}$  and the version of obedience in Lemma SA.1.2 is used:*

$$\begin{aligned} & \sup_{\sigma} u_s(\sigma, \tau^*), \\ & \text{s.t. } u_r(\sigma, \tau^*) > \underline{u}_r^* \text{ and, } \text{supp}(\sigma) \subseteq A_0, \end{aligned}$$

*Proof of Theorem SA.1.2.* Let  $\bar{\sigma}$  attain the value for the sender of the following program

$$\begin{aligned} & \max_{\sigma} u_s(\sigma, \tau^*), \\ & \text{s.t. } u_r(\sigma, \tau^*) \geq \underline{u}_r^* \text{ and, } \text{supp}(\sigma) \subseteq A_0. \end{aligned}$$

Let  $\underline{\sigma}$  be an obedient uninformative experiment, so that  $u_r(\underline{\sigma}, \tau^*) = \underline{u}_r^*$ . Observe that  $\text{supp}(\underline{\sigma}) \subseteq A_0$ . Assume that  $u_r(\bar{\sigma}, \tau^*) > u_r(\underline{\sigma}, \tau^*)$ . There exists an obedient experiment  $\tilde{\sigma}$  such that  $\text{supp}(\tilde{\sigma}) = A_0$  and  $u_r(\tilde{\sigma}, \tau^*) > \underline{u}_r^*$ .<sup>26</sup> Define a sequence of experiments  $\underline{\sigma}_n = (1 - \epsilon_n)\underline{\sigma} + \epsilon_n\tilde{\sigma}$  where  $\epsilon_n > 0$  with  $\epsilon_n \rightarrow 0$  as  $n$  goes to infinity. If the sender offers the ambiguous experiment  $((\bar{\sigma}, \underline{\sigma}_n), (\mu, 1 - \mu))$  for small enough  $\epsilon_n$ , the receiver is strongly obedient since the worst payoff is  $u_r(\underline{\sigma}_n, \tau^*)$ , obedience is preserved under convex combinations of experiments, and  $\text{supp}(\bar{\sigma}) \subseteq A_0 = \text{supp}(\underline{\sigma}_n)$ . As we can choose  $\mu$  arbitrarily close to 1 and  $\epsilon_n$  arbitrarily close to 0, we approach the value of the program in Theorem SA.1.2. Furthermore, it is not possible for the sender to do better than this (i.e., have a higher supremum), since the receiver's payoff from any obedient experiment (and thus from any

<sup>26</sup>For any  $a \in A_0$ , fix some  $q_a \in \Delta(\text{supp}(p))$  under which  $a$  is optimal for the receiver. There exists a  $\beta_a \in (0, 1)$  and a  $q \in \Delta(\text{supp}(p))$  such that  $p = \beta_a q_a + (1 - \beta_a)q'_a$ . Let  $a'_a \in A_0$  denote an action that is optimal for the receiver under  $q'_a$ . Applying this argument to all  $a \in A_0$  to construct a set  $\cup_{a \in A_0} \{q_a, q'_a\}$  whose convex hull contains  $p$  in its interior. Since each probability distribution in the set can be thought of as a Bayesian posterior, this interior convex combination is a Bayes plausible distribution over the posteriors and thus, by Kamenica and Gentzkow (2011), corresponds to an obedient  $\tilde{\sigma}$  with  $\text{supp}(\tilde{\sigma}) = A_0$ . Finally, since  $u_r(\hat{\sigma}, \tau^*) > \underline{u}_r^*$ , there exists an  $a \in A_0$  such that  $\sum_{\omega} q_a(\omega) u_r(a, \omega) > \underline{u}_r^*$ . Thus,  $u_r(\tilde{\sigma}, \tau^*) > \underline{u}_r^*$ .

obedient ambiguous experiment) is at least  $\underline{u}_r^*$  and the strong version of obedience requires that all experiments in the support of  $\mu$  recommend actions in  $A_0$ .

If  $u_r(\bar{\sigma}, \tau^*) = u_r(\underline{\sigma}, \tau^*)$ , we need to slightly modify the construction to guarantee obedience. The idea is to mix  $\bar{\sigma}$  with a bit of  $\tilde{\sigma}$  to guarantee a unique worst payoff, i.e.,  $u_r((1 - 2\epsilon_n)\bar{\sigma} + 2\epsilon_n\tilde{\sigma}, \tau^*) > u_r(\underline{\sigma}_n, \tau^*)$  for all  $\epsilon_n > 0$ . As  $\epsilon_n$  approaches 0 and  $\mu$  approaches 1, the payoff for the sender approaches the value of the program in the theorem.  $\square$

## SA.2 The insufficiency of binary ambiguous experiments

**Proposition SA.2.1.** *It is not always sufficient to consider only binary ambiguous experiments in searching for either a strict benefit from ambiguity or optimal ambiguous persuasion.*

We provide a detailed sketch of the proof. The full proof is available in [Cheng et al. \(2024\)](#).

*Sketch of Proof of Proposition SA.2.1.* The proof is by construction. We first show an example in which the only optimal ambiguous experiments are more than binary. A modification of this example is then used to provide an example in which the sender may strictly benefit from ambiguous communication even when no binary ambiguous experiment benefits the sender.

### Example in which all optimal ambiguous experiments are more than binary.

Suppose  $\phi_s(x) = x$  and  $\phi_r(x) = \ln(x + 5)$ . Let  $\Omega = \{\omega_1, \omega_2\}$ , with equal prior probabilities  $p = (1/2, 1/2)$ . There are five actions  $\{a_1, a_2, b_1, b_2, b_3\}$  and the payoff matrix is

$(u_s, u_r)$	$\omega_1$	$\omega_2$
$a_1$	3, 3	0, 0
$a_2$	-1, -1	3, 3
$b_1$	0, 4	-1, -2
$b_2$	0, 2	1, 2
$b_3$	-2, -4	1, 4

The optimal Bayesian persuasion experiment is

$$\begin{aligned} \sigma_a(a_1|\omega_1) &= 4/5, & \sigma_a(a_2|\omega_1) &= 1/5; \\ \sigma_a(a_1|\omega_2) &= 2/5, & \sigma_a(a_2|\omega_2) &= 3/5. \end{aligned}$$

Notice that

$$u_s(\sigma_a, \tau^*) = u_r(\sigma_a, \tau^*) = 2.$$



Let  $\sigma_{11}, \sigma_{12}, \sigma_{21}$  and  $\sigma_{22}$  denote the extreme experiments where  $\sigma_{ij}$  recommends  $a_i$  and  $a_j$  deterministically in states  $\omega_1$  and  $\omega_2$ , respectively. Notice that these extreme experiments are all Pareto-ranked:

$$\begin{aligned} u_s(\sigma_{11}, \tau^*) &= u_r(\sigma_{11}, \tau^*) = 3/2; \\ u_s(\sigma_{12}, \tau^*) &= u_r(\sigma_{12}, \tau^*) = 3; \\ u_s(\sigma_{21}, \tau^*) &= u_r(\sigma_{21}, \tau^*) = -1/2; \\ u_s(\sigma_{22}, \tau^*) &= u_r(\sigma_{22}, \tau^*) = 1. \end{aligned}$$

Consider the following splitting of  $\sigma_a$ ,

$$\sigma_a = \frac{1}{5}\sigma_{11} + \frac{3}{5}\sigma_{12} + \frac{1}{5}\sigma_{21}.$$

It can be verified that for  $\hat{\sigma} = (\sigma_{11}, \sigma_{12}, \sigma_{21})$  and  $\hat{\mu}$  such that  $\sum_{\theta} em_{\theta}^{(\hat{\sigma}, \hat{\mu})} \sigma_{\theta} = \sigma_a$ ,  $(\hat{\sigma}, \hat{\mu})$  is an obedient ambiguous experiment yielding the sender a payoff of  $159/70 = 2.27143$ , strictly higher than the 2 under Bayesian persuasion. Therefore, any optimal ambiguous experiment must involve ambiguity and thus be at least binary.

As  $\phi_s(x) = x$  and  $\phi_r(x) = \ln(x + 5)$ , by Proposition 2, in any optimal ambiguous experiment, there cannot exist any further Pareto-ranked splitting of any experiment in the collection.

Observe that  $\sigma_a$  is the only incentive-compatible experiment that never recommends any of the  $b$  actions. Furthermore,  $\sigma_a$  cannot be split into a convex combination of two extreme experiments. Thus, any binary splitting of  $\sigma_a$  must involve at least one non-extreme experiment. However, since all these extreme experiments are Pareto-ranked, there must exist a Pareto-ranked splitting of any such non-extreme experiment (into extreme experiments). Therefore, any binary ambiguous experiment constructed from splittings of  $\sigma_a$  cannot be optimal.

The proof goes on to show that an optimal ambiguous experiment in this example also cannot be a binary ambiguous experiment that is constructed from a splitting of any other incentive-compatible experiment (in particular, any recommending a  $b$  action with a positive probability).

### **Example in which ambiguous communication benefits the sender, but does not do so when restricted to binary ambiguous experiments**

Suppose  $\phi_s(x) = x$  and  $\phi_r(x) = \ln(x + 5)$ . Let  $\Omega = \{\omega_1, \omega_2\}$  and the prior  $p$  be uniform. There are seven actions  $\{a_1, a_2, b_1, b_2^+, b_2^-, b_3, c\}$ . Let the payoff matrix be, for some  $x > 2$ ,

$(u_s, u_r)$	$\omega_1$	$\omega_2$
$a_1$	3, 3	0, 0
$a_2$	-1, -1	3, 3
$b_1$	0, 4	0, -2
$b_2^-$	0, 5/2	0, 1
$b_2^+$	0, 5/4	0, 9/4
$b_3$	0, -4	0, 4
$c$	$x, 7/4$	$x, 7/4$

The only differences from the previous example are the addition of  $c$  and the replacement of  $b_2$  by  $b_2^-$  and  $b_2^+$ . Let  $\sigma_c$  denote the experiment that recommends action  $c$  deterministically in both states. Because  $x > 2$ , the optimal Bayesian persuasion experiment is  $\sigma_c$ , yielding the sender a payoff of  $x$ . The proof then shows the existence of  $x > 2$  such that the sender's payoff from  $(\hat{\sigma}, \hat{\mu})$  is strictly higher than  $x$  but the sender's payoff from any binary ambiguous experiment is lower than  $x$ . The replacement of  $b_2$  by  $b_2^-$  and  $b_2^+$  serves to make  $\sigma_c$  obedient only at the prior  $p$ , which helps simplify the calculations in the proof.  $\square$

### SA.3 A Local Argument for the Receiver's Gain from Pareto-Ranked Splittings of Obedient Experiments

Let  $\sigma^*$  be obedient and such that a pareto-ranked splitting of it exists. Then for small enough  $\epsilon > 0$ , one can always find  $\bar{\sigma}$  and  $\underline{\sigma}$  such that  $(\bar{\sigma}, \underline{\sigma}, 1/2)$  is a Pareto-ranked splitting of  $\sigma^*$  satisfying

$$u_r(\bar{\sigma}, \tau^*) = u_r(\sigma^*, \tau^*) + \epsilon, \quad u_r(\underline{\sigma}, \tau^*) = u_r(\sigma^*, \tau^*) - \epsilon.$$

Let  $\sigma = (\bar{\sigma}, \underline{\sigma})$  and let

$$\mu_{\bar{\sigma}} = \frac{\phi'_r(u_r(\underline{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))}.$$

Then by Lemma 1,  $(\sigma, \mu)$  is an obedient ambiguous experiment. To show that the receiver's payoff is higher under  $\sigma$  than under  $\sigma^*$ , we split the receiver's payoff change into two parts.

First, the receiver's payoff change from replacing  $\sigma^*$  by  $\mu_{\bar{\sigma}}\bar{\sigma} + (1 - \mu_{\bar{\sigma}})\underline{\sigma}$  is

$$\begin{aligned} & \frac{\phi'_r(u_r(\underline{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} (u_r(\sigma^*, \tau^*) + \epsilon) + \frac{\phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} (u_r(\sigma^*, \tau^*) - \epsilon) - u_r(\sigma^*) \\ &= \frac{\phi'_r(u_r(\underline{\sigma}, \tau^*)) - \phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} \epsilon \geq 0. \end{aligned}$$

Notice the change is non-negative and first-order in  $\epsilon$ .

Second, the receiver's payoff change when facing the ambiguous experiment  $(\sigma, (\mu_{\bar{\sigma}}, 1 - \mu_{\bar{\sigma}}))$  when moving from ambiguity neutrality to ambiguity aversion  $\phi_r$  is

$$\begin{aligned} & \phi_r^{-1} \left( \frac{\phi'_r(u_r(\underline{\sigma}), \tau^*)}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} \phi_r(u_r(\sigma^*, \tau^*) + \epsilon) + \frac{\phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} \phi_r(u_r(\sigma^*, \tau^*) - \epsilon) \right) \\ & - \frac{\phi'_r(u_r(\underline{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} (u_r(\sigma^*) + \epsilon) + \frac{\phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} (u_r(\sigma^*) - \epsilon). \end{aligned}$$

Next, we take the first-order Taylor expansion of the first term so that any residual is of order  $\epsilon^2$ . Observe that (ignoring the term of order  $\epsilon^2$ )

$$\begin{aligned} & \frac{\phi'_r(u_r(\underline{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} \phi_r(u_r(\sigma^*, \tau^*) + \epsilon) + \frac{\phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} \phi_r(u_r(\sigma^*, \tau^*) - \epsilon) \\ & = \frac{\phi'_r(u_r(\underline{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} (\phi_r(u_r(\sigma^*, \tau^*)) + \phi'_r(u_r(\sigma^*, \tau^*)) \epsilon) \\ & \quad + \frac{\phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} (\phi_r(u_r(\sigma^*, \tau^*)) - \phi'_r(u_r(\sigma^*, \tau^*)) \epsilon) \\ & = \phi_r(u_r(\sigma^*, \tau^*)) + \phi'_r(u_r(\sigma^*, \tau^*)) \frac{\phi'_r(u_r(\underline{\sigma}, \tau^*)) - \phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} \epsilon \end{aligned}$$

Applying  $\phi_r^{-1}$  to the above term and again taking the first-order Taylor expansion and ignoring the term of order  $\epsilon^2$  yields

$$\begin{aligned} & \phi_r^{-1} \left( \phi_r(u_r(\sigma^*, \tau^*)) + \phi'_r(u_r(\sigma^*, \tau^*)) \frac{\phi'_r(u_r(\underline{\sigma}, \tau^*)) - \phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} \epsilon \right) \\ & = \phi_r^{-1}(\phi_r(u_r(\sigma^*, \tau^*))) + (\phi_r^{-1})'(\phi_r(u_r(\sigma^*, \tau^*))) \phi'_r(u_r(\sigma^*, \tau^*)) \frac{\phi'_r(u_r(\underline{\sigma}, \tau^*)) - \phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} \epsilon \\ & = u_r(\sigma^*) + \frac{\phi'_r(u_r(\underline{\sigma}, \tau^*)) - \phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} \epsilon, \end{aligned}$$

where the last equality follows from

$$(\phi_r^{-1})'(\phi_r(u_r(\sigma^*, \tau^*))) = \frac{1}{\phi'_r(\phi_r^{-1}(\phi_r(u_r(\sigma^*, \tau^*))))} = \frac{1}{\phi'_r(u_r(\sigma^*, \tau^*))}.$$

Thus, after ignoring the terms of order  $\epsilon^2$ , the receiver's payoff change when moving from ambiguity neutrality to ambiguity aversion is

$$u_r(\sigma^*, \tau^*) + \frac{\phi'_r(u_r(\underline{\sigma}, \tau^*)) - \phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} \epsilon - u_r(\sigma^*, \tau^*) - \frac{\phi'_r(u_r(\underline{\sigma}, \tau^*)) - \phi'_r(u_r(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*)) + \phi'_r(u_r(\bar{\sigma}, \tau^*))} \epsilon = 0.$$

In other words, the additional cost to the receiver of this  $\epsilon$ -ambiguous experiment because the receiver is ambiguity averse is of order  $\epsilon^2$ .

Therefore, for a small enough splitting of  $\sigma^*$ , the receiver's payoff increase from the increase,  $\mu_{\bar{\sigma}} - 1/2$ , in weight on the better experiment  $\bar{\sigma}$  is first-order while their payoff decrease due to bearing the resulting ambiguity is at most second-order.

## SA.4 Proofs of auxiliary results from “Persuasion with Ambiguous Communication”

### SA.4.1 Proofs of Proposition 3 and Corollary 2

*Proof of Proposition 3.* Throughout, we view a splitting  $(\lambda_\theta, \sigma_\theta)_{\theta \in \Theta}$  of some  $\sigma \in \Sigma^*$  as a finitely supported distribution in  $\Delta(\Sigma)$ . With a slight abuse of notation, we write  $\lambda$  for the distribution,  $\Delta_{\text{simple}}(\Sigma)$  for the set of finitely supported distributions on  $\Sigma$ , and  $\mathbb{E}_\lambda$  for the expectation operator with respect to  $\lambda$ . Since ambiguous communication benefits the sender,  $\{\lambda \in \Delta_{\text{simple}}(\Sigma) : \mathbb{E}_\lambda[\sigma] \in \Sigma^*, \mathbb{E}_\lambda[\Phi_{u_s^{BP}}(\sigma)] \in (0, \infty)\} \neq \emptyset$  by Corollary 5. Since  $\Sigma^*$  is a convex set and has a non-empty interior, any point in  $\Sigma^*$  can be approached by points in the interior of  $\Sigma^*$ . As the expectations in the above set are continuous in  $\lambda$ , this implies that  $\{\lambda \in \Delta_{\text{simple}}(\Sigma) : \mathbb{E}_\lambda[\sigma] \in \text{int } \Sigma^*, \mathbb{E}_\lambda[\Phi_{u_s^{BP}}(\sigma)] \in (0, \infty)\} \neq \emptyset$ . Furthermore, since  $\text{int } \Sigma^* \times (0, \infty)$  is open in the natural product topology, this set is open.  $\square$

*Proof of Corollary 2.* By Proposition 3, there exists a non-empty open set of ambiguous experiments that benefit the sender. Fix one. Since  $\mathbb{E}_\lambda[\Phi_{u_s^{BP}}(\sigma)]$  is continuous in  $\phi'_r$ , this experiment continues to benefit the sender under small perturbations of  $\phi'_r$ . Finally, since  $\phi_r$  is concave, small perturbations of  $\phi_r$  imply small perturbations of  $\phi'_r$  (Rockafellar, 1970, Theorem 25.7, p. 248).  $\square$

### SA.4.2 Proof of Theorem A.1

*Proof of Theorem A.1.* Fix any  $u \in \mathbb{R}$ . Let  $(\sigma_\theta, \lambda_\theta)_{\theta \in \Theta}$  be feasible for the maximization problem  $\Phi^*(u)$ . Suppose that there exists a pair  $(\sigma_\theta, \sigma_{\theta'})$  with  $\lambda_\theta > 0$  and  $\lambda_{\theta'} > 0$  and such that there exists a  $\lambda \in (0, 1)$  for which,  $\Phi_u(\lambda\sigma_\theta + (1 - \lambda)\sigma_{\theta'}) > \lambda\Phi_u(\sigma_\theta) + (1 - \lambda)\Phi_u(\sigma_{\theta'})$ .

Then,  $(\sigma_\theta, \lambda_\theta)_{\theta \in \Theta}$  cannot be a solution to the maximization problem  $\Phi^*(u)$ . This can be seen from the following construction of a strict improvement satisfying the constraints in that problem: If  $\frac{\lambda_\theta}{\lambda} \leq \frac{\lambda_{\theta'}}{1 - \lambda}$ , then replacing  $\sigma_\theta$  by the merged experiment  $\lambda\sigma_\theta + (1 - \lambda)\sigma_{\theta'}$  and replacing  $\lambda_\theta$  by  $\hat{\lambda}_\theta = \frac{\lambda_\theta}{\lambda}$  and  $\lambda_{\theta'}$  by  $\hat{\lambda}_{\theta'} = \lambda_{\theta'} - (1 - \lambda)\frac{\lambda_\theta}{\lambda}$  yields such an improvement. If instead  $\frac{\lambda_\theta}{\lambda} > \frac{\lambda_{\theta'}}{1 - \lambda}$ , then replacing  $\sigma_{\theta'}$  by the merged experiment  $\lambda\sigma_\theta + (1 - \lambda)\sigma_{\theta'}$  and replacing  $\lambda_{\theta'}$  by  $\hat{\lambda}_{\theta'} = \frac{\lambda_{\theta'}}{1 - \lambda}$  and  $\lambda_\theta$  by  $\hat{\lambda}_\theta = \lambda_\theta - \lambda\frac{\lambda_{\theta'}}{1 - \lambda}$  is such an improvement.

To prove (i), towards a contradiction, suppose in the solution there exists  $(\sigma_\theta, \sigma_{\theta'}) \in \Sigma_+(u) \times \Sigma_+(u)$  with  $u_s(\sigma_\theta, \tau^*) \neq u_s(\sigma_{\theta'}, \tau^*)$  and  $\phi'_r(u_r(\sigma_\theta, \tau^*)) \neq \phi'_r(u_r(\sigma_{\theta'}, \tau^*))$ , but they are not Pareto-ranked, i.e.,  $u_s(\sigma_\theta, \tau^*) > u_s(\sigma_{\theta'}, \tau^*) > u$ , and  $u_r(\sigma_\theta, \tau^*) < u_r(\sigma_{\theta'}, \tau^*)$ . Then for any  $\lambda \in (0, 1)$ , we can show  $\Phi_u(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}) > \lambda\Phi_u(\sigma_\theta) + (1-\lambda)\Phi_u(\sigma_{\theta'})$ . To see this, observe that

$$\begin{aligned}
\Phi_u(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}) &= \frac{\phi_s(u_s(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}, \tau^*))} \\
&\geq \lambda \frac{\phi_s(u_s(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\sigma_\theta, \tau^*))} + (1-\lambda) \frac{\phi_s(u_s(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\sigma_{\theta'}, \tau^*))} \\
&\geq \lambda \frac{\lambda\phi_s(u_s(\sigma_\theta, \tau^*)) + (1-\lambda)\phi_s(u_s(\sigma_{\theta'}, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\sigma_\theta, \tau^*))} \\
&\quad + (1-\lambda) \frac{\lambda\phi_s(u_s(\sigma_\theta, \tau^*)) + (1-\lambda)\phi_s(u_s(\sigma_{\theta'}, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\sigma_{\theta'}, \tau^*))} \tag{SA.4.1} \\
&= \lambda\Phi_u(\sigma_\theta) + \lambda(1-\lambda) \frac{\phi_s(u_s(\sigma_{\theta'}, \tau^*)) - \phi_s(u_s(\sigma_\theta, \tau^*))}{\phi'_r(u_r(\sigma_\theta, \tau^*))} \\
&\quad + (1-\lambda)\Phi_u(\sigma_{\theta'}) + (1-\lambda)\lambda \frac{\phi_s(u_s(\sigma_\theta, \tau^*)) - \phi_s(u_s(\sigma_{\theta'}, \tau^*))}{\phi'_r(u_r(\sigma_{\theta'}, \tau^*))} \\
&> \lambda\Phi_u(\sigma_\theta) + (1-\lambda)\Phi_u(\sigma_{\theta'}),
\end{aligned}$$

where the first inequality follows from concavity of  $1/\phi'_r$  and positivity of  $\phi_s(u_s(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}, \tau^*)) - \phi_s(u)$ , the second inequality follows from concavity of  $\phi_s$ , and the last strict inequality follows from  $u_r(\sigma_\theta, \tau^*) < u_r(\sigma_{\theta'}, \tau^*)$  and  $\phi'_r(u_r(\sigma_\theta, \tau^*)) \neq \phi'_r(u_r(\sigma_{\theta'}, \tau^*))$ .

The proof of (ii) is the same as the proof of (i), except that the first inequality in the chain (SA.4.1) now follows from convexity of  $1/\phi'_r$  and  $\phi_s(u_s(\lambda\sigma_\theta + (1-\lambda)\sigma_{\theta'}, \tau^*)) - \phi_s(u) < 0$ .  $\square$

### SA.4.3 Proof of Theorem A.2

*Proof of Theorem A.2.* Fix any  $u \in \mathbb{R}$ . Let  $(\sigma_\theta, \lambda_\theta)_{\theta \in \Theta}$  be feasible for  $(\Phi^*(u))$ . Suppose that there exist  $\sigma_\theta$  satisfying  $\lambda_\theta > 0$  and two experiments  $\sigma$  and  $\sigma'$  such that  $\sigma_\theta = \lambda\sigma + (1-\lambda)\sigma'$  for some  $\lambda \in (0, 1)$  and  $\Phi_u(\lambda\sigma + (1-\lambda)\sigma') < \lambda\Phi_u(\sigma) + (1-\lambda)\Phi_u(\sigma')$ , then  $(\sigma_\theta, \lambda_\theta)_{\theta \in \Theta}$  cannot be a solution to  $(\Phi^*(u))$ . This follows by noting that splitting  $\sigma_\theta$  into  $\sigma$  with probability  $\lambda\lambda_\theta$  and  $\sigma'$  with probability  $(1-\lambda)\lambda_\theta$  induces a strict improvement.

To show (i), if  $1/\phi'_r(\cdot)$  is concave, towards a contradiction, we have

$$\begin{aligned}
\Phi_u(\lambda\bar{\sigma} + (1-\lambda)\underline{\sigma}) &= \frac{\phi_s(u_s(\lambda\bar{\sigma} + (1-\lambda)\underline{\sigma}, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\lambda\bar{\sigma} + (1-\lambda)\underline{\sigma}, \tau^*))} \\
&\leq \lambda \frac{\phi_s(u_s(\lambda\bar{\sigma} + (1-\lambda)\underline{\sigma}, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\bar{\sigma}, \tau^*))} + (1-\lambda) \frac{\phi_s(u_s(\lambda\bar{\sigma} + (1-\lambda)\underline{\sigma}, \tau^*)) - \phi_s(u)}{\phi'_r(u_r(\underline{\sigma}, \tau^*))}
\end{aligned}$$

$$\begin{aligned}
&= \lambda \Phi_u(\bar{\sigma}) + \lambda \frac{\phi_s(u_s(\lambda \bar{\sigma} + (1-\lambda)\underline{\sigma}, \tau^*)) - \phi_s(u_s(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\bar{\sigma}, \tau^*))} \\
&\quad + (1-\lambda) \Phi_u(\underline{\sigma}) + (1-\lambda) \frac{\phi_s(u_s(\lambda \bar{\sigma} + (1-\lambda)\underline{\sigma}, \tau^*)) - \phi_s(u_s(\underline{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*))} \\
&\leq \lambda \Phi_u(\bar{\sigma}) + \lambda(1-\lambda) \frac{\phi'_s(u_s(\bar{\sigma}, \tau^*))}{\phi'_r(u_r(\bar{\sigma}, \tau^*))} (u_s(\underline{\sigma}, \tau^*) - u_s(\bar{\sigma}, \tau^*)) \\
&\quad + (1-\lambda) \Phi_u(\underline{\sigma}) + \lambda(1-\lambda) \frac{\phi'_s(u_s(\underline{\sigma}, \tau^*))}{\phi'_r(u_r(\underline{\sigma}, \tau^*))} (u_s(\bar{\sigma}, \tau^*) - u_s(\underline{\sigma}, \tau^*)) \\
&< \lambda \Phi_u(\bar{\sigma}) + (1-\lambda) \Phi_u(\underline{\sigma}),
\end{aligned}$$

where the first inequality follows from concavity of  $1/\phi'_r$  and  $\phi_s(u_s(\lambda \bar{\sigma} + (1-\lambda)\underline{\sigma}, \tau^*)) \leq \phi_s(u)$ , the second inequality from concavity of  $\phi_s$  and the third inequality from the supposition.

The proof of (ii) is the same as the proof of (i), except that the first inequality now follows from convexity of  $1/\phi'_r$  and  $\phi_s(u_s(\lambda \bar{\sigma} + (1-\lambda)\underline{\sigma}, \tau^*)) > \phi_s(u)$ .  $\square$

#### SA.4.4 Proof of Lemma A.5

*Proof of Lemma A.5.* Define, for all integers  $k \in [1, n]$ ,

$$\mathcal{S}_k = \sum_{i=1}^k x_i \mu_i \left[ \sum_{j \neq i: j=1}^k \mu_j (y_i - y_j) \right].$$

Notice that when  $k = n$ , we have

$$\begin{aligned}
\mathcal{S}_n &= \sum_{i=1}^n x_i \mu_i \left[ \sum_{j=1; j \neq i}^n \mu_j (y_i - y_j) \right] \\
&= \sum_{i=1}^n x_i \mu_i \left[ (1 - \mu_i) y_i - \sum_{j=1; j \neq i}^n \mu_j y_j \right] \\
&= \sum_{i=1}^n x_i \mu_i \left[ y_i - \sum_{j=1}^n \mu_j y_j \right] \\
&= \left( \sum_{j=1}^n \mu_j y_j \right) \left( \sum_{i=1}^n x_i \frac{\mu_i y_i}{\sum_{j=1}^n \mu_j y_j} - \sum_{i=1}^n x_i \mu_i \right).
\end{aligned}$$

Since  $\left( \sum_{j=1}^n \mu_j y_j \right) > 0$ , it suffices to show  $\mathcal{S}_n < 0$ . We prove this by induction. Observe that

$\mathcal{S}_1 = 0$ . For  $k \geq 1$ ,

$$\begin{aligned}
\mathcal{S}_{k+1} &= \sum_{i=1}^{k+1} x_i \mu_i \left[ \sum_{j=1; j \neq i}^{k+1} \mu_j (y_i - y_j) \right] \\
&= \sum_{i=1}^k x_i \mu_i \left[ \sum_{j \neq i; j=1}^k \mu_j (y_i - y_j) \right] + \sum_{i=1}^k x_i \mu_i [\mu_{k+1} (y_i - y_{k+1})] + \\
&\quad x_{k+1} \mu_{k+1} \sum_{j=1}^k [\mu_j (y_{k+1} - y_j)] \\
&= \sum_{i=1}^k x_i \mu_i \left[ \sum_{j \neq i; j=1}^k \mu_j (y_i - y_j) \right] + \sum_{i=1}^k \mu_i \mu_{k+1} (x_i - x_{k+1}) (y_i - y_{k+1}) \\
&= \mathcal{S}_k + \underbrace{\sum_{i=1}^k \mu_i \mu_{k+1} (x_i - x_{k+1}) (y_i - y_{k+1})}_{\leq 0}.
\end{aligned}$$

For  $k = j^* - 1$ ,

$$\sum_{i=1}^{j^*-1} \mu_i \mu_{j^*} (x_i - x_{j^*}) (y_i - y_{j^*}) \leq \mu_{i^*} \mu_{j^*} (x_{i^*} - x_{j^*}) (y_{i^*} - y_{j^*}) < 0.$$

Therefore,  $0 = \mathcal{S}_1 > \mathcal{S}_{j^*} \geq \mathcal{S}_n$ . □

## SA.4.5 Proof of Lemma A.6

*Proof of Lemma A.6.* Define

$$\underline{\sigma}^\lambda = \frac{1}{1-\lambda} \sigma - \frac{\lambda}{1-\lambda} \hat{\sigma} \tag{SA.4.2}$$

where  $\lambda \in (0, 1)$ . Observe that if  $\underline{\sigma}^\lambda$  is a well-defined experiment, then  $\lambda \hat{\sigma} + (1-\lambda) \underline{\sigma}^\lambda = \sigma$ , and  $u_s(\underline{\sigma}^\lambda, \tau^*) < u_s(\sigma, \tau^*)$ ,  $u_r(\underline{\sigma}^\lambda, \tau^*) < u_r(\sigma, \tau^*)$ , so that  $(\hat{\sigma}, \underline{\sigma}^\lambda, \lambda)$  is a Pareto-ranked splitting of  $\sigma$ .

It remains to show that there exists  $\lambda \in (0, 1)$  such that  $\underline{\sigma}^\lambda$  is indeed an experiment. In other words, for each  $\omega$ ,  $\underline{\sigma}^\lambda(\cdot|\omega)$  must be a probability distribution over actions.

If  $|\text{supp}(\sigma(\cdot|\omega))| = 1$ , then  $\text{supp}(\hat{\sigma}(\cdot|\omega)) \subseteq \text{supp}(\sigma(\cdot|\omega))$  implies  $\text{supp}(\hat{\sigma}(\cdot|\omega)) = \text{supp}(\sigma(\cdot|\omega))$ . It follows that  $\underline{\sigma}^\lambda(\cdot|\omega) = \sigma(\cdot|\omega)$  for all  $\lambda \in (0, 1)$ , and is thus a distribution over actions.

If  $|\text{supp}(\sigma(\cdot|\omega))| > 1$ , embed  $\sigma(\cdot|\omega)$  into the Euclidean space  $\mathbb{R}^{|\text{supp}(\sigma(\cdot|\omega))|}$  and notice that  $\sigma(\cdot|\omega)$  is in the relative interior of the probability simplex  $\Delta(\text{supp}(\sigma(\cdot|\omega)))$ . Thus there exists  $\epsilon_\omega > 0$  such that for all  $x \in \mathbb{R}^{|\text{supp}(\sigma(\cdot|\omega))|}$  with  $\sum_i x_i = 1$ , if  $\|x - \sigma(\cdot|\omega)\| < \epsilon_\omega$ , then  $x \in \Delta(\text{supp}(\sigma(\cdot|\omega)))$ . Since  $\text{supp}(\hat{\sigma}(\cdot|\omega)) \subseteq \text{supp}(\sigma(\cdot|\omega))$ , one has  $\hat{\sigma}(\cdot|\omega) \in \Delta(\text{supp}(\sigma(\cdot|\omega)))$  as



well. Then, for all  $\lambda \in (0, 1)$ ,  $\underline{\sigma}^\lambda(\cdot|\omega) \in \mathbb{R}^{|\text{supp}(\sigma(\cdot|\omega))|}$  and  $\sum_a \underline{\sigma}^\lambda(a|\omega) = 1$ . Moreover, there exists  $\lambda_\omega > 0$  such that for all  $\lambda \in (0, \lambda_\omega)$ ,  $\|\underline{\sigma}^\lambda(\cdot|\omega) - \sigma(\cdot|\omega)\| < \epsilon_\omega$ , and thus  $\underline{\sigma}^\lambda(\cdot|\omega) \in \Delta(\text{supp}(\sigma(\cdot|\omega)))$ , making it a distribution over actions.

Because  $\Omega$  is finite,  $\underline{\lambda}(\hat{\sigma}, \sigma) \equiv \min_{\omega: |\text{supp}(\sigma(\cdot|\omega))| > 1} \lambda_\omega > 0$ . Therefore, for all  $\lambda \in (0, \underline{\lambda}(\hat{\sigma}, \sigma))$ ,  $\underline{\sigma}^\lambda$  is a well-defined experiment.  $\square$

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