Revealed Convex Preference

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Abstract

Convexity of preferences is a pervasive assumption in economic analysis, so it is important to understand precisely how this property is revealed in data. In this paper, we develop a method to test for convexity given partial preference information in the form of a pair of weak and strict (incomplete) orders. We characterize the pairs that can be extended to convex preferences and provide an efficient algorithm to verify this property. Requiring the convex preference to align with an order satisfying independence (e.g., the Euclidean order or first-order stochastic dominance) does not complicate the test. We apply our results to choices over risky or uncertain prospects and to consumer demand with linear or nonlinear budget sets, with either finitely or infinitely many observations (as in Afriat (1967), Matzkin (1991), Forges and Minelli (2009), and Reny (2015)); in this setting, we identify precisely when demand data are rationalizable by a convex and increasing preference.

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1 Introduction

The convexity assumption in individual preferences, which posits that mixtures of choices are preferred to extremes, plays a foundational role in economic analysis. It ensures the existence of equilibrium prices in competitive markets and Nash equilibria in games and enables the application of convex optimization tools to address economic problems. Despite its significance, however, few existing studies have attempted to develop practical methods for testing this assumption. In this paper, we aim to fill this gap by proposing a systematic method for testing convexity. By leveraging the observed choices of a decision maker (DM)—which encode partial information about her preferences—we propose conditions that can be easily tested for determining whether a convex preference relation consistent with these choices exists.

We model known information about the DM's preference as a pair of binary relations $p = (\trianglerighteq, \succ)$ on the space of alternatives X, which we call an *order pair*. The relation $\trianglerighteq$ records weak preference information and $\succ$ records strict preference information: $x \trianglerighteq y$ (respectively, $x \succ y$) means that the DM weakly (respectively, strictly) prefers x to y. This framework accommodates both observed choice behavior and additional restrictions on preference. For example, in the n-dimensional consumption space, if a consumer is observed to choose x from a budget set A and the researcher additionally requires strict monotonicity on the consumer's preference, then the data imply $x \trianglerighteq y$ for all $y \in A$, while strict monotonicity can be imposed by taking $\succ$ to be the strict Euclidean order $>_n$.[1]

A preference $\succsim^*$ *extends* an order pair $p = (\trianglerighteq, \succ)$, or is a *completion* of p, if $\trianglerighteq \subseteq \succsim^*$ and $\succ \subseteq \succ^*$, where $\succ^*$ is the asymmetric part of $\succsim^*$. A preference is a *convex completion* of p if it is convex and extends p. Our main result, Theorem 1, characterizes the order pairs that admit convex completions: an order pair admits a convex completion if and only if it satisfies the *separating property* (SP) on every finite menu.[2] The key idea of SP is to use convexity to construct, for each finite menu, the largest set of alternatives that must be weakly preferred to the menu's least preferred alternative.

Specifically, fix a finite menu A and let x^* be the DM's least preferred alternative in A. If the DM's preference is convex, then every alternative in the convex hull of A, denoted A^0, must be weakly preferred to x^*. We then use the weak component $\trianglerighteq$ of the order pair p to expand A^0: if $x \trianglerighteq y$ for some $y \in A^0$, then x must also be weakly preferred to x^*. Taking the convex hull after adding these alternatives yields a larger set A^1. Iterating this procedure generates an increasing sequence $(A^k)_{k\geq 0}$. The set $B = \bigcup_{k=0}^{\infty} A^k$ is the maximal set whose elements must all be weakly preferred to x^*. Hence, we cannot have $x^* \succ x$ for any $x \in B$. This motivates SP: p satisfies SP on A if there exists $x^* \in A$ such that

[1] For any two n-dimensional vectors x and y, $x >_n y$ if $x \neq y$ and for all $k \in 1, ..., n, x_k \geq y_k$.

[2] A menu is a nonempty subset of alternatives.

$x^* \not\succ x$ for all $x \in B$. Theorem 1 states that p admits a convex completion if and only if it satisfies SP on every finite menu contained in $G_p := \{x \in X : x \succ y \text{ for some } y\}$.

By Theorem 1, when G_p is finite, checking whether p admits a convex completion requires checking SP on up to $2^{|G_p|}$ menus contained in G_p. Theorem 2 proposes an algorithm that simplifies this task: to verify that all menus contained in a finite set A satisfy SP, it suffices to check SP along a decreasing sequence (in the sense of set inclusion) of menus in A. Consequently, when G_p is finite, verifying whether p admits a convex completion requires at most $|G_p|$ SP checks.

In many applications, we seek to impose additional conditions on the DM's preference. For instance, we may require the DM's preference over consumption bundles to respect the Euclidean order and her preference over lotteries to respect first- or second-order stochastic dominance. Such requirements can be expressed as preorders and incorporated directly into the order pair. Once imposed, the corresponding set G_p may become infinite, which would seem to require infinitely many SP checks. Theorem 3 shows that if the imposed preorder satisfies *independence*—as is the case for the Euclidean order and for first- and second-order stochastic dominance—then we can revise p to a new order pair q such that it suffices to check whether q satisfies SP on all finite menus contained in a particular set H. In the applications we consider, H is finite, so additionally requiring the DM's preference to align with a preorder with the independence property does not complicate the testing.

We apply Theorem 3 to test whether DMs have convex preferences in various settings. In Section 4.1, we consider a finite data set $\{(x^t, B^t)\}_{t \in T}$ in the n-dimensional consumption space. Each observation (x^t, B^t) includes a budget set B^t and a bundle x^t chosen by the consumer in B^t. We construct an order pair based on the consumer's choices and the strict monotonicity of the consumer's preference. We show that the order pair satisfies SP on every finite menu contained in $\{x^t\}_{t \in T}$ if and only if consumer's choices are consistent with the maximization of a strictly increasing and convex preference. We then study the testing of strictly increasing, convex, and *symmetric* preferences in the same context.

In Section 4.2, we apply Theorem 3 to test the convexity of DMs' preferences over risky or uncertain prospects. Convex preferences over risky prospects can arise when the DM entertains multiple utility functions and seeks to make cautious or conservative choices.[3] Convex preferences over state-dependent prospects—i.e., acts (Savage, 1954; Anscombe and Aumann, 1963)—are also widely used to model ambiguity aversion.[4] In these settings, the DM's preference is often required to be consistent with a preorder such as first- or second-order stochastic dominance, or state-wise monotonicity; each

[3] See, for instance, Maccheroni (2002) and Cerreia-Vioglio et al. (2015).

[4] See Gilboa and Schmeidler (1989), Klibanoff et al. (2005), Maccheroni et al. (2006), Strzalecki (2011), Cerreia-Vioglio et al. (2011), etc.

satisfies independence. A preorder with the independence property can naturally arise as the part of the DM's preference about which she is confident (Dubra et al., 2004). We study data consisting of finitely many strict comparisons $\{(x^t, y^t)\}_{t\in T}$, where for each $t \in T$ the DM strictly prefers x^t to y^t. We show that when the imposed preorder admits a multi-utility representation (Dubra et al., 2004) with finitely many utility functions, continuity comes as a free lunch: if there exists a convex preference consistent with the data and the preorder, then there also exists a convex and continuous preference.

In the consumer choice setting, when the data set contains finitely many observations with linear budget sets,[5] Afriat (1967) shows that if the observed choices pass the rationality test, then they can be rationalized by a *continuous* and *convex* preference. Reny (2015) extends this insight to infinite data sets, showing that convexity can still be preserved even though continuity may fail. A parallel result for *strictly* convex rationalizations, under the condition that the complement of each budget set is convex, is established by Matzkin (1991).

In Section 5, we generalize the convexity conclusions of Afriat (1967) and Reny (2015). We identify a condition under which an order pair $p = (\trianglerighteq, \succ)$ admits a completion if and only if it admits a *convex* completion. Specifically, we call p *pre-convex* if (i) for every alternative x such that $x \trianglerighteq x'$ for some $x' \neq x$, the set $\{z : x \ntrianglerighteq z\}$ is convex, and (ii) for every alternative y, the set $\{z : y \nsucc z\}$ is convex. Theorem 8 shows that a pre-convex p admits a convex completion precisely when it admits a completion. We then show how the choice data studied by Afriat (1967) and Reny (2015) can be systematically translated into pre-convex order pairs, thereby recovering their convexity results as special cases of our theorem. We further apply Theorem 8 to characterize when a data set $\mathcal{O} = \{(A^t, B^t)\}_{t\in T}$ is *exactly* rationalizable by a convex preference under an assumption on the budget sets analogous to that in Matzkin (1991). In particular, we provide conditions for the existence of a convex preference $\succsim$ such that, for each $t \in T$, $A^t = \max(B^t; \succsim)$.

In Section 6, we provide additional discussion and extensions of our results. We first study preference identification under the convexity assumption. Using our main characterization, we derive conditions under which an alternative x is weakly preferred to, or strictly preferred to, another alternative y for *every* convex preference consistent with the data. We then present a parallel characterization for order pairs that admit *strictly convex* completions.

The remainder of the paper is organized as follows. Section 2 introduces basic concepts and notation. Section 3 presents our main characterization theorem and an accompanying algorithm for empirical testing. Section 4 extends the framework by requiring convex completions to be consistent with a preorder that satisfies independence, with applications

[5] Formally, a linear budget set takes the form $\{x \in \mathbb{R}^n_+ : x \cdot p^t \leq I^t\}$, where $p^t \in \mathbb{R}^n_{++}$ denotes the price vector and $I^t > 0$ is income.

to consumer choice and to decisions under risk and uncertainty. Section 5 identifies conditions under which convexity imposes no additional testable implications beyond rationality. Section 6 provides further discussion and extensions. The Appendix contains all proofs and additional results omitted from the main text.

1.1 Related Literature

This paper contributes to the revealed preference literature by providing a unified framework for testing convex preferences.[6] Classical works, such as Richter (1966), Afriat (1967), and Fishburn (1976), pioneered the characterization of rationalizable choices, i.e., choices consistent with the maximization of a preference. Forges and Minelli (2009) characterize rationalizable choices made in non-linear budget sets in the consumer choice setting. More recently, Nishimura et al. (2017) developed a general framework to study choices that are rationalizable by a continuous preference that extends a given order. Similarly, we examine the case where the DM's preference is required to be aligned with a given preorder, and show that when the preorder satisfies independence, testing for convexity can be significantly simplified.

A key contribution of our analysis lies in identifying a condition under which imposing convexity on the DM's preference entails no additional restriction beyond rationality. While analogous findings appear in the existing literature (Afriat, 1967; Matzkin, 1991; Reny, 2015), our theorem generalizes existing results on recovering convex preferences by relying solely on the ordinal properties of the order pairs. This distinction enables our result to be applied to more general choice environments.

In the consumer choice setting, Cherchye et al. (2014) study the testing of convex and strictly increasing preferences. They propose replacing each budget set with its minimal co-convex hull and demonstrate that this approach reduces the testing to simply checking whether the consumer's choices are rationalizable or not. Furthermore, they show that when the budget sets are finite unions of polyhedral convex sets, their approach is practically useful. In contrast, the condition we develop does not rely on assumptions about the budget sets and is directly implementable. For more details and discussions, see Section 4.1.

Our paper also connects to the literature on preference extension. The foundational characterizations of completable incomplete preferences are provided by Szpilrajn (1930) and Suzumura (1976). Subsequent research in topological order theory has examined conditions under which incomplete preferences can be extended to continuous preferences, as explored in Levin (1983), Herden (1989), Herden and Pallack (2002), and Nishimura

[6] See Chambers and Echenique (2016) for a comprehensive review of the revealed preference literature.

et al. (2017), among others. Additionally, Caradonna and Chambers (2024) investigate the problem of extending incomplete preferences to invariant preferences.

Among the literature on preference extension, our paper is most closely related to Scapparone (1999), Demuynck (2009), and Bossert and Sprumont (2009). Scapparone (1999) provides a necessary and sufficient condition under which a *strict* order can be extended to a convex preference. Our main theorem generalizes this result by also considering weak orders. Demuynck (2009) presents a general method for extending incomplete preferences and applies it to characterize those that can be extended to preferences satisfying a slightly stronger condition than convexity. Thus, the condition of Demuynck (2009) is sufficient but not necessary for an incomplete preference to admit a convex completion. Bossert and Sprumont (2009) investigates the problem of extending a weak order to a strictly convex and strictly increasing preference in the consumption space. Relating to Bossert and Sprumont (2009), we characterize all order pairs that allow for strictly convex completions in Section 6.2 and generalize their characterization result in Proposition 3. In addition to characterizing convex-completable orders, our paper also provides detailed guidelines for simplifying the test, which is not the focus of the above papers.

2 Preliminaries

Let a nonempty and convex set X be the space of alternatives. A *binary relation* $\succsim$ is a subset of $X \times X$ for which we write $x \succsim y$ for $(x, y) \in \succsim$. The convolution of two binary relations $\succsim$ and $\trianglerighteq$, denoted by $\succsim \diamond \trianglerighteq$, is a binary relation defined such that $x \succsim \diamond \trianglerighteq z$ if and only if $x \succsim y \trianglerighteq z$ for some $y \in X$. A binary relation $\succsim$ is *reflexive* if for all $x \in X$, $x \succsim x$; *complete* if for all $x, y \in X$, either $x \succsim y$ or $y \succsim x$; *transitive* if for all $x, y, z \in X$, $x \succsim y$ and $y \succsim z$ imply $x \succsim z$; and *asymmetric* if for all $x, y \in X$, $x \succsim y$ implies not $y \succsim x$. The *symmetric* and *asymmetric* parts of a given binary relation $\succsim$ are binary relations, usually denoted by $\sim$ and $\succ$ respectively, such that $x \sim y$ if $x \succsim y$ and $y \succsim x$, and $x \succ y$ if $x \succsim y$ and not $y \succsim x$.[7] A *preorder* is a reflexive and transitive binary relation. A *partial order* is a asymmetric and transitive binary relation. A *preference* is a complete preorder. We say that a preference $\succsim$ allows for a utility representation if there exists a utility function $u : X \to \mathbb{R}$ such that for all $x, y \in X$, $x \succsim y$ if and only if $u(x) \geq u(y)$. Such a utility function u is said to *represent* $\succsim$. Throughout the paper, we will use the notation $\trianglerighteq_I$ to denote the minimal reflexive binary relation, i.e., $\trianglerighteq_I = \{(x, x) : x \in X\}$. A *menu* is referred to as a nonempty subset of X. For any menu A and preference $\succsim$, denote by $\max(A; \succsim)$ the set of $\succsim$-optimal alternatives in A, that is, $\max(A; \succsim) = \{x \in A : x \succsim y \text{ for all } y \in A\}$.

[7] We also use $\triangleright$ and $\bowtie$ to denote the asymmetric and symmetric parts of $\trianglerighteq$, respectively.

For any $A \subseteq X$, denote by $co(A)$ the *convex hull* of A, i.e., the smallest convex set that contains A.[8] A preference $\succsim$ is said to be *convex* if for all $x, y \in X$ and $t \in [0, 1]$, $x \succsim y$ implies $tx + (1-t)y \succsim y$. There are three other equivalent characterizations of the convexity of $\succsim$: (i) for every *finite menu* A of X, if y is the $\succsim$-worst alternative in A, then for all $x \in co(A)$, $x \succsim y$; (ii) for all $x \in X$, the upper counter set $\{y \in X : y \succsim x\}$ is convex; (iii) for all $x \in X$, the strict upper counter set $\{y \in X : y \succ x\}$ is convex. A preference $\succsim$ is *strictly convex* if for all distinct $x, y \in X$ and $t \in (0, 1)$, $x \succsim y$ implies $tx + (1-t)y \succ y$.

3 Convex Completion

In this section, we introduce our primitive, order pairs, which capture weak and strict preference information of DMs, and characterize those that can be extended to convex preferences. We then provide an algorithm to simplify the test for convexity.

3.1 Main Result

A pair of binary relations $(\trianglerighteq, \succ)$ on X is an *order pair* if $\trianglerighteq$ is reflexive. The relation $\trianglerighteq$ captures weak preference information: $x \trianglerighteq y$ indicates that the DM weakly prefers x to y. The relation $\succ$ comprises strict preference information: $x \succ y$ means that the DM strictly prefers x to y. [9]

To see the generality of our framework, consider the following two examples.

Example 1. Consider the classic revealed preference problem studied by Afriat (1967). A data set is given by $\{(x^t, p^t)\}_{t \in T}$, where for each t, $x^t \in \mathbb{R}^n_+$ is the observed choice bundle by the consumer, and $p^t \in \mathbb{R}^n_{++}$ is the price vector under which the consumer purchases x^t. Under the local non-satiation assumption, for each t, x^t should be weakly preferred to any bundle y that is affordable (i.e., $y \cdot p^t \leq x^t \cdot p^t$) and strictly preferred to any bundle z that is strictly affordable (i.e., $z \cdot p^t < x^t \cdot p^t$).[10] Define

$$\trianglerighteq = \{(x, y) : x = x^t \text{ and } y \cdot p^t \leq x^t \cdot p^t \text{ for some } t\},$$

$$\succ = \{(x, y) : x = x^t \text{ and } y \cdot p^t < x^t \cdot p^t \text{ for some } t\}.$$

The order pair $p = (\trianglerighteq, \succ)$ captures the information we reveal from the choices. □

[8] Alternatively, an alternative x is in $co(A)$ if and only if there exist finite alternatives $\{x^k\}_{k=1}^n \subseteq A$ and $\{\alpha^k\}_{k=1}^n \subseteq (0, 1]$ with $\sum_{k=1}^n \alpha^k = 1$ such that $x = \sum_{k=1}^n \alpha^k x^k$.

[9] Similar primitive is adopted in Caradonna and Chambers (2024). While Caradonna and Chambers (2024) require $\succ$ to be a subset of $\trianglerighteq$, we do not require so.

[10] The local non-satiation assumption on the consumer's preference states that for any bundle $x \in \mathbb{R}^n_+$ and any $\epsilon > 0$, there exists another bundle y with $|y - x| < \epsilon$ such that y is strictly preferred to x.

Example 2. Consider a participant in the lab who chooses option x^t from the binary menu $\{x^t, y^t\}$ for each $t \in T$, indicating that those choices reflect her strict preference. If in addition, we know that a preorder $\trianglerighteq^*$ is part of the participant's preference (Nishimura et al., 2017), then the order pair p can be constructed as

$$p = (\bowtie^*, \triangleright^* \cup \{(x^t, y^t)\}_{t \in T}),$$

where $\bowtie^*$ and $\triangleright^*$ are the symmetric and asymmetric parts of $\trianglerighteq^*$, respectively. □

Below, we define convex completions of order pairs.

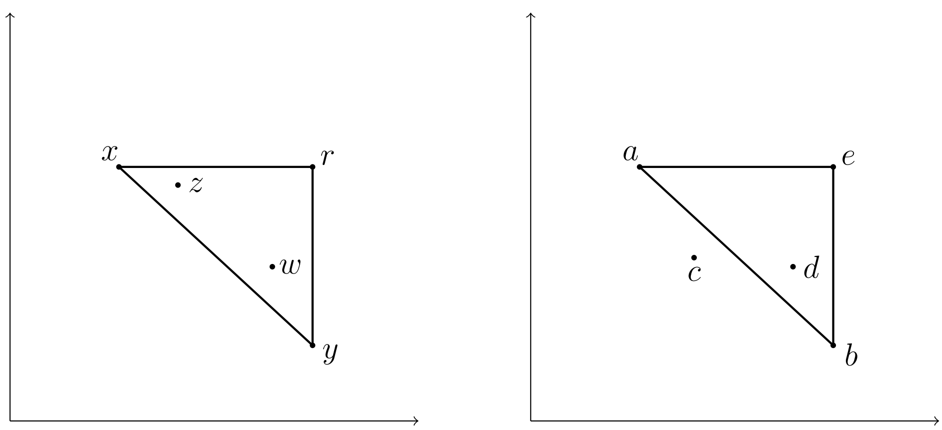
Definition 1. *A binary relation $\succsim^*$ extends an order pair $p = (\trianglerighteq, \succ)$ if $\trianglerighteq \subseteq \succsim^*$ and $\succ \subseteq \succ^*$. The binary relation $\succsim^*$ is a completion of p if it is a preference and extends p. The preference $\succsim^*$ is a convex (strictly convex) completion of p if it is a convex (strictly convex) preference and is a completion of p.*

If $\succsim^*$ is a convex completion of p, then both the weak ($\succsim^*$) and strict ($\succ^*$) parts of the preference are consistent with the corresponding orders of p. When p is constructed based on the DM's observed choices, checking whether p admits a convex completion amounts to verifying the standard economic assumption that the DM optimizes a convex preference. To develop intuition about which order pairs admit convex completions, consider the following example.

Example 3. Let X be the 2-dimensional consumption space $\mathbb{R}^2_+$. Consider an order pair $(\trianglerighteq, \succ)$ such that

$$r \trianglerighteq x,\ x \succ z,\ y \succ w,$$

where the alternatives are depicted in Figure 1(a).[11] We show that such a pair has no



(a) no convex completion (b) convex completion exists

Figure 1: Existence of Convex Completions

[11] While $\trianglerighteq$ is reflexive (i.e., $\trianglerighteq_I \subseteq \trianglerighteq$), we omit the trivial part $\trianglerighteq_I$ for notational simplicity.

convex completion. Suppose to the contrary that a convex preference $\succsim^*$ extends it. Let $\underline{x}$ be the $\succsim^*$-worst alternative in $\{x, y, r\}$. By the convexity of $\succsim^*$, for all $\hat{x} \in co(\{x, y, r\})$, we have $\hat{x} \succsim^* \underline{x}$. Thus, for all $\hat{x} \in \{x, y, z, w, r\}$, we have $\hat{x} \succsim^* \underline{x}$. Since $x \succ z$ implies $x \succ^* z$, we have $\underline{x} \neq x$. Similarly, $y \succ w$ implies $\underline{x} \neq y$. By $r \trianglerighteq x \succ z$ and the transitivity of $\succsim^*$, we have $r \succ^* z$, which implies $\underline{x} \neq r$. This leads to a contradiction.

Within the same space X, consider another order pair $(\trianglerighteq', \succ')$ with

$$e \trianglerighteq' a,\ a \succ' c,\ b \succ' d,$$

where the alternatives are depicted in Figure 1(b). For this pair, the previous contradiction no longer arises: If $(\trianglerighteq', \succ')$ allows for a convex completion $\succsim^*$, then in $\{a, b, e\}$, a can serve as the $\succsim^*$-worst alternative. This is because the only alternative that is known to be strictly worse than a is c, which lies outside of the convex hull of $\{a, b, e\}$.

In fact, we can explicitly construct a convex completion $\succsim^*$ (with $\sim^*$ denoting its symmetric part) of $(\trianglerighteq', \succ')$ as follows. Let $A = co(\{a, b, e\})$ and $B = co(\{b, e\})$. For all $x', y' \in B$ and $z' \in X \backslash B$, let $x' \sim^* y' \succ^* z'$. For all $x', y' \in A \backslash B$ and $z' \in X \backslash A$, let $x' \sim^* y' \succ^* z'$. For all $x', y' \in X \backslash A$, let $x' \sim^* y'$. Essentially, $\succsim^*$ ranks alternatives in B the best, alternatives in $A \backslash B$ the second best, and alternatives in $X \backslash A$ the worst. The preference $\succsim^*$ is convex, because for all $x' \in X$, the upper counter set $\{y' \in X : y' \succsim^* x'\}$ equals either X, A, or B—all of which are convex. The preference $\succsim^*$ extends the pair $(\trianglerighteq', \succ')$, as it induces the preference $b \sim^* e \succ^* a \sim^* d \succ^* c$, consistent with $(\trianglerighteq', \succ')$. □

Given an order pair $p = (\trianglerighteq, \succ)$, for any $x \in X$, define

$$x_p^{\downarrow\downarrow} = \{y \in X : x \succ y\} \tag{1}$$

as the set of alternatives that are strictly worse than x. The intuition from the preceding example is that if an order pair $p = (\trianglerighteq, \succ)$ admits a convex completion $\succsim^*$, then for any finite menu A, all alternatives in $co(A)$ are weakly better than the $\succsim^*$-worst alternative in A (call it x). Since alternatives in $x_p^{\downarrow\downarrow}$ are strictly worse than x, the two sets $co(A)$ and $x_p^{\downarrow\downarrow}$ must be disjoint. Extending this observation, we next identify the largest set B such that for every convex completion $\succsim^*$ of p, all alternatives in B are weakly preferred to the $\succsim^*$-worst alternative in A. Hence, B must be disjoint from $x_p^{\downarrow\downarrow}$ for some $x \in A$.

Fix an order pair $p = (\trianglerighteq, \succ)$ and a finite menu A. Suppose that p admits a convex completion $\succsim^*$ with $\succsim^*$-worst alternative in A denoted $\underline{x}$. Define

$$\Pi_p^0(A) = co(A).$$

By convexity of $\succsim^*$, every alternative in $\Pi_p^0(A)$ is weakly preferred to $\underline{x}$ under $\succsim^*$.

Recursively define for each $k \in \mathbb{N}$:

$$\Pi_p^{k+1}(A) = co\left(\bigcup_{x \in \Pi_p^k(A)} \{y : y \trianglerighteq x\}\right).$$

The set $\Pi_p^{k+1}(A)$ represents the convex hull of all alternatives that are known to be weakly preferred (under $\trianglerighteq$) to at least one alternative in $\Pi_p^k(A)$.

Observe that if $x \succsim^* \underline{x}$ holds for all $x \in \Pi_p^k(A)$, then by $\trianglerighteq \subseteq \succsim^*$ and convexity of $\succsim^*$, this property extends to all $x \in \Pi_p^{k+1}(A)$. Since the initial case $\Pi_p^0(A)$ satisfies $x \succsim^* \underline{x}$ for all $x \in \Pi_p^0(A)$, induction implies this to hold for all $\Pi_p^k(A)$, $k \in \mathbb{N}$.

Define the limit set:

$$\Pi_p(A) := \bigcup_{k=0}^{\infty} \Pi_p^k(A).$$

This set $\Pi_p(A)$ is maximal with respect to two properties: (i) its construction relies only on the weak preference information $\trianglerighteq$ and the convexity of $\succsim^*$, and (ii) all its alternatives x satisfy $x \succsim^* \underline{x}$. Moreover, since each $\Pi_p^k(A)$ is convex and the sequence $(\Pi_p^k(A))_{k=0}^{\infty}$ is nested increasing under set inclusion, $\Pi_p(A)$ is also convex.

Since $\Pi_p(A)$ consists of alternatives that are weakly preferred to $\underline{x}$ under $\succsim^*$, we have $\Pi_p(A) \cap \underline{x}_p^{\downarrow\downarrow} = \emptyset$. While the convex completion $\succsim^*$ is unknown *a priori*, the construction of $\Pi_p(A)$ depends only on $\trianglerighteq$ and the convexity of $\succsim^*$, without requiring explicit knowledge of $\succsim^*$ itself. Similarly, although the $\succsim^*$-worst alternative $\underline{x}$ in A depends on the specific preference $\succsim^*$, the finiteness of menu A guarantees the existence of such an alternative. These observations motivate the following necessary condition for $(\trianglerighteq, \succ)$ to admit a convex completion.

Definition 2. *An order pair $p = (\trianglerighteq, \succ)$ satisfies the separating property (SP) on a finite menu A if there is $x \in A$ such that $x_p^{\downarrow\downarrow} \cap \Pi_p(A) = \emptyset$.*

The following example demonstrates how to identify $\Pi_p(A)$ and checking SP.

Example 4. Let X be the 2-dimensional consumption space $\mathbb{R}_+^2$. Consider an order pair $p = (\trianglerighteq, \succ)$ such that:

$$a \trianglerighteq r,\ b \trianglerighteq r,\ c \trianglerighteq r,\ x \succ d,\ y \succ d,\ z \succ d,$$

where the alternatives are depicted in Figure 2.

We demonstrate that p violates SP on menu $A = \{x, y, z\}$. First, $\Pi_p^0(A) = co(A)$. Since $r \in co(A)$, the set $\Pi_p^1(A)$ expands to include all alternatives weakly preferred to r under $\trianglerighteq$. Consequently, $\Pi_p^1(A)$ additionally contains a, b, and c, yielding $\Pi_p^1(A) = co(\{a, b, c\})$.

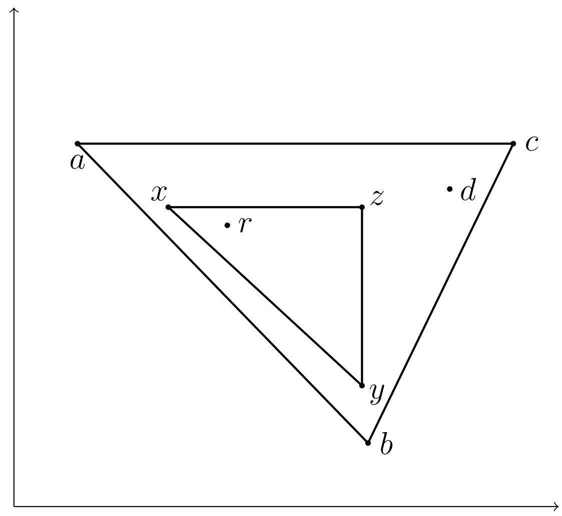


Figure 2: Violation of SP

As no further alternatives satisfy the inclusion criteria, $\Pi_p(A) = co(\{a, b, c\})$. However, SP is violated on A because $d \in \Pi_p(A)$, and $x \succ d$, $y \succ d$ and $z \succ d$ all hold. □

To state our main characterization result, we define

$$G_p = \{x \in X : x \succ y \text{ for some } y\} \tag{2}$$

as the set of alternatives that are strictly better than at least one alternative. Our main theorem states that to ensure p to have a convex completion, it suffices the check whether p satisfies SP on every finite menu *contained in* G_p.

Theorem 1. *Consider an order pair p, the following statements are equivalent.*

(i) *p admits a convex completion.*

(ii) *p satisfies SP on every finite menu.*

(iii) *p satisfies SP on every finite menu contained in G_p.*

To see why statements (ii) and (iii) are equivalent, note that for any finite menu A, if A contains some alternative $x \notin G_p$, then $x_p^{\downarrow\downarrow} = \emptyset$ directly implies $\Pi_p(A) \cap x_p^{\downarrow\downarrow} = \emptyset$. Hence, p automatically satisfies SP on A. Notably, if $G_p = \emptyset$, meaning that the strict binary relation $\succ$ is trivial, then the order pair has a trivial convex completion, under which all alternatives are indifferent. We defer a proof sketch of the implication from (ii) to (i) to the end of Section 3.2, after introducing the algorithm that simplifies testing whether p satisfies SP on every menu contained in a finite set A. Below, we have several remarks on Theorem 1.

Remark 1. A given order pair $p = (\trianglerighteq, \succ)$ admits a convex completion if and only if $q = (\trianglerighteq \setminus \succ, \succ)$ does.[12] Practically, it is more convenient to check SP for q rather than for

[12] To ensure that q is an order pair, one can alternatively define q as $q = (\trianglerighteq_I \cup (\trianglerighteq \setminus \succ), \succ)$.

p, since q contains less weak preference relations.

Remark 2. The condition stated in Theorem 1 *cannot* be weakened to requiring that the order pair satisfies SP only on every *singleton* menu: A pair may satisfy SP on all singleton menus yet fail to admit a convex completion. For a counterexample, consider the pair p in Example 4. One can verify that for every $\bar{x} \in X$, $\Pi_p(\{\bar{x}\}) \cap \bar{x}_p^{\downarrow\downarrow} = \emptyset$ holds. Nevertheless, as demonstrated, p does not admit any convex completion.

Remark 3. An equivalent characterization to that in Theorem 1, which avoids the existence quantifier, is that for all finite menu A, $A \not\subseteq \{x : x \succ y \text{ for some } y \in \Pi_p(A)\}$. The condition stated in the theorem is more useful for conducting empirical tests for convexity, as will be demonstrated in Sections 3.2 and 4.

Remark 4. When the order pair $p = (\trianglerighteq_I, \succ)$, i.e., when the weak binary relation carries no preference information, the condition in Theorem 1 reduces to the requirement that for every finite menu A,

$$x_p^{\downarrow\downarrow} \cap co(A) = \emptyset \text{ for some } x \in A. \tag{3}$$

Scapparone (1999) studies such an order pair p and shows that it admits a convex completion if and only if for every finite menu B,

$$B \not\subseteq co\left(\{x : x \succ y \text{ for some } y \in B\}\right). \tag{4}$$

To see that the two characterization conditions are equivalent, first consider some finite menu A that violates condition (3). For each $x \in A$, pick $d(x) \in co(A)$ such that $x \succ d(x)$. Define $B = \{d(x)\}_{x \in A}$. It follows that $A \subseteq \{x : x \succ y \text{ for some } y \in B\}$. Since $B \subseteq co(A)$, B violates condition (4).

Conversely, consider some finite menu B that violates condition (4). For each $y \in B$, let $H(y)$ be a finite menu such that $y \in co(H(y))$ and for all $x \in H(y)$, there exists $y' \in B$ with $x \succ y'$. Define $A = \bigcup_{y \in B} H(y)$. Since $B \subseteq co(A)$, A violates condition (3).

Remark 5. Suzumura (1976) provides a characterization condition for incomplete preferences—which contain both weak and strict components—to admit completions.[13] Our main result, Theorem 1, can be viewed as a counterpart of the extension result of Suzumura (1976) for convex preferences. Following Theorem 3 of Suzumura (1976), a necessary and sufficient condition for an order pair $p = (\trianglerighteq, \succ)$ to admit a completion is that p is *consistent*, that is, there exists *no* cycle $(x^k)_{k=1}^n$ with $n \geq 2$ and $x^1 = x^n$ such that $x^{k+1}(\trianglerighteq \cup \succ)x^k$ for all $k \in \{1, \ldots, n-1\}$ and $x^1 \succ x^n$.[14]

[13] More concretely, Suzumura (1976) considers a single binary relation $\trianglerighteq$ but requires the preference $\succsim$ to extend the order pair $(\trianglerighteq, \triangleright)$.

[14] An order pair $p = (\trianglerighteq, \succ)$ is also said to be consistent on a set A if it contains no cycle in A.

By Theorem 1, if p satisfies SP on every finite menu, it must be consistent. We offer a direct proof as follows: Suppose, for contradiction, that the consistency condition fails for some sequence $(x^k)_{k=1}^n$ with $n \geq 2$. Define the set

$$K = \{k \in \{2, \dots, n\} : x^k \succ x^{k-1}\},$$

so that for all $k \in \{2, \dots, n\} \backslash K$, we have $x^k \trianglerighteq x^{k-1}$. Consider the menu $A = \{x^k\}_{k \in K \cup \{1\}}$. By our iterative inclusion of $\trianglerighteq$-better alternatives, we have $\{x^k\}_{k=1}^n \subseteq \Pi_p(A)$. However, for every $k \in K \cup \{1\}$, either $x^k \succ x^{k-1}$ (if $k \geq 2$), or $x^k \succ x^n$ (if $k = 1$), indicating that p violates SP on A, a contradiction.

Remark 6. Demuynck (2009) provides the characterization condition for an order pair $p = (\trianglerighteq, \triangleright)$ to allow for a completion that satisfies a stronger property than convexity. Particularly, a preference $\succsim$ satisfies the property considered by Demuynck (2009) if it is convex and additionally satisfies that for all finite menu $A = \{x^1, ..., x^n\}$, alternative y, and $(\alpha^i)_{i=1}^n \in (0,1)^n$ with $\sum_{i=1}^n \alpha^i = 1$, if for all $x^i \in A$, $x^i \succsim y$, and for some $x^j \in A$, $x^j \succ y$, then $\sum_{i=1}^n \alpha^i x^i \succ y$.[15] To see the difference between this property and convexity in terms of preference extension, consider the following example.

Let the order pair $p = (\trianglerighteq, \triangleright)$ be such that $\trianglerighteq = \trianglerighteq_I \cup \{(x, y), (x, \frac{1}{2}x + \frac{1}{2}y), (y, \frac{1}{2}x + \frac{1}{2}y), (\frac{1}{2}x + \frac{1}{2}y, y)\}$, where x and y are distinct alternatives. By the order pair, x is strictly better than both y and $\frac{1}{2}x + \frac{1}{2}y$ while the latter two are equally good. By Theorem 1, p allows for a convex completion, since $G_p = \{x\}$ and $\Pi_p(x) = \{x\}$ does not contain y nor $\frac{1}{2}x + \frac{1}{2}y$. However, p does not admit a completion that satisfies the stronger property of Demuynck (2009), since $x \triangleright y$ should imply that $\frac{1}{2}x + \frac{1}{2}y$ is strictly better than y, which contradicts the condition $y \trianglerighteq \frac{1}{2}x + \frac{1}{2}y$. Hence, the characterization condition of Demuynck (2009) is sufficient but not necessary for convex completions.[16]

3.2 Algorithm for Checking SP

In this section, we show that verifying whether $p = (\trianglerighteq, \succ)$ satisfies SP on all submenus within a given finite menu A reduces to checking SP on a descending sequence (in the set-inclusion sense) of menus in A. This reduction decreases the number of required SP checks from exponential ($2^{|A|}$) to linear (at most $|A|$) in the menu size. Consequently, when the set G_p defined in (2) is finite, determining whether p admits a convex completion requires checking SP at most $|G_p|$ times. Below, we present the algorithm.

Algorithm I. For any order pair p and finite menu A, set $A^0 = A$ and $k = 1$.

[15] Note that this requirement is weaker than strict convexity, as strict convexity implies $\sum_{i=1}^n \alpha^i x^i \succ y$ without the condition that $x^j \succ y$ for some j.

[16] See Section 3.2 of Demuynck (2009) for more details of the characterization condition.

START. Let $A^k = \{x \in A^{k-1} : x_p^{\downarrow\downarrow} \cap \Pi_p(A^{k-1}) \neq \emptyset\}$. Consider the following mutually exclusive cases:

(a) $A^k = \emptyset$: Stop and output "SP holds".

(b) $\emptyset \neq A^k \subsetneq A^{k-1}$: Go to START with $k = k + 1$.

(c) $\emptyset \neq A^k = A^{k-1}$: Stop and output "SP fails". □

In Algorithm I, observe that the procedure terminates in one of two ways: either at some iteration k with $A^k = \emptyset$, or at some iteration k with $A^k \neq \emptyset$ and $A^k = A^{k-1}$. Consequently, the algorithm either outputs "SP holds" or "SP fails".

To understand the algorithm, recall that if p admits a convex completion $\succsim^*$, then for any alternative x that is $\succsim^*$-worst in A^{k-1}, it must satisfy $x_p^{\downarrow\downarrow} \cap \Pi_p(A^{k-1}) = \emptyset$. The algorithm constructs A^k by eliminating all such candidates for the $\succsim^*$-worst alternative in A^{k-1}. Notably, if p satisfies SP on all submenus of A, then at each iteration k, the set of such candidates is not empty, and we have $A^k \subsetneq A^{k-1}$. Therefore, if p satisfies SP on all submenus of A, then the algorithm outputs "SP holds". The following theorem establishes the converse.

Theorem 2. *For any order pair $p = (\trianglerighteq, \succ)$ and finite menu A, run Algorithm I for p on A. The following statements are equivalent.*

(i) *p satisfies SP on every submenu of A.*

(ii) *Algorithm I outputs "SP holds".*

(iii) *There is a convex preference $\succsim^*$ on X such that:*

$$\forall x, y \in X, x \trianglerighteq y \text{ implies } x \succsim^* y,$$
$$\forall x \in A, y \in X, x \succ y \text{ implies } x \succ^* y.$$

Proof of Theorem 2. We have demonstrated the implication from (i) to (ii). To see why (iii) implies (i), note that if a preference $\succsim^*$ satisfying the conditions in (iii) exists, each submenu of A must contain a $\succsim^*$-worst alternative, and following reasoning parallel to the previous section, SP necessarily holds. The remaining implication from (ii) to (iii) follows from an explicit construction of $\succsim^*$.

By (ii), the algorithm identifies a strictly decreasing sequence $(A^k)_{k=0}^n$ of submenus of A with $A^n = \emptyset$. The idea for constructing the preference $\succsim^*$ is to construct a sequence of nested convex sets in X, such that alternatives that are included in smaller sets are ranked higher. Specifically, for each $x \in X$, define

$$k^*(x) = \begin{cases} -1, & \text{if } x \notin \Pi_p(A^0), \\ \max\{k : x \in \Pi_p(A^k)\}, & \text{otherwise.} \end{cases}$$

Since $A^n = \emptyset$, for each x, we have $k^*(x) \in \{-1, ..., n-1\}$. Let $\succsim^*$ be the preference represented by k^*.

We verify that $\succsim^*$ satisfies the conditions in statement (iii). Convexity holds since each upper contour set $\{y \in X : y \succsim^* x\}$ equals either X or $\Pi_p(A^{k^*(x)})$, both convex. For $\trianglerighteq$-consistency, if $x \trianglerighteq y$, then $k^*(x) \geq k^*(y)$ (trivially when $k^*(y) = -1$, and otherwise because $x \trianglerighteq y$ implies $x \in \Pi_p(A^{k^*(y)})$). For $\succ$-consistency, if $x \succ y$ with $x \in A$, then there exists k such that $x \in A^k$ and $x_p^{\downarrow\downarrow} \cap \Pi_p(A^k) = \emptyset$, indicating that $y \notin \Pi_p(A^k)$ (since $y \in x_p^{\downarrow\downarrow}$), and thus $k^*(x) \geq k > k^*(y)$, i.e., $x \succ^* y$. $\square$

By the proof of Theorem 2, when G_p is finite, we can explicitly construct a quasi-concave utility function u, following Algorithm I, such that the preference it represents extends the order pair p.

With Theorem 2, we sketch the proof for the implication from (ii) to (i) in Theorem 1. Define the collection of order pairs $\mathcal{P}$ such that $q = (\trianglerighteq_q, \succ_q) \in \mathcal{P}$ if $\trianglerighteq \subseteq \trianglerighteq_q$, $\succ_q \subseteq \succ$, and q satisfies SP on every finite menu. Define an order $\subseteq$ on $\mathcal{P}$ such that for any two order pairs $q = (\trianglerighteq_q, \succ_q)$ and $\hat{q} = (\trianglerighteq_{\hat{q}}, \succ_{\hat{q}})$ in $\mathcal{P}$, $q \subseteq \hat{q}$ if and only if $\trianglerighteq_q \subseteq \trianglerighteq_{\hat{q}}$ and $\succ_q \subseteq \succ_{\hat{q}}$. By Zorn's lemma, there exists a $\subseteq$-maximal element $r = (\trianglerighteq_r, \succ_r)$ in $\mathcal{P}$, i.e., there is no $q \in \mathcal{P}$ with $q \neq r$ and $r \subseteq q$.

The next step is to show that $\trianglerighteq_r$ is complete. Suppose to the contrary, then there exist distinct $x, y \in X$ such that neither $x \trianglerighteq_r y$ nor $y \trianglerighteq_r x$. Thus the two order pairs $(\trianglerighteq_{p^*} \cup \{(x,y)\}, \succ_{p^*})$ and $(\trianglerighteq_{p^*} \cup \{(y,x)\}, \succ_{p^*})$ must violate SP on some finite menus A and B, respectively. However, since r satisfies SP on every finite menu contained in the finite set $A \cup B \cup \{x, y\}$, Theorem 2 implies that there exists a convex preference $\succsim^*$ such that $\trianglerighteq_r \subseteq \succsim^*$ and, for all $\hat{x} \in A \cup B \cup \{x, y\}$ and $\hat{y} \in X$,

$$\hat{x} \succ_r \hat{y} \;\Rightarrow\; \hat{x} \succ^* \hat{y},$$

where $\succ^*$ denotes the asymmetric part of $\succsim^*$. The convexity of $\succsim^*$ ensures that the order pair $(\succsim^*, \succ^*)$ satisfies SP on every finite menu contained in $A \cup B \cup \{x, y\}$. Since either $\trianglerighteq_r \cup \{(x,y)\} \subseteq \succsim^*$ or $\trianglerighteq_r \cup \{(y,x)\} \subseteq \succsim^*$, it follows that either $(\trianglerighteq_r \cup \{(x,y)\}, \succ_r)$ or $(\trianglerighteq_r \cup \{(y,x)\}, \succ_r)$ satisfies SP on every finite menu contained in $A \cup B \cup \{x, y\}$, contradicting the choice of A and B. Hence $\trianglerighteq_r$ is complete. By a similar argument, we can show that for any distinct $x, y \in X$, $x \trianglerighteq_r y$ implies either $y \trianglerighteq_r x$ or $x \succ_r y$.

What remains is to show that $\trianglerighteq_r$ is transitive by its maximality under $\subseteq$. Moreover, since r satisfies SP on every finite menu, $x \succ_r y$ and $y \trianglerighteq_r x$ cannot simultaneously occur. It follows that $\succ_r$ coincides with the asymmetric part of $\trianglerighteq_r$. That is, $(\trianglerighteq_r, \succ_r) = (\trianglerighteq_r, \triangleright_r)$, and $\trianglerighteq_r$ is the desired convex preference that we seek to recover.

4 Convex Rationalization with a Preorder

We have shown that testing whether an order pair p admits a convex completion amounts to checking whether it satisfies SP on every finite menu in G_p. When p is constructed based on the DM's choices observed in finitely many menus, G_p is usually finite. Algorithm I then performs the test with at most $|G_p|$ rounds of SP checks.

However, in many applications, aside from convexity, we may also want the DM's preference to extend some given orders (Nishimura et al., 2017). This may result in an infinite set G_p. For example, consider a consumer who weakly prefers x to y and z to w, where $x, y, z, w \in X = \mathbb{R}^n_+$ are consumption bundles. If we seek to test whether the consumer has a convex and *strictly increasing* preference, then we should start with the order pair

$$p = (\{(x, y), (z, w)\} \cup \trianglerighteq_I, >_n),$$

where $>_n$ denotes the n-dimensional strict Euclidean order. The set G_p is infinite in this example, and a direct application of Theorem 1 indicates that infinitely many SP checks are needed to determine whether p admits a convex completion.

In this section, we show that when the additional orders we impose on the DM's preferences satisfy *independence* (which holds for the consumer example above), we can simplify the problem of testing convexity and reduce the SP checks to finitely many rounds. We begin with the following definition.

Definition 3. *A binary relation $\succsim$ satisfies independence if for all $x, y, z \in X$ and $t \in (0, 1]$, $x \succsim y$ if and only if $tx + (1 - t)z \succsim ty + (1 - t)z$.*

Independence is a fundamental axiom characterizing preferences over lotteries that admit expected utility representations. We refer to a binary relation (respectively, preorder, partial order) that satisfies independence as an I-binary relation (respectively, I-preorder, I-partial order). Below, we investigate the testing of the convexity of the DM's preference when it is required to extend an *I-preorder* and show that this additional requirement adds *no complexity* on the testing.

Theorem 3. *Consider an order pair $p = (\trianglerighteq \cup \sim^*, \succ \cup \succ^*)$, where $\trianglerighteq$ is reflexive, and $\sim^*$ and $\succ^*$ are the symmetric and asymmetric parts of an I-preorder $\succsim^*$. Define*

$$\begin{aligned} H &= \{x \in X : x \trianglerighteq y \textit{ for some } y \neq x\} \cup \{x \in X : x \succ y \textit{ for some } y\}, \\ \succ^{**} &= \{(x, y) \in H \times X : x \succ y \textit{ or } x \succ \diamond \succ^* y \textit{ or } x \trianglerighteq \diamond \succ^* y\}. \end{aligned} \tag{5}$$

If p violates SP on some finite menu A, then there exists a finite menu $B \subseteq H$ on which $q = (\trianglerighteq \cup \sim^*, \succ^{**})$ violates SP. The order pair p admits a convex completion if and only if q satisfies SP on every finite menu contained in H.

By Theorem 3, checking whether p admits a convex completion reduces to verifying whether the revised order pair q satisfies SP on all finite menus contained in H. Notably, H is independent of the I-preorder $\succsim^*$, which makes it practically feasible to test for convex completions of I-preorders in applications with finite-choice data.

We examine two special cases of Theorem 3. First, extending an I-partial order $\succ^*$ can be equivalently framed as extending the I-preorder $\succsim^* = \succ^* \cup \trianglerighteq_I$, to which Theorem 3 applies directly.[17] Specifically, the order pair $p = (\trianglerighteq, \succ \cup \succ^*)$ admits a convex completion if and only if $q = (\trianglerighteq, \succ^{**})$ satisfies SP on every finite menu contained in H, where H and $\succ^{**}$ are as defined in equation (5). In the consumer example introduced earlier in this section, to test if $p = (\{(x,y),(z,w)\} \cup \trianglerighteq_I, >_n)$ admits a convex completion, we need only verify that $q = (\{(x,y),(z,w)\} \cup \trianglerighteq_I, >_n^{**})$ satisfies SP on menus in $\{x, z\}$, with:

$$>_n^{**} = \{(x,x') : x >_n x' \text{ or } y >_n x'\} \cup \{(z,z') : z >_n z' \text{ or } w >_n z'\}.$$

Algorithm I requires at most two rounds of SP checks for this verification.

Second, when the order pair to be extended is $p = (\sim^*, \succ \cup \succ^*)$, where $\sim^*$ and $\succ^*$ are the symmetric and asymmetric parts of an I-preorder $\succsim^*$, the condition in Theorem 3 can be simplified further. By equation (5), H reduces to $\{x \in X : x \succ y \text{ for some } y\}$, $\succ^{**}$ reduces to $\{(x,y) \in H \times X : x \succ y \text{ or } x \succ \diamond \succ^* y \text{ or } x \succ^* y\}$, and the revised order pair simplifies to $q = (\sim^*, \succ^{**})$. Since $\sim^*$ satisfies independence, for any finite menu A we have $\Pi_q(A) = \{x \in X : x \sim^* y \text{ for some } y \in co(A)\}$.[18] Hence, q satisfies SP on A if and only if the following set

$$\begin{aligned}
&\Big\{x \in A : x_q^{\downarrow\downarrow} \cap \Pi_q(A) = \emptyset\Big\} \\
=&\Big\{x \in A : \{y \in X : x \succ^{**} \hat{x} \sim^* y \text{ for some } \hat{x} \in X\} \cap co(A) = \emptyset\Big\} \\
=&\Big\{x \in A : \{y \in X : x \succ \diamond \succsim^* y \text{ or } x \succ^* y\} \cap co(A) = \emptyset\Big\}
\end{aligned}$$

is not empty. Hence, if q satisfies SP on A, then we must have

$$\Big\{x \in A : \{y \in X : x \succ \diamond \succsim^* y\} \cap co(A) = \emptyset\Big\} \neq \emptyset. \tag{6}$$

It turns out the converse is also true: If every finite menu $A \subseteq H$ satisfies the weaker (and simpler) condition (6), then p admits a convex completion. We state this as a proposition below.

[17] When we say that $\succ^*$ is a partial order or an I-partial order, we also allow for the possibility that $\succ^*$ is a null binary relation, i.e., $\succ^* = \emptyset$.

[18] It suffices to show that $\bar{A} = \{x \in X : x \sim^* y \text{ for some } y \in co(A)\}$ is convex. For any $x, y \in \bar{A}$ and $\alpha \in [0,1]$, pick $\hat{x}, \hat{y} \in co(A)$ with $x \sim^* \hat{x}$ and $y \sim^* \hat{y}$. The independence of $\sim^*$ implies $\alpha x + (1-\alpha)y \sim^* \alpha\hat{x} + (1-\alpha)\hat{y} \in co(A)$. Hence, $\alpha x + (1-\alpha)y \in \bar{A}$, establishing convexity of $\bar{A}$.

Proposition 1. *Let $\sim^*$ and $\succ^*$ be the symmetric and asymmetric parts of an I-preorder $\succsim^*$. The order pair $p = (\sim^*, \succ \cup \succ^*)$ admits a convex completion if and only if condition (6) holds for all finite menu contained in $H = \{x \in X : x \succ y \text{ for some } y \in X\}$.*

When H is finite, it suffices to check whether condition (6) holds for at most $|H|$ menus contained in H, following an iterative procedure similar to Algorithm I. Starting with $H^0 = H$, we check whether H^0 satisfies condition (6). If it does not, we stop; otherwise we define $H^1 = \left\{x \in H^0 : \{y \in X : x \succ \diamond \succsim^* y\} \cap co(H^0) \neq \emptyset\right\}$ and repeat the test on H^1. This yields a strictly decreasing sequence $H^0 \supsetneq H^1 \supsetneq \cdots \supsetneq H^k$ of subsets of H, and the process terminates within $|H|$ steps.

4.1 Convex Rationalization with Non-linear Budget Sets

The literature provides conditions characterizing rational consumer behavior across a variety of settings. The classic case is studied by Afriat (1967), where the data record consumer choices from *linear* budget sets. Rationalization with general budget sets is subsequently investigated by Forges and Minelli (2009) and Nishimura et al. (2017), among others. In this section, we allow for general budget sets and our focus is to characterize consumers' choice behavior that can be rationalized by *convex* preferences.

Let the space of alternatives be $X = \mathbb{R}^n_+$ with each dimension denoting a good. An alternative $x \in X$ represents a bundle of goods, where for each $k \in \{1, ..., n\}$, x_k denotes the amount of good k.[19]

A *consumption data set* is a non-empty collection $\mathcal{O} = \{(x^t, B^t)\}_{t \in T}$, where for $t \in T$, the tuple (x^t, B^t) is called an observation with $x^t \in B^t \subseteq X$. The observation t is interpreted as that the consumer chooses x^t in budget set B^t.

Definition 4. *A consumption data set $\mathcal{O} = \{(x^t, B^t)\}_{t \in T}$ can be rationalized (or is rationalizable) by a preference $\succsim$ if for every $t \in T$ and $y^t \in B^t$, $x^t \succsim y^t$.*

A preference $\succsim$ rationalizes the consumption data set if, for each observation t, the consumer's choice x^t is $\succsim$-optimal in the budget set B^t. This preference is also called a rationalization of the data set. If a utility function u represents the preference, the data set is also said to be rationalized by u.

We are interested in the rationalizability by a convex and *strictly increasing* preference. Given a consumption data set $\mathcal{O} = \{(x^t, B^t)\}_{t \in T}$, define the *revealed* weak and strict revealed preference relations $\trianglerighteq_{\mathcal{O}}$ and $\succ_{\mathcal{O}}$ respectively as follows:

(i) $x \trianglerighteq_{\mathcal{O}} y$ if either $x = y$ or there exists $t \in T$ such that $x = x^t$ and $y \in B^t$, and
(ii) $x \succ_{\mathcal{O}} y$ if there exists $t \in T$ and $z \in B^t$ such that $x = x^t$ and $z >_n y$.

[19] We use superscripts to distinguish different alternatives. When an alternative is an n-dimensional vector, we use subscript to denote its value on the corresponding dimension.

The weak relation $\trianglerighteq_{\mathcal{O}}$ is reflexive and directly revealed from the consumer's choices. The strict relation $\succ_{\mathcal{O}}$ combines the observed choices with the Euclidean order.

In the setting of Afriat (1967), each observation consists of a chosen bundle x^t and a price vector $p^t \in \mathbb{R}^n_{++}$, and the budget set is linear and given by $B^t = \{x : x \cdot p^t \leq x^t \cdot p^t\}$. It follows that $x^t \trianglerighteq_{\mathcal{O}} y$ if $y \cdot p^t \leq x^t \cdot p^t$, and $x^t \succ_{\mathcal{O}} y$ if $y \cdot p^t < x^t \cdot p^t$. Our definitions of the weak and strict revealed preference relations coincide with the standard ones in the literature. We have following theorem for the rationalization condition of $\mathcal{O}$.

Theorem 4. *A consumption data set $\mathcal{O} = \{(x^t, B^t)\}_{t \in T}$ can be rationalized by a strictly increasing and convex preference if and only if $p = (\trianglerighteq_{\mathcal{O}}, \succ_{\mathcal{O}})$ satisfies SP on all finite menu $A \subseteq \{x^t\}_{t \in T}$.*

Note that by definition, the data set $\mathcal{O}$ can be rationalized by a strictly increasing and convex preference if and only if the order pair $(\trianglerighteq_{\mathcal{O}}, >_n)$ admits a convex completion. Since $\succ_{\mathcal{O}} = \trianglerighteq_{\mathcal{O}} \diamond >_n$, and the set $\{x \in X : x \trianglerighteq_{\mathcal{O}} y \text{ for some } y \neq x\}$ coincides with $\{x^t\}_{t \in T}$, Theorem 3 implies Theorem 4.

By Theorem 2, when $\{x^t\}_{t \in T}$ is a finite set, it suffices to run Algorithm I for p on $\{x^t\}_{t \in T}$ to verify the rationalizability of the data set, which involves no more than $|T|$ SP checks. We illustrate this procedure with more details throug the following example.

Example 3. Let $X = \mathbb{R}^2_+$. The consumption data set $\mathcal{O} = \{(x^t, B^t)\}_{t=1}^3$ is depicted in Figure 3: Each budget set B^t contains bundles on or below the budget line, and x^t lies on the budget line. Applying Theorems 2 and 4, we run Algorithm I for $p = (\trianglerighteq_{\mathcal{O}}, \succ_{\mathcal{O}})$ on $A = \{x^1, x^2, x^3\}$. We have $\Pi_p(A) = co(A)$, and for each t, $\{y : x^t \succ_{\mathcal{O}} y\}$ equals the interior of B^t. Observe that $co(A)$ does not intersect the interiors of B^2 and B^3, and thus p satisfies SP on A. By Algorithm I, it remains to check whether p satisfies SP on the singleton menu $\{x^1\}$, which holds trivially.

If the consumer's choice in B^3 is y^3 instead of x^3 (as shown in Figure 3), then the modified data set $\mathcal{O}^* = \{(x^1, B^1), (x^2, B^2), (y^3, B^3)\}$ has no convex and strictly increasing rationalizations. This is because $p^* = (\trianglerighteq_{\mathcal{O}^*}, \succ_{\mathcal{O}^*})$ violates SP on $\{x^2, y^3\}$: The line segment connecting x^2 and y^3 intersects the interiors of both B^2 and B^3. However, since every chosen bundle does not lie in other budget sets, Theorem 2 of Nishimura et al. (2017) implies that $\mathcal{O}^*$ is rationalizable by a strictly increasing preference. □

Continuous Rationalization and Connection to Cherchye et al. (2014). Unlike the Afriat's theorem, a finite consumption dataset $\mathcal{O}$ with a strictly increasing, convex rationalization may not admit a strictly increasing, convex, and *continuous* rationalization. We show this through the following example.

Example 4. Let $X = \mathbb{R}^2_+$. Consider a data set $\mathcal{O} = \{(x^t, B^t)\}_{t \in \{1,2\}}$ as depicted in Figure 4: Each budget set contains alternatives on or below its budget line. The chosen bundles

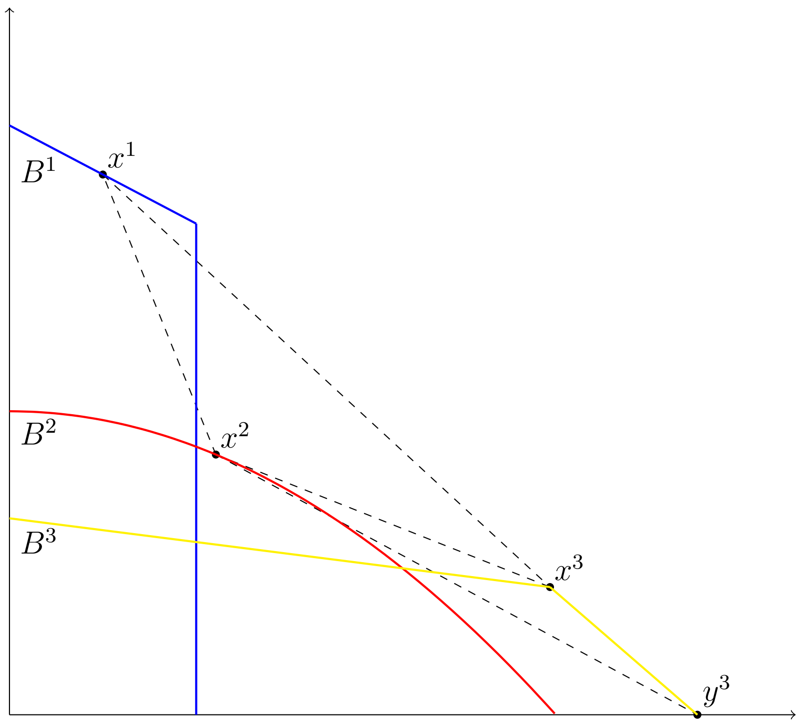


Figure 3: Checking SP with Consumption Data

are $x^1 = (0, 3)$ and $x^2 = (6, 0)$, respectively. The bundle y^1 is on the boundary of B^1 and lies on the line segment joining x^1 and x^2.

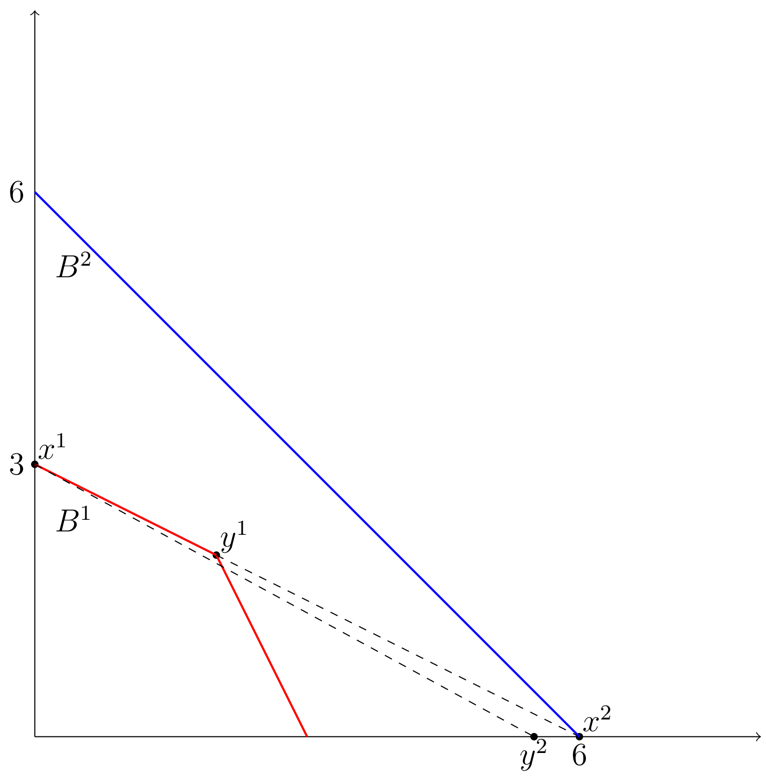


Figure 4: Continuity and Convexity

The data set has a strictly increasing and convex rationalization, as it is rationalizable by a strictly increasing quasi-concave utility function u defined as follows:

$$u(x) = \begin{cases} x_1 + 2x_2, & \text{if } x_1 + 2x_2 \leq 6 \text{ and } x \neq (6,0), \\ x_1 + x_2 + 100, & \text{otherwise.} \end{cases}$$

However, the data set cannot be rationalized by any strictly increasing, convex and continuous preference: If such a preference $\succsim$ exists, then we have $x^2 \succ x^1$. By continuity of $\succsim$, any bundle (e.g., y^2) sufficiently close to x^2 also satisfies $y^2 \succ x^1$. Thus, any convex combination of x^1 and y^2 must be weakly better than x^1, contradicting to the fact that some of them are in the interior of B^1. □

Cherchye et al. (2014) examine the same problem studied in this section, i.e., testing whether a consumption data set $\mathcal{O} = \{(x^t, B^t)\}_{t\in T}$ is rationalizable by a strictly increasing and convex preference. They assume that T is finite and each B^t is closed, monotone, and has nonempty interior.[20] Theorem 3 of Cherchye et al. (2014) claims that the following three statements are equivalent: (i) The data set $\mathcal{O}$ can be rationalized by a locally non-satiated and quasi-concave utility function; (ii) there exists a *co-convex hull* C^t for each observation t such that the new consumption data set $\mathcal{O}^* = \{(x^t, C^t)\}_{t\in T}$ allows for a utility rationalization that is locally non-satiated;[21] (iii) the data set can be rationalized by a strictly increasing, concave and continuous utility function.

When statement (ii) holds, the nice properties of the co-convex hulls ensure that $\mathcal{O}^*$ has a utility rationalization with the desired properties—and so does $\mathcal{O}$. Nevertheless, statement (i) *cannot* imply statement (ii). To see this, consider Example 4. Assume that C^1 is the co-convex hull for observation 1. Denote by D^1 the complement of C^1. The set D^1 contains no bundle z that is strictly below the line segment x^1x^2, since otherwise, for some $\epsilon > 0$ that is close to 0, the line segment zw^ϵ is contained in D^1 (where $w^\epsilon = x^1 + (0, \epsilon)$) and intersects with B^1, which is a contradiction. Hence, all bundles strictly below the line segment x^1x^2 are contained in C^1. Since C^1 is closed, we have $x^2 \in C^1$. This leads to a contradiction, as x^2 is revealed to be strictly better than x^1 and should not be contained in any budget set in which x^1 is chosen.

Instead of looking for the co-convex hulls, our testing method is more direct. To illustrate, consider a finite consumption data set $\mathcal{O} = \{(x^t, B^t)\}_{t\in T}$ and assume that for each t, the budget set can be written as $B^t = \{x \in \mathbb{R}^n_+ : f^t(x) \leq 0\}$ for some continuous and strictly increasing function $f^t : \mathbb{R}^n_+ \to \mathbb{R}$. For any menu $A \subseteq \{x^t\}_{t\in T}$, through the iterative inclusion process for constructing $\Pi_p(A)$, we can find some $\bar{A} \subseteq \{x^t\}_{t\in T}$ with $A \subseteq \bar{A}$ such that $\Pi_p(A) = co(\bar{A})$. Hence, to test whether $(\trianglerighteq_{\mathcal{O}}, \succ_{\mathcal{O}})$ satisfies SP on A, it

[20] A budget set B^t is monotone if $x \in B^t$ implies $\{y : x >_n y\} \subseteq B$.

[21] The budget set C^t is a co-convex hull for observation t if it is closed and monotone, $\mathbb{R}^n_+ \backslash C^t$ is a convex set, $B^t \subseteq C^t$, and for all $x >_n x^t$, $x \notin C^t$.

suffices to check whether there exists some $x \in A$ such that for all t with $x^t = x$, $co(\bar{A})$ does not intersect with the interior of B^t. Since the intersection is non-empty if and only if $\min_{x \in co(\bar{A})} f^t(x) < 0$, the testing reduces to solving a finite collection of minimization problems with the constraint sets being polytopes.

Testing Symmetric Preferences. We end this section by examining another important property—symmetry—that is often imposed on the consumer's preference when the bundles represent contingent consumptions. In this context, each dimension corresponds to a possible future state of the world. For a given alternative x, the value on the k-th dimension, x_k, represents the contingent consumption of the consumer when the k-th state of the world is realized.

If multiple states are deemed equally likely by the consumer, then it is natural to assume the consumer's preference to be *symmetric* among those states. Formally, let $X = \mathbb{R}^n_+$ and $N = \{1, ..., n\}$. For any preference $\succsim$ over X and nonempty $\hat{N} \subseteq N$, we say that $\succsim$ is $\hat{N}$-*symmetric* if for any bijection $\phi : \hat{N} \to \hat{N}$ and $x \in X$, we have $x \sim \phi(x)$, where the bundle $\phi(x)$ is defined such that for all $k \in \hat{N}$, $(\phi(x))_k = x_{\phi(k)}$, and for all $k \notin \hat{N}$, $(\phi(x))_k = x_k$. With an $\hat{N}$-symmetric preference, exchanging payoffs across states in $\hat{N}$ does not affect the desirability of the bundle. For any collection $\mathcal{S} = \{N^i\}_{i=1}^m$ of mutually disjoint nonempty subsets of N, we say that $\succsim$ is $\mathcal{S}$-symmetric if for every i, $\succsim$ is N^i-symmetric.

For a given consumption data set $\mathcal{O} = \{(x^t, B^t)\}_{t \in T}$ and a collection of mutually disjoint nonempty subsets $\mathcal{S} = \{N^i\}_{i=1}^m$ of N, we construct a $\mathcal{S}$-symmetric consumption data set $\mathcal{O}_\mathcal{S}$ such that $(x, B) \in \mathcal{O}_\mathcal{S}$ if and only if there exists $t \in T$, a subset N^i, and a bijection $\phi : N^i \to N^i$ such that $x = \phi(x^t)$ and $B = \{\phi(y) : y \in B^t\}$. With the notations defined above, the symmetric rationalizability of a given data set can be tested as follows.

Theorem 5. *Consider a consumption data set $\mathcal{O}$ and a collection of mutually disjoint nonempty subsets $\mathcal{S} = \{N^i\}_{i=1}^m$ of N. The following statements are equivalent:*

(i) *The data set $\mathcal{O}$ has a strictly increasing, convex and $\mathcal{S}$-symmetric rationalization.*

(ii) *The data set $\mathcal{O}_\mathcal{S}$ has a strictly increasing and convex rationalization.*

(iii) *The data set $\mathcal{O}_\mathcal{S}$ has a strictly increasing, convex and $\mathcal{S}$-symmetric rationalization.*

By statement (ii) of the theorem, the testing reduces to checking whether $(\trianglerighteq_{\mathcal{O}_\mathcal{S}}, \succ_{\mathcal{O}_\mathcal{S}})$ satisfies SP on every finite menu contained in the set of chosen bundles. When $\mathcal{O}$ is finite, the testing is practically manageable.

4.2 Convex Completion under Risk and Uncertainty

Convex preferences constitute an important class for modeling DMs' behavior under risk and uncertainty. In lottery choice domains, Cerreia-Vioglio et al. (2015) propose the cautious expected utility model, where a DM evaluates each lottery using its minimal certainty equivalence across a set of expected utilities—with convexity capturing the DM's cautiousness. In choice domains with ambiguity, convexity reflects a preference for hedging (Gilboa and Schmeidler, 1989), or alternatively, aversion to ambiguity, which resolves the Ellsberg paradox (Ellsberg, 1961). This section develops tests for convex preferences in these contexts.

Let $L = [a, b] \subseteq \mathbb{R}$ be the space of monetary payoffs. Denote by $\Delta(L)$ the set of all distributions (lotteries) over L. Alternatively, we can consider each lottery as a cumulative distribution function $F : L \to \mathbb{R}$ such that $F(a) \geq 0$, $F(b) = 1$, and F is non-decreasing and right-continuous.

To accommodate contexts with ambiguity, we also consider a nonempty and finite state space S. A function $f : S \to \Delta(L)$ is called an act. With an act f, the DM will receive the lottery $f(s)$ when state s is realized. The notation F is also used to denote the constant act that maps each state $s \in S$ to the lottery F. Denote by $\mathcal{F}$ the set of all acts. When we do not explicitly specify, the space of alternatives X is either $\Delta(L)$ or $\mathcal{F}$.

We consider scenarios (e.g., in the lab) in which DMs choose from pairs of lotteries or acts. A *choice data set* is a collection $\mathcal{C} = \{(x^t, y^t)\}_{t \in T}$, where in each choice problem $t \in T$, the DM faces the binary menu $\{x^t, y^t\} \subseteq X$ and chooses option x^t.

Definition 5. *A choice data set $\mathcal{C} = \{(x^t, y^t)\}_{t \in T}$ is rationalized by a preference $\succsim$ over X if for each $t \in T$, $x^t \succ y^t$.*

Our notion of rationalization of $\mathcal{C}$ interprets the DM's choice of x^t over y^t as a strict preference. Except for the observed choices, the researcher may also want the DM's preference to extend some orders that satisfy independence. Examples are given below.

First-Order Stochastic Dominance. Let $X = \Delta(L)$. A lottery F first-order stochastically dominates another lottery G, denoted $F \succ^{FOSD} G$, if $F(x) \leq G(x)$ for all $x \in L$ and $F(x) < G(x)$ for some $x \in L$. Equivalently, $F \succ^{FOSD} G$ holds if and only if $F - G \leq 0$ and $F - G \neq 0$. The first-order stochastic dominance relation forms an I-partial order. Preferences aligning with this order strictly favor greater monetary payoffs over lesser ones—a fundamental rationality postulate in domains of risk. □

Second-Order Stochastic Dominance. Let $X = \Delta(L)$. A lottery F second-order stochastically dominates another lottery G, denoted $F \succ^{SOSD} G$, if

$$\int_a^x F(t)\,dt \leq \int_a^x G(t)\,dt, \quad \forall x \in [a, b],$$

with strict inequality for some $x \in [a, b]$. This relation forms an I-partial order. When $F \succ^{SOSD} G$, F is *less risky* than G in terms of mean-preserving spreads. Thus, preferences consistent with $\succ^{SOSD}$ exhibit risk aversion. □

Multiple Expected Utilities. Instead of being an objective criterion, orders satisfying independence may also arise as a part of the DM's preference over lotteries that the DM *subjectively* deems confident. Dubra et al. (2004) consider a continuous I-preorder $\succsim^*$ and show that it allows for multi-utility representation, where there exists a set of continuous utility functions $\mathcal{U}$ mapping from L to $\mathbb{R}$ such that for all $F, G \in X$, $F \succsim^* G$ if and only if

$$\forall u \in \mathcal{U}, \int_L u(x)dF(x) \geq \int_L u(x)dG(x).$$

In this case, we say that $\succsim^*$ is represented by $\mathcal{U}$. To interpret, the DM behaves as if she considers each utility in $\mathcal{U}$ to be a potential candidate for decision making but is uncertain about which one to rely on. Consequently, if under every utility function, F delivers a higher expected utility than G, the DM can confidently choose F over G. The DM's preference, if exists, should respect this I-preorder. □

Monotone Preference over Acts. Let $X = \mathcal{F}$. Consider a DM whose preference over constant acts, i.e., over $\Delta(L)$, is known to be $\succsim$ which admits an expected utility representation $u : L \to \mathbb{R}$ such that for all $F, G \in \Delta(L)$, $F \succsim G$ if and only if $\int_L u(x)dF(x) \geq \int_L u(x)dG(x)$. The preference $\succsim$ naturally induces an I-preorder $\succsim^*$ over $\mathcal{F}$ such that for all $f, g \in \mathcal{F}$, $f \succsim^* g$ if and only if for all $s \in S$, $f(s) \succsim g(s)$. The DM's preference modeled in the literature is usually assumed to monotone, i.e., consistent with the I-preorder $\succsim^*$. □

The following theorem characterizes choice data sets that are rationalizable by convex preferences.

Theorem 6. *Consider a choice data set $\mathcal{C} = \{(x^t, y^t)\}_{t\in T}$ and an I-preorder $\succsim^*$ on X. The following statements are equivalent.*

(i) *There is a convex preference $\succsim$ on X that rationalizes $\mathcal{C}$ and extends $(\sim^*, \succ^*)$.*

(ii) *For all nonempty and finite $A \subseteq \{x^t\}_{t\in T}$,*

$$\text{there is } x \in A \text{ such that } \left(\bigcup_{r\in T:\, x^r = x} \{y \in X : y^r \succsim^* y\} \right) \cap co(A) = \emptyset. \tag{7}$$

Theorem 6 follows directly from Proposition 1. Moreover, when T is finite, it suffices to verify the condition (7) for at most $|T|$ menus in $\{x^t\}_{t\in T}$. The following example illustrates how to apply Theorem 6 to test whether a given choice dataset admits a convex rationalization.

Example 5. Let $L = [0, 1]$ and $X = \Delta(L)$. Consider a choice data set $\mathcal{C} = \{(F^t, G^t)\}_{t\in\{1,2\}}$ such that for all $x \in [0, 1]$, $G^1(x) = G^2(x) = G(x)$ and $F^1(x) = 2x - F^2(x)$, where F^1 and

G are given by:

$$F^1(x) = \begin{cases} 2x, \ x \in [0, \frac{1}{8}], \\ \frac{1}{4}, \ x \in (\frac{1}{8}, \frac{1}{2}], \\ 2x - \frac{3}{4}, \ x \in (\frac{1}{2}, \frac{7}{8}], \\ 1, \ x \in (\frac{7}{8}, 1], \end{cases} \qquad G(x) = \begin{cases} \frac{4}{5}x, \ x \in [0, \frac{1}{4}], \\ \frac{6}{5}x - \frac{1}{10}, \ x \in (\frac{1}{4}, \frac{3}{4}], \\ \frac{4}{5}x + \frac{1}{5}, \ x \in (\frac{3}{4}, 1]. \end{cases}$$

First, we claim that $\mathcal{C}$ can be rationalized by a convex preference that aligns with the first-order stochastic dominance relation. Consider first $T = \{1, 2\}$. For $x = \frac{1}{4}$, we have $F^1(x) > G(x)$ and $F^2(x) > G(x)$, and for $x = \frac{3}{4}$, we have $F^1(x) < G(x)$ and $F^2(x) < G(x)$. Hence, for all $t \in [0, 1]$, $tF^1 + (1 - t)F^2$ is not equal to G nor first-order stochastically dominated by G. Thus, either F^1 or F^2 makes condition (7) hold for $\{F^1, F^2\}$. Following Algorithm I, the next set to check for condition (7) is the empty set, and we are done.

However, the data set cannot be rationalized by a convex preference that aligns with second-order stochastic dominance relation. This is because for $T = \{1, 2\}$, $\frac{1}{2}F^1 + \frac{1}{2}F^2$ is the uniform distribution and is second-order stochastically dominated by G. □

In the case where $X = \Delta(L)$ and the DM is known to have a *continuous* I-preorder $\succsim^*$ that has multi-utility representation $\mathcal{U}$, nicer properties for the convex rationalizations can be obtained. In particular, when both T and $\mathcal{U}$ are finite, *continuity* of the preorder can be preserved without cost from the convex rationalization.

Theorem 7. *Let $X = \Delta(L)$. Consider a choice data set $\mathcal{C} = \{(F^t, G^t)\}_{t \in T}$ and continuous I-preorder $\succsim^*$ on $\Delta(L)$ that is represented by $\mathcal{U}$ with each $u \in \mathcal{U}$ continuous and strictly increasing. If both T and $\mathcal{U}$ are finite, then the following statements are equivalent.*

(i) *$\mathcal{C}$ has a convex rationalization that extends $(\sim^*, \succ^*)$.*

(ii) *$\mathcal{C}$ has a convex and continuous rationalization that extends $(\sim^*, \succ^*)$.*

(iii) *For all nonempty and finite $A \subseteq \{F^t\}_{t \in T}$,*

$$\text{there is } F \in A \text{ such that } \left(\bigcup_{r \in T : F^r = F} \{G \in X : G^r \succsim^* G\} \right) \cap co(A) = \emptyset. \tag{8}$$

5 Rationalization and Convex Rationalization

The seminal result of Afriat (1967) establishes that for a finite consumption data set with linear budget sets, if it has a strictly increasing rationalization, then the rationalization can additionally be continuous and convex. Reny (2015) investigates infinite data sets with linear budget sets and shows that while continuity may fail, convexity remains to be an implication of strictly increasing rationalization.

In this section, we link our characterization of convex completions to this fundamental insight and provide a condition under which imposing convexity on the rationalization entails no loss of generality. We then apply the theorem to extend the results of Afriat (1967) and Reny (2015) for consumer choice data and derive a characterization of rationalization in the sense of Richter (1966).

5.1 Pre-Convexity

In this section, we provide a condition under which convexity of the rationalization comes at no cost. We begin by introducing the following definition.

Definition 6. *An order pair $p = (\trianglerighteq, \succ)$ is weakly pre-convex if for all $x \in X$ with $x \trianglerighteq x'$ for some $x' \neq x$, the set $\{z : x \ntrianglerighteq z\}$ is convex. Given a set A, p is pre-convex on A if it is weakly pre-convex, and for all $x \in A$, the set $\{z : x \not\succ z\}$ is convex. The order pair p is pre-convex if it is pre-convex on X.*

Consider an order pair $p = (\succsim, \succ)$, where $\succsim$ is a preference. If $\succsim$ is convex, then for all $x \in X$, both $\{z : x \not\succsim z\} = \{z : z \succ x\}$ and $\{z : x \not\succ z\} = \{z : z \succsim x\}$ are convex sets, and thus p is pre-convex. Therefore, our notion of pre-convexity generalizes the standard definition of convexity to order pairs.

Theorem 8. *Consider an order pair $p = (\trianglerighteq, \succ)$ and a menu A. If p pre-convex on A and consistent, then p satisfies SP on all finite menus in A. In particular, if p is pre-convex, then p admits a convex completion if and only if it is consistent.*

Recall that p is consistent (Suzumura, 1976) if there is no cycle $(x^k)_{k=1}^n$ with $n \geq 2$ and $x^1 = x^n$ such that $x^{k+1}(\trianglerighteq \cup \succ)x^k$ for all $k \in \{1, \ldots, n-1\}$ and $x^1 \succ x^n$. This condition is equivalent to the existence of a completion of p. Thus, for a pre-convex order pair, it admits a completion if and only if it admits a convex completion.

Our pre-convexity condition is closely related in spirit to the co-convexity condition proposed by Matzkin (1991). Specifically, Matzkin (1991) considers a data set $\mathcal{O} = \{(x^t, B^t)\}_{t \in T}$ where, for each $t \in T$, $B^t \subseteq \mathbb{R}^n_+$ is a budget set and $x^t \in B^t$ is the observed chosen bundle. Each B^t is assumed to be *co-convex*, meaning that it is closed and its complement in $\mathbb{R}^n_+$ is convex. Under co-convexity, Matzkin (1991) shows that the standard revealed-preference axiom—equivalent to the existence of a preference under which each x^t is the unique best bundle in B^t—further implies the existence of such a preference that is additionally strictly convex. In this sense, co-convexity makes strict convexity a "free lunch." Note that the key feature of the co-convexity condition is that, for each $t \in T$, the set of alternatives outside the budget set B^t—that is, those not revealed to be strictly worse than x^t—forms a convex set. Our pre-convexity condition shares the same idea.

5.2 A Generalization of Afriat (1967) and Reny (2015)

In this section, we fix X to be the consumption space $\mathbb{R}^n_+$ and generalize the results of of Afriat (1967) and Reny (2015).

Consider a possibly infinite consumption data set $\mathcal{O} = \{(x^t, B^t)\}_{t\in T}$. We assume each budget set B^t is monotone, i.e., $x \in B^t$ implies $\{y : x >_n x\} \subseteq B^t$.[22] For each $t \in T$, define the dominated set $\hat{B}^t = \{x \in X : y >_n x \text{ for some } y \in B^t\}$. Since B^t is monotone, we have $\hat{B}^t \subseteq B^t$.

Recall that in Section 4.1, we define the weak and strict revealed preference relations $\trianglerighteq_{\mathcal{O}}$ and $\succ_{\mathcal{O}}$ such that $x \trianglerighteq_{\mathcal{O}} y$ ($x \succ_{\mathcal{O}} y$) if for some $t \in T$, $x = x^t$ and $y \in B^t$ ($y \in \hat{B}^t$). We say that $\mathcal{O}$ satisfies the generalized axiom of revealed preference (GARP, (Afriat, 1967; Varian, 1982)) if the order pair $(\trianglerighteq_{\mathcal{O}}, \succ_{\mathcal{O}})$ is consistent on $\{x^t\}_{t\in T}$.

Theorem 9. *Consider a data set $\mathcal{O} = \{(x^t, B^t)\}_{t\in T}$. If for each $t \in T$, the two sets $\mathbb{R}^n_+ \backslash B^t$ and $\mathbb{R}^n_+ \backslash \hat{B}^t$ are both convex, then the following statements are equivalent:*

(i) *$\mathcal{O}$ satisfies GARP.*

(ii) *$\mathcal{O}$ is rationalizable by a strictly increasing preference.*

(iii) *$\mathcal{O}$ is rationalizable by a strictly increasing and convex preference.*

Theorem 9 generalizes the results in Afriat (1967) and Reny (2015). In their framework, each budget set B^t is linear with $B^t = \{x \in \mathbb{R}^n_+ : x \cdot p^t \leq x^t \cdot p^t\}$ for some price vector $p^t \in \mathbb{R}^n_{++}$. It follows that $\hat{B}^t = \{x \in \mathbb{R}^n_+ : x \cdot p^t < x^t \cdot p^t\}$. Therefore, the complements of B^t and $\hat{B}^t$ are given respectively by

$$\{y : y \cdot p^t > x^t \cdot p^t\} \text{ and } \{y : y \cdot p^t \geq x^t \cdot p^t\},$$

both of which are convex. By Theorem 9, with linear budget sets, requiring the rationalization to be convex incurs no additional cost, regardless of whether it is finite or infinite.

More generally, consider a data set where each budget set takes the form $B^t = \{x \in \mathbb{R}^n_+ : f^t(x) \leq 0\}$ for some quasi-concave, continuous and strictly increasing function f^t. It can be shown that $\hat{B}^t = \{x \in \mathbb{R}^n_+ : f^t(x) < 0\}$. The quasi-concavity of f^t implies that both $\{x \in \mathbb{R}^n_+ : f^t(x) > 0\}$ and $\{x \in \mathbb{R}^n_+ : f^t(x) \geq 0\}$ are convex sets. Therefore, the data set has a strictly increasing rationalization if and only if it has a strictly increasing and convex rationalization. However, since we allow T to be infinite, the continuity of the

[22] This assumption is not necessary for our analysis, but it simplifies the notations and make them consistent with the literature.

rationalization is not guaranteed by the continuity of each f^t.[23]

More on empirical testing. For data sets where not all budget sets satisfy the primitive assumption in Theorem 9, GARP remains necessary but may not be sufficient for convex rationalizability of the data sets. Nevertheless, Theorem 8 can still be useful to simplify the testing of convexity. We demonstrate this through the following example.

Example 5. Let $X = \mathbb{R}^2_+$. Consider a consumption data set $\mathcal{O} = \{(x^t, B^t)\}_{t\in\{1,2,3,4\}}$ depicted in Figure 5: each budget line is a concrete one with the corresponding chosen bundle x^t on it and the budget set B^t containing all bundles on or below it. First, note that x^1 is revealed to be strictly preferred to x^3 and x^2 is revealed to be strictly preferred to both x^4 and x^3. Hence, GARP holds. To test whether the data set can be rationalized by a strictly increasing and convex preference, we run Algorithm I for $p = (\trianglerighteq_{\mathcal{O}}, \succ_{\mathcal{O}})$ on menu $\{x^1, x^2, x^3, x^4\}$. Note that p satisfies SP on $\{x^1, x^2, x^3, x^4\}$ due to x^3. By Algorithm I, it remains to check SP on submenus of $\{x^1, x^2, x^4\}$. However, since the remaining budget sets are linear, it follows from Theorem 8 and GARP that SP holds. Thus, the testing procedure is simplified. $\square$

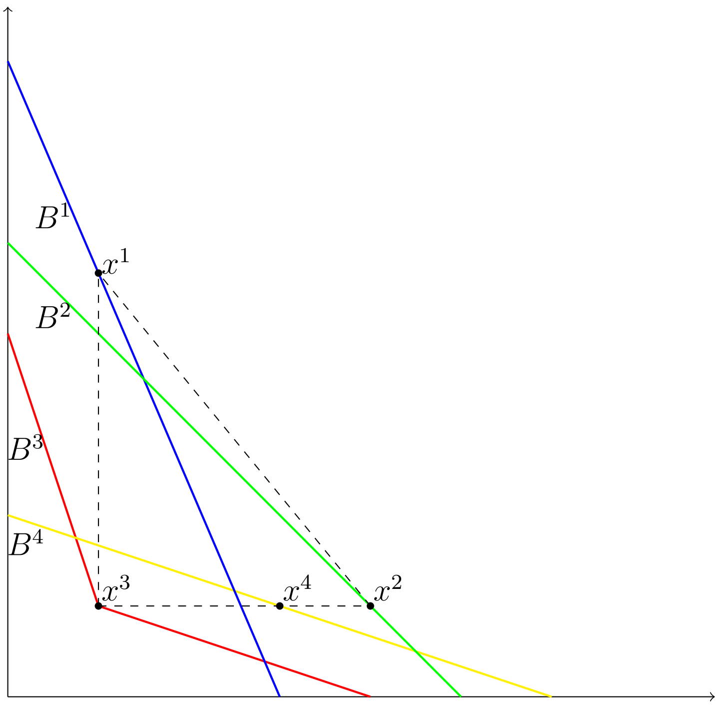


Figure 5: Simplification of Checking SP

[23] When T is finite, by the standard technique from the revealed preference literature, we can explicitly construct the utility function u that rationalizes the data set: We can find positive numbers $\{\lambda^t\}_{t\in T}$ and real numbers $\{v^t\}_{t\in T}$ such that u is given by:

$$u(x) = \min_{t\in T} \left(v^t + \lambda^t f^t(x)\right), \forall x \in \mathbb{R}^n_+.$$

The function u is strictly increasing, quasi-concave, and continuous.

5.3 Exact Rationalization

The solution concept in Afriat (1967) requires each chosen bundle to be *one of* the consumer's optima within the budget set, allowing for other optimal bundles. An alternative solution concept, investigated by Richter (1966) and Matzkin (1991), among others, demands that the chosen bundles *exactly coincide* with the set of optima. We adopt the second solution concept and investigate data sets that have convex rationalizations.

A *general* consumption data is a nonempty collection $\mathcal{O} = \{(A^t, B^t)\}_{t\in T}$ such that for all $t \in T$, $\emptyset \neq A^t \subseteq B^t \subseteq \mathbb{R}^n_+$, and for all distinct $t, t' \in T$, $B^t \neq B^{t'}$. We also maintain the assumption that each budget set is monotone. To interpret, in each budget set B^t, bundles in A^t are chosen by the consumer. The notion of exact rationalization, defined as follows, posits that A^t coincides with the set of all optimal bundles in B^t.

Definition 7. *A general consumption data set $\mathcal{O} = \{(A^t, B^t)\}_{t\in T}$ is exactly rationalizable by a preference $\succsim$ if for all $t \in T$, $A^t = \max(B^t; \succsim)$.*

We are interested in data sets that are exactly rationalizable by strictly increasing and convex preferences. Such a data set necessarily satisfies two conditions. First, the data set must be exactly rationalizable by a strictly increasing preference. This can be characterized by the Congruence axiom (Richter, 1966): Define $\trianglerighteq^*_{\mathcal{O}}$ such that $x \trianglerighteq^*_{\mathcal{O}} y$ if there is $t \in T$ with $x \in A^t$ and $y \in B^t$, and $\succ^*_{\mathcal{O}}$ such that $x \succ^*_{\mathcal{O}} y$ if there is $t \in T$ with $x \in A^t$ and $y \in B^t \backslash A^t$. The data set $\mathcal{O}$ satisfies the Congruence axiom if for all $x^{t_1} \trianglerighteq^*_{\mathcal{O}} x^{t_2} \trianglerighteq^*_{\mathcal{O}} \ldots \trianglerighteq^*_{\mathcal{O}} x^{t_n}$, if $x^{t_n} \in A^{t_n}$ and $x^{t_1} \in B^{t_n}$, then $x^{t_1} \in A^{t_n}$. This axiom is satisfied if and only if the order pair $(\trianglerighteq^*_{\mathcal{O}}, \succ^*_{\mathcal{O}})$ is consistent on $\cup_{t\in T} A^t$. Thus, following a similar proof to that of Theorem 2 in Nishimura et al. (2017), we can show that the Congruence axiom is sufficient and necessary for $\mathcal{O}$ to be exactly rationalizable by a strictly increasing preference.

Second, the data set should be *convex valued*, that is, for all $t \in T$, $co(A^t) \cap B^t = A^t$. This is implied by the convexity of the rationalization: If a convex preference $\succsim$ exactly rationalizes $\mathcal{O}$, then bundles in $co(A^t) \cap B^t$ are weakly $\succsim$-better than some bundle in A^t, meaning that they are optimal in B^t and should be chosen. Our next theorem states that under an analogous assumption to that in Theorem 9, the two conditions are also sufficient for the exact rationalization of $\mathcal{O}$ by a convex and strictly increasing preference.

Theorem 10. *Consider a general consumption data set such that for all $t \in T$, $\mathbb{R}^n_+ \backslash B^t$ is a convex set, and for each $x \in A^t$, $\{x\} \cup \left(\mathbb{R}^n_+ \backslash B^t\right)$ is a convex set. The following two statements are equivalent:*

(i) *$\mathcal{O}$ is convex valued and satisfies the Congruence axiom.*

(ii) *$\mathcal{O}$ is convex valued and exactly rationalizable by a strictly increasing preference.*

(iii) *$\mathcal{O}$ is exactly rationalizable by a strictly increasing and convex preference.*

When each budget set B^t of $\mathcal{O}$ is linear, the set $\mathbb{R}^n_+ \backslash B^t$ is convex. If each A^t are on the boundary of B^t, then the set $\{x\} \cup \left(\mathbb{R}^n_+ \backslash B^t\right)$ is also convex for all $x \in A^t$. In this case, by Theorem 10, $\mathcal{O}$ is exactly rationalizable by a strictly increasing and convex preference if and only if each A^t is convex, and $(\trianglerighteq^*_{\mathcal{O}}, \succ^*_{\mathcal{O}})$ is consistent on $\cup_{t \in T} A^t$.

Matzkin (1991) considers a general data set $\mathcal{O}$ where for each $t \in T$, B^t is closed and has a convex complement in $\mathbb{R}^n_+$, and A^t is a singleton set contained in $B^t \backslash \hat{B}^t$. Since each A^t is a singleton set, $\mathcal{O}$ is convex valued. Furthermore, since $A^t \subseteq B^t \backslash \hat{B}^t$, the set $A^t \cup \left(\mathbb{R}^n_+ \backslash B^t\right)$ is also convex.[24] Matzkin (1991) shows that this data set is exactly rationalizable by a strictly increasing, strictly convex and continuous preference if and only if it satisfies the Congruence axiom. Theorem 10 is a parallel result in which we allow an infinite number of observations (so continuity of the rationalizing preference is not guaranteed), focus on convexity rather than strict convexity, and permit the consumer to choose multiple alternatives from a given budget set. Figure 6 depicts a general consumption data set with two observations where $A^1 = \{x^1, y^1\}$ and $A^2 = \{x^2, y^2\}$. It can be easily verified that it satisfies the conditions in Theorem 10 and thus is exactly rationalizable in our sense.

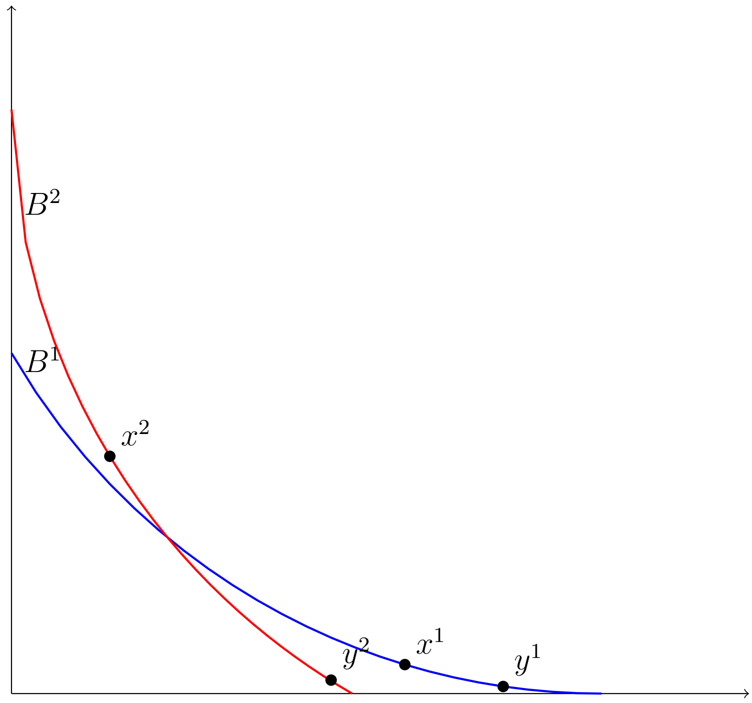


Figure 6: Exact Rationalization with Multiple Chosen Bundles

[24] To see this, let $A^t = \{x^t\}$. Suppose to the contrary that $A^t \cup \left(\mathbb{R}^n_+ \backslash B^t\right)$ is not convex. Then there exists $y \in \mathbb{R}^n_+ \backslash B^t$ and $\lambda \in (0,1)$ such that $\lambda x^t + (1-\lambda)y \in B^t$. Pick $\epsilon \in \mathbb{R}^n_{++}$ such that $y - \epsilon \in \mathbb{R}^n_+ \backslash B^t$. We have $\lambda(x^t + ((1-\lambda)/\lambda)\epsilon) + (1-\lambda)(y-\epsilon) = \lambda x^t + (1-\lambda)y \in B^t$. This is a contradiction: $x^t \in B^t \backslash \hat{B}^t$ implies $x^t + ((1-\lambda)/\lambda)\epsilon \in \mathbb{R}^n_+ \backslash B^t$, and the convexity of $\mathbb{R}^n_+ \backslash B^t$ implies $\lambda x^t + (1-\lambda)y \in \mathbb{R}^n_+ \backslash B^t$.

6 Discussions

In this section, we first discuss how to use our characterization to obtain identification results of the DM's preference. We then provide a full characterization for order pairs that admit strictly convex completions.

6.1 Identification

We fix an order pair $p = (\trianglerighteq, \succ)$ and assume that p allows for a convex completion throughout this section. Given the assumption that the DM's true preference is convex, we aim to obtain more information regarding the DM's preference. For this purpose, we have the following definition.

Definition 8. *For any two alternatives x and y, x is identified to be weakly preferred to y if for every convex preference $\succsim$ that extends p, we have $x \succsim y$, and x is identified to be strictly preferred to y if for every convex preference $\succsim$ that extends p, we have $x \succ y$.*

Our notion of "identified weakly preferred" and "identified strictly preferred" is robust as it requires the identified preference relation to be true for every possible candidate of the DM's true preference. To identify such relations, we follow the approach developed by Varian (1982).

Specifically, Varian (1982) considers the choice environment studied by Afriat (1967), where each observation consists of a bundle x^t and a positive price vector p^t. He poses the following question: for a given bundle x that is not chosen in any budget set of the data set, what is the set of price vectors under which x could possibly be chosen without leading to a contradiction with the existing observed choices? He then suggests to take each price vector p, expand the data set to include (x, p), and verify whether the expanded data set passes the test of rationality.

In our problem, to determine whether x *must* be weakly preferred to y under every possible convex preference of the DM, we need to exclude the possibility that y is strictly preferred to x. To do so, we can include the relation $y \succ x$ into the strict component of the order pair. It then follows that x is identified as weakly preferred to y if and only if the expanded order pair violates SP on some finite menu. In a similar manner, we can identify whether x must be strictly preferred to y. The following theorem gives the characterization results.

Theorem 11. *Consider an order pair p that admits a convex completion. For any $x, y \in X$, we have the following identification results.*

(i) *x is identified as weakly preferred to y if and only if for some finite menu A:*

$$x \in \Pi_p(A) \text{ and } \{y\} = \{z \in A : \Pi_p(A) \cap z_p^{\downarrow\downarrow} = \emptyset\}.$$

(ii) *x is identified as strictly preferred to y if and only if for some finite menu A:*

$$y \notin A,\ x \in \Pi_p(A) \text{ and } \{y\} = \{z \in A \cup \{y\} : \Pi_p(A \cup \{y\}) \cap z_p^{\downarrow\downarrow} = \emptyset\}.$$

To understand Theorem 11, note that if y is the only alternative in menu A that satisfies $\Pi_p(A) \cap y_p^{\downarrow\downarrow} = \emptyset$, then it is the sole candidate for the worst alternative in this menu. Hence, based on our analysis in Section 3.1, each alternative in $\Pi_p(A)$ is weakly better than y, and each alternative in $A \backslash \{y\}$ is strictly better than y, which implies that each alternative in $\Pi_p(A \backslash \{y\})$ is strictly better than y. These are precisely the two statements above.

6.2 Strictly Convex Completion

In this section, we provide a characterization of order pairs that allow for strictly convex completions. For this purpose, we introduce the following notations.

Given a convex set $A \subseteq X$, $x \in A$ is an extreme point of A if we cannot find $z, w \in A \backslash \{x\}$ and $t \in (0,1)$ such that $tz + (1-t)w = x$. Denote by $ex(A)$ the set of all extreme points of A. Given a binary relation $\trianglerighteq$, let $Tr(\trianglerighteq)$ be the transitive closure of $\trianglerighteq$: $x\, Tr(\trianglerighteq)\, y$ if there is a finite sequence of alternatives $(x^k)_{k=1}^m$ $(m \geq 2)$ such that $x^1 = y$, $x^m = x$, and for all $k \in \{1, \ldots, m-1\}$, $x^{k+1} \trianglerighteq x^k$. Given an order pair $p = (\trianglerighteq, \succ)$, for any $x \in X$, let $\Gamma_p(x) = \{y : y\, Tr(\trianglerighteq)\, x\}$, and for any $A \subseteq X$, let $\Gamma_p(A) = \bigcup_{x \in A} \Gamma_p(x)$ and $\Lambda_p(A) = co(\Gamma_p(A))$. Note that $\Gamma_p(A) \subseteq \Lambda_p(A) \subseteq \Pi_p(A)$.

Definition 9. *An order pair p satisfies the transitive separation property (TSP) on a finite menu A if there exists $x \in A \cap ex(\Lambda_p(A))$ such that $\Lambda_p(A) \cap x_p^{\downarrow\downarrow} = \emptyset$.*

If an order pair $p = (\trianglerighteq, \succ)$ admits a strictly convex completion $\succsim^*$, then p satisfies TSP on every finite menu. To see this, consider a finite menu A. Let $x \in A$ be the $\succsim^*$-worst alternative. Similar to our analysis in Section 3.1, we can show that the set $\Lambda_p(A)$ must be contained in the upper contour set $\{y : y \succsim^* x\}$ of x and satisfies $\Lambda_p(A) \cap x_p^{\downarrow\downarrow} = \emptyset$. By the strict convexity of $\succsim^*$, x is an extreme point of $\{y : y \succsim^* x\}$ and, thus, an extreme point of $\Lambda_p(A)$.[25] Our next theorem asserts that the converse direction is also true.

Theorem 12. *An order pair p admits a strictly convex completion if and only if it satisfies TSP on every finite menu.*

We investigate the implications of Theorem 12 in two special cases. First, when $p = (\trianglerighteq_I, \succ \cup \succ^*)$ for some I-partial order $\succ^*$, by Theorem 12, it admits a strictly convex

[25] Suppose to the contrary that $x = tz + (1-t)w$ for some $t \in (0,1)$ and $z, w \in \{y : y \succsim^* x\}$ with $x \notin \{z, w\}$. By the strict convexity of $\succsim^*$, we have either $x \succ^* y$ or $x \succ^* z$, leading to a contradiction.

completion if and only if for all finite menu A,

$$\text{there is } x \in ex(co(A)) \text{ such that for all } y \in co(A), \text{ not } x \succ \cup \succ^* y. \tag{9}$$

If condition (9) fails for a finite menu A, then it also fails for its submenu $\hat{A} = ex(co(A))$. Hence, for each $x \in \hat{A}$, there is $y \in co(\hat{A})$ such that either $x \succ y$ or $x \succ^* y$. By Lemma 8 and a simple induction, we can find a menu $B \subseteq \hat{A}$ such that for all $x \in B$, $x \succ y$ for some $y \in X$ and $x \succ \diamond \succ^* z$ for some $z \in co(B)$. This leads to the following proposition.

Proposition 2. *Given an order pair $p = (\trianglerighteq_I, \succ \cup \succ^*)$ where $\succ^*$ is an I-partial order, it admits a strictly convex completion if and only if $q = (\trianglerighteq_I, \succ^{**})$ satisfies TSP on every finite menu contained in $H = \{x : x \succ y$ for some $y \neq x\}$, where $\succ^{**} = \{(x, y) \in H \times X : x \succ y$ or $x \succ \diamond \succ^* y\}$.*

We apply Proposition 2 to the consumer choice setting. Consider a (possibly infinite) data set $\mathcal{O} = \{(x^t, p^t)\}_{t \in T}$ such that for each t, $x^t \in \mathbb{R}^n_+$ is the bundle consumed by the consumer and $p^t \in \mathbb{R}^n_{++}$ is the price vector. The data set $\mathcal{O}$ is said to be *strictly rationalizable* if there is a preference $\succsim$ over $\mathbb{R}^n_+$ such that for each t, x^t is the *unique* $\succsim$-optimal bundle in the budget set $B^t = \{x \in \mathbb{R}^n_+ : x \cdot p^t \leq x^t \cdot p^t\}$. It is shown that $\mathcal{O}$ is strictly rationalizable if and only if it satisfies Strong Axiom of Revealed Preference (SARP, (Houthakker, 1950)).[26] Notably, SARP implies that for every finite menu $A \subseteq \{x^t\}_{t \in T}$, there is $x \in A$ such that for all $t \in T$ with $x = x^t$, we have

$$\forall y \in A \backslash \{x\}, \; x \cdot p^t < y \cdot p^t.$$

It follows that x is an extreme point of $co(A)$, and for all $t \in T$ with $x = x^t$, the budget set B^t intersects with $co(A)$ at only x. Hence, by Proposition 2, SARP is also sufficient for $\mathcal{O}$ to be strictly rationalized by a strictly increasing and strictly convex preference.[27]

The second case we consider is when $p = (\trianglerighteq, \succ)$ with $\succ$ being an I-partial order. We have the following result.

Proposition 3. *Consider an order pair $p = (\trianglerighteq, \succ)$ such that $\succ$ is a an I-partial order. Let $W_p = \{x : y \trianglerighteq x$ for some $y \neq x\}$. The following three statements are equivalent:*

(i) *The order pair p admits a strictly convex completion.*

(ii) *The order pair p satisfies TSP on every finite menu in W_p.*

(iii) *For any finite index set $N = \{1, ..., n\}$ and $f, g : N \to X$, if* (i) *$g(N) \subseteq W_p$,* (ii) *$f(N) = ex(co(f(N)))$,* (iii) *for all $k \in N$, $f(k) \trianglerighteq g(k)$, $f(k) \neq g(k)$,* (iv) *for all $k \in N$*

[26] In this specific setting, SARP means that for all nonempty and finite sequences of mutually distinct $\{x^{t_s}\}_{s=1}^n$ with $n \geq 2$, if for all $k \in \{1, \ldots, n-1\}$, $x^{t_{k+1}} \cdot p^{t_{k+1}} \geq x^{t_k} \cdot p^{t_{k+1}}$, then $x^{t_1} \cdot p^{t_1} < x^{t_n} \cdot p^{t_1}$.

[27] The pair $p = (\trianglerighteq_I, \succ \cup \succ^*)$ can be defined as $\succ = \{(x, y) : x = x^t$ and $y \in B^t \backslash \{x^t\}$ for some $t \in T\}$ and $\succ^* = >_n$. It follows that $\{x : x \succ y$ for some $y \neq x\} = \{x^t\}_{t \in T}$ and $\succ \diamond \succ^* \subseteq \succ$.

and $x \in co(f(N))$*,* $f(k) \not\succ x$*, and* (v) *for all* $k \in N$*, there is* $y \in co(f(N))$ *such that* $g(k) = y$ *or* $g(k) \succ y$*, then there is nonempty* $Q \subseteq N$ *such that* $g(Q) \subseteq f(Q)$*.*[28]

Statement (iii) of Proposition 3 corresponds to the No Extended Dominance axiom of Bossert and Sprumont (2009) (henceforth BS) and reduces to BS's No Strong Inclusion axiom when $\succ = \emptyset$. While BS restrict attention to the case where $X = \mathbb{R}^n$, $\{(x, y) : x \trianglerighteq y \text{ and } x \neq y\}$ is finite, and $\succ$ is the strict Euclidean order, Proposition 3 demonstrates that their characterization for strictly convex completions continues to hold in the infinite case and for more general convex domains and I-partial orders. Moreover, our characterization in statement (ii) simplifies that in Theorem 12 for this special case, as it requires checking only finite menus contained in W_p.[29]

Appendix

Proof of Theorem 1. We have shown the equivalence between statements (ii) and (iii), and that (i) implies (ii). It remains to show that statement (ii) implies (i). Consider an order pair $p = (\trianglerighteq_p, \succ_p)$ such that p satisfies SP on every finite menu. We say that an order pair $q = (\trianglerighteq_q, \succ_q)$ extends p, denoted by $p \subseteq q$, if $\trianglerighteq_p \subseteq \trianglerighteq_q$ and $\succ_p \subseteq \succ_q$. Let $\mathcal{P}$ be the set of all order pairs that extend p and satisfy SP on every finite menu. Endowed with the extension order $\subseteq$, $(\mathcal{P}, \subseteq)$ forms a partially ordered set. A subset $\mathcal{Q} \subseteq \mathcal{P}$ is called a chain of $(\mathcal{P}, \subseteq)$ if for all $q, \hat{q} \in \mathcal{Q}$, either $q \subseteq \hat{q}$ or $\hat{q} \subseteq q$.

Lemma 1. *Consider an order pair* $q = (\trianglerighteq_q, \succ_q)$ *and a finite menu* A*. For any nonempty* $B \subseteq A$*, if for all* $x \in B$*,* $x_q^{\downarrow\downarrow} \cap \Pi_q(A) \neq \emptyset$*, then there exists a finite sequence of finite menus* $(A^k)_{k=0}^n$ *such that* (i) $A^0 = A$*,* (ii) *for all* $k \in \{1, ..., n\}$ *and* $x \in A^k$*, there exists* $y \in co(\cup_{l=0}^{k-1} A^l)$ *with* $x \trianglerighteq_q y$*, and* (iii) *for all* $x \in B$*,* $x_q^{\downarrow\downarrow} \cap co(\cup_{k=0}^n A^k) = \emptyset$*.*

Proof. We prove the lemma for the case where $B = \{x\}$; the proof for the case where $|B| \geq 2$ is similar. Since $x_q^{\downarrow\downarrow} \cap \Pi_q(A) \neq \emptyset$, there exists $y^* \in \Pi_q(A)$ such that $x \succ_q y^*$. Therefore, it suffices to show that for all $y \in \Pi_q(A)$, there exists a finite sequence of finite menus $(A^k)_{k=0}^n$ such that $A^0 = A$, $y \in A^n$, and for all $k \in \{1, \ldots, n\}$ and $z \in A^k$, there exists $w \in co\left(\bigcup_{l=0}^{k-1} A^l\right)$ with $z \trianglerighteq_q w$. We call such a sequence a *finite path* from A to y. If $y \in \Pi_q^0(A) = co(A)$, then the sequence $(A^k)_{k=0}^1$ with $A^0 = A$ and $A^1 = \{y\}$ is a finite path from A to y. Inductively, suppose that for all $y \in \Pi_q^k(A)$, a finite path from A to y exists. We will show that it is also true for each $y \in \Pi_q^{k+1}(A)$. Since $y \in \Pi_q^{k+1}(A)$, we can find $(\alpha^i)_{i=1}^m \in (0, 1)^m$ with $\sum_{i=1}^m \alpha^i = 1$, $\{z^1, \ldots, z^m\} \subseteq X$ and $\{y^1, \ldots, y^m\} \subseteq \Pi_q^k(A)$ such

[28] For any $Q \subseteq N$, we adopt the standard notation $f(Q) = \{f(k) : k \in Q\}$ and $g(Q) = \{g(k) : k \in Q\}$.

[29] One might wonder whether we can revise statement (ii) of the proposition to that p satisfies TSP on every finite menu A contained in $\{x : x \trianglerighteq y \text{ for some } y \neq x\}$. The answer is no. To see this, consider four distinct alternatives x, y, z, and w with $x = \frac{3}{4}z + \frac{1}{4}w$ and $y = \frac{1}{4}z + \frac{3}{4}w$. Let $\trianglerighteq = \trianglerighteq_I \cup \{(z, x), (w, y)\}$ and $\succ = \emptyset$. The only violation of TSP occurs on $A = \{x, y\}$, which is not a subset of $\{\hat{x} : \hat{x} \trianglerighteq \hat{y} \text{ for some } \hat{y} \neq \hat{x}\}$.

that $y = \sum_{i=1}^{m} \alpha^i z^i$, and for all $i \in \{1, \ldots, m\}$, $z^i \trianglerighteq_q y^i$. For each y^i, by our induction hypothesis, let $(A^{i,j})_{j=0}^{l_i}$ be the finite path from A to y^i. Pick an integer L such that for all $i \in \{1, \ldots, m\}$, $L \geq l_i$. For each $i \in \{1, \ldots, m\}$ and $j > l_i$, let $A^{i,j} = A^{i,l_i}$. By our construction, the sequence $(A^{i,j})_{j=0}^{L}$ remains to be a finite path from A to y^i. It follows that $(A^k)_{k=0}^{L+2}$ is a finite path from A to y, where $A^{L+2} = \{y\}$, $A^{L+1} = \{z^1, \ldots, z^m\}$, and for all $k \leq L$, $A^k = \bigcup_{i=1}^{m} A^{i,k}$. □

Lemma 2. *Every chain $\mathcal{Q}$ of $(\mathcal{P}, \subseteq)$ has a least upper bound.*

Proof. Consider a chain $\{(\trianglerighteq_q, \succ_q)\}_{q \in \mathcal{Q}}$. Define $o = (\trianglerighteq_o, \succ_o)$ such that $\trianglerighteq_o = \cup_{q \in \mathcal{Q}} \trianglerighteq_q$ and $\succ_o = \cup_{q \in \mathcal{Q}} \succ_q$. It suffices to show that $o = (\trianglerighteq_o, \succ_o)$ satisfies SP on every finite menu. Suppose to the contrary that there is some finite menu A such that o violates SP on A. It follows that for all $x \in A$, we have $x_o^{\downarrow\downarrow} \cap \Pi_p(A) \neq \emptyset$. By Lemma 1, we can find a finite sequence of finite menus $(A^k)_{k=0}^{n}$ such that (i) $A^0 = A$, (ii) for all $k \in \{1, ..., n\}$ and $x \in A^k$, there exists $y \in co(\cup_{l=0}^{k-1} A^l)$ with $x \trianglerighteq_o y$, and (iii) for all $x \in A$, there exists $y \in co(\cup_{k=0}^{n} A^k)$ such that $x \succ_o y$. Without loss of generality, assume $n \geq 1$. Condition (ii) is equivalent to that for all $k \in \{1, ..., n\}$ and $x \in A^k$, there exists $y \in co(\cup_{l=0}^{k-1} A^l)$ and $q(x) \in \mathcal{Q}$ with $x \trianglerighteq_{q(x)} y$. Condition (iii) is equivalent to that for all $x \in A$, there exists $y \in co(\cup_{k=0}^{n} A^k)$ and $r(x) \in \mathcal{Q}$ such that $x \succ_{r(x)} y$. Since $\mathcal{Q}$ is a chain, the finite set $\{r(x)\}_{x \in A} \cup \{q(x)\}_{x \in \cup_{k=1}^{n} A^k}$ contains a $\subseteq$-maximal order pair q^*. Therefore, for all $k \in \{1, ..., n\}$ and $x \in A^k$, there exists $y \in co(\cup_{l=0}^{k-1} A^l)$ with $x \trianglerighteq_{q^*} y$, and for all $x \in A$, there exists $y \in co(\cup_{k=0}^{n} A^k)$ with $x \succ_{q^*} y$. We conclude that $co(\cup_{k=0}^{n} A^k) \subseteq \Pi_{q^*}(A)$, and for all $x \in A$, there exists $y \in \Pi_{q^*}(A)$ such that $x \succ_{q^*} y$. This contradicts with the fact that q^* satisfies SP on every finite menu. □

We say that a preference $\succsim^*$ extends an order pair $(\trianglerighteq, \succ)$ on a menu A if for all x, y, $x \trianglerighteq y$ implies $x \succsim^* y$, and for all x, y with $x \in A$, $x \succ y$ implies $x \succ^* y$.

Lemma 3. *Let $q = (\trianglerighteq_q, \succ_q)$ be an element of $\mathcal{P}$. If for some $x, y \in X$, neither $x \trianglerighteq_q y$ nor $y \trianglerighteq_q x$, then either $r = (\trianglerighteq_r, \succ_r)$ or $w = (\trianglerighteq_w, \succ_w)$ is an element in $\mathcal{P}$, where $\trianglerighteq_r = \trianglerighteq_q \cup \{(x, y)\}$, $\trianglerighteq_w = \trianglerighteq_q \cup \{(y, x)\}$ and $\succ_r = \succ_w = \succ_q$.*

Proof. Suppose to the contrary that r violates SP on a finite menu A, and w violates SP on a finite menu B. Let $C = A \cup B \cup \{x, y\}$. Since q satisfies SP on every finite menu, and in particular on every menu contained in C, by Theorem 2, there exists a convex preference $\succsim$ that extends q on C. It follows that $\succsim$ extends either r or w on C. By Theorem 2, either r or w satisfies SP on every menu contained in C, which leads to a contradiction. Therefore, either r or w satisfies SP on every finite menu, that is, either $r \in \mathcal{Q}$ or $w \in \mathcal{Q}$. □

The next lemma can be proved similarly to Lemma 3, and its proof is omitted.

Lemma 4. *Let $q = (\trianglerighteq_q, \succ_q)$ be an element of $\mathcal{P}$. If for some $x, y \in X$, $x \trianglerighteq_q y$, $y \not\trianglerighteq_q x$ and $x \not\succ_q y$, then either $(\trianglerighteq_q \cup \{(y, x)\}, \succ_q)$ or $(\trianglerighteq_q, \succ_q \cup \{(x, y)\})$ is an element of $\mathcal{P}$.*

Lemma 5. *If $q = (\trianglerighteq_q, \succ_q)$ is an element of $\mathcal{P}$, then $r = (\trianglerighteq_r, \succ_r)$ is also an element of $\mathcal{P}$, where $\succ_r = \succ_q$, and $\trianglerighteq_r$ is the transitive closure of $\trianglerighteq_q$.*

Proof. For all finite menu A, $\Pi_q(A) = \Pi_r(A)$. Thus, r satisfies SP on every finite menu. □

Lemma 6. *Let $q = (\trianglerighteq_q, \succ_q)$ be an element of $\mathcal{P}$. If $y \succ_q x$, then $x \not\trianglerighteq_q y$.*

Proof. Suppose to the contrary that $y \succ_q x$ and $x \trianglerighteq_q y$. It follows that $x \in \Pi_q(\{y\})$. Thus, q violates SP on $\{y\}$, which is a contradiction. □

Since every chain in $\mathcal{P}$ has a least upper bound, by the Zorn's lemma, there exists a $\subseteq$-maximal element $p^* = (\trianglerighteq^*, \succ^*)$ in $\mathcal{P}$ that extends $p = (\trianglerighteq_p, \succ_p)$. By Lemmas 3 and 5, along with the $\subseteq$-maximality of p^*, $\trianglerighteq^*$ must be complete and transitive. By Lemma 4 and the $\subseteq$-maximality of p^*, if $x \not\trianglerighteq^* y$, then $y \succ^* x$. By Lemma 6, if $y \succ^* x$, then $x \not\trianglerighteq^* y$. Therefore, $\succ^*$ corresponds to the asymmetric part $\triangleright^*$ of $\trianglerighteq^*$. This means that $\trianglerighteq^*$ is a completion of p. To show that $\trianglerighteq^*$ is convex, note that for all $x \in X$, its upper contour set $\{y : y \trianglerighteq^* x\}$ is a subset of $\Pi_{p^*}(\{x\})$. Moreover, since $\Pi_{p^*}(\{x\}) \cap \{y : x \triangleright^* y\} = \emptyset$ by SP, we have $\Pi_{p^*}(\{x\}) \subseteq \{y : y \trianglerighteq^* x\}$. Hence, $\{y : y \trianglerighteq^* x\} = \Pi_{p^*}(\{x\})$ is convex. Therefore, $\trianglerighteq^*$ is a convex preference. □

Proof of Theorem 3. **Step 1.** We first consider the case where $\sim^* = \trianglerighteq_I$. Let H and $\succ^{**}$ be given by equation (5). The pair p reduces to $(\trianglerighteq, \succ \cup \succ^*)$, and q reduces to $(\trianglerighteq, \succ^{**})$. We show that if p violates SP on some finite menu A, then q violates SP on some finite menu $B \subseteq H$. Since $\succsim^*$ is an I-preorder, $\succ^*$ is an I-partial order. We start with the following two lemmas.

Lemma 7. *Let $\succ^*$ be an I-partial order. For all $\{x^k, y^k\}_{k=1}^n \subseteq X$ and $\{\lambda^k\}_{k=1}^n \subseteq (0, +\infty)$ with $\sum_{k=1}^n \lambda^k = 1$, if for all $k \in \{1, ..., n\}$, either $x^k \succ^* y^k$ or $x^k = y^k$ with the former holds for at least one k, then*

$$\sum_{k=1}^n \lambda^k x^k \succ^* \sum_{k=1}^n \lambda^k y^k.$$

Proof. Without loss of generality, assume that for all $k \in \{1, ..., m\}$ $(m \geq 1)$, $x^k \succ^* y^k$, and for all $k \in \{m+1, ..., n\}$, $x^k = y^k$. Since $\succ^*$ satisfies independence, $x^1 \succ^* y^1$ implies $\sum_{k=1}^n \lambda^k x^k \succ^* \lambda^1 y^1 + \sum_{k=2}^n \lambda^k x^k$. By induction, we have

$$\lambda^1 y^1 + \sum_{k=2}^n \lambda^k x^k \succ^* \lambda^1 y^1 + \lambda^2 y^2 + \sum_{k=3}^n \lambda^k x^k \succ^* ... \succ^* \sum_{k=1}^m \lambda^k y^k + \sum_{k=m+1}^n \lambda^k x^k.$$

The transitivity of $\succ^*$ implies $\sum_{k=1}^n \lambda^k x^k \succ^* \sum_{k=1}^m \lambda^k y^k + \sum_{k=m+1}^n \lambda^k x^k = \sum_{k=1}^n \lambda^k y^k$. □

Lemma 8. *Let $\succ^*$ be an I-partial order. Consider a finite menu A and $x \in A$ such that for some $y \in co(A)$, $x \succ^* y$. Let $B = A \backslash \{x\}$. For all $z \in X$ and $w \in co(A)$,*

(i) *if $z \succ^* w$, then there is $\hat{w} \in co(B)$ such that $z \succ^* \hat{w}$,*

(ii) *if $z \succ w$, then there is $\hat{w} \in co(B)$ such that $z \succ \hat{w}$ or $z \succ \diamond \succ^* \hat{w}$,*

(iii) *if $z \succ \diamond \succ^* w$, then there is $\hat{w} \in co(B)$ such that $z \succ \diamond \succ^* \hat{w}$,*

(iv) *if $z \trianglerighteq \diamond \succ^* w$, then there is $\hat{w} \in co(B)$ such that $z \trianglerighteq \diamond \succ^* \hat{w}$.*

Proof. Since $y \in co(A)$, there is $\lambda \in (0,1]$ and $\hat{y} \in co(A \backslash \{x\})$ such that $y = \lambda x + (1-\lambda)\hat{y}$. Note that $\lambda \neq 1$, since otherwise we have $x \succ^* x$, violating the asymmetry of $\succ^*$. By the independence of $\succ^*$, we have $x \succ^* \hat{y}$. Next, consider any $z \in X$ and $w \in co(A)$. Pick $\bar{w} \in co(A \backslash \{x\})$ and $t \in [0,1]$ such that $w = tx + (1-t)\bar{w}$. If $t = 0$, then all the statements trivially hold. Consider the case where $t > 0$. Let $\hat{w} = t\hat{y} + (1-t)\bar{w} \in co(A \backslash \{x\})$. By the independence of $\succ^*$, we have $w \succ^* \hat{w}$. Therefore, statements (i)-(iv) hold. □

We proceed to prove our claim for the case where $\sim^* = \trianglerighteq_I$. We construct the finite menu $B \subseteq H$ on which q violates SP as follows. The trivial case is when p violates SP on a finite menu $A \subseteq H$: For all $x \in H$ and $y \in X$, if $x \succ^* y$, then $x \trianglerighteq x \succ^* y$, which implies $x \succ^{**} y$. Hence, q violates SP on $B = A$.

The non-trivial case is when p violates SP on a finite menu $A \not\subseteq H$. Consider the binary partition $\{A^H, A^N\}$ of A such that $A^H = A \cap H$ and $A^N = A \backslash H \neq \emptyset$. Since p violates SP on A, by Lemma 1, there exists a finite collection of mutually disjoint finite menus $(D^l)_{l=0}^L$ (for which we let $D = \cup_{l=0}^n D^l$) such that (i) $D^0 = A$, and for all $l \in \{0, ..., L-1\}$ and $x \in D^{l+1}$, $x \notin co\left(\cup_{j=0}^l D^j\right)$ and $x \trianglerighteq y$ for some $y \in co\left(\cup_{j=0}^l D^j\right)$, (ii) for all $x \in A^N$, there exists $f(x) \in co(D)$ such that $x \succ^* f(x)$, (iii) for all $x \in A^H$, there exists $g(x) \in co(D)$ such that either $x \succ g(x)$ or $x \succ^* g(x)$. For all $l \geq 1$, by the definition of H, we have $D^l \subseteq H$. In the remaining part of the proof, we fix a selection $\{f(x)\}_{x \in A^N}$ and $\{g(x)\}_{x \in A^H}$ such that (ii) and (iii) hold. Note that for all $x \in A^H$, $x \succ^{**} g(x)$.

Construct a binary partition (E^l, F^l) of D^l for every $l \geq 1$ (if $L \geq 1$) such that:

(a) for all $x \in E^l$, there exists some $g(x) \in co(D)$ with $x \succ^{**} y$, and

(b) for all $x \in F^l$, there exists some $y \in co\left(A^H \cup \left(\cup_{j=1}^{l-1} D^j\right)\right)$ with $x \trianglerighteq y$.

To see why such a binary partition exists, note that for all $l \geq 1$ and $x \in D^l$, if x does not satisfies (b), then by (iv), there exists $\lambda \in (0,1]$, $z \in co(A^N)$ and $w \in co\left(A^H \cup \left(\cup_{j=1}^{l-1} D^j\right)\right)$ such that $x \trianglerighteq \lambda z + (1-\lambda)w$. Let $z = \sum_{y \in A^N} \alpha^y y$, where $\{\alpha^y\}_{y \in A^N} \subseteq [0,1]$ and $\sum_{y \in A^N} \alpha^y = 1$. By Lemma 7, we have $x \succ^{**} \lambda(\sum_{y \in A} \alpha^y f(y)) + (1-\lambda)w \in co(D)$. Therefore, x satisfies (a). Fix the selection $\{g(x)\}_{x \in E^l}$ for each l such that (a) holds. For all $x \in A^H \cup (\cup_{l=1}^L E^l)$, we have $x \succ^{**} g(x) \in co(D)$.

To proceed, define $B = A^H \cup \left(\cup_{l=1}^L E^l\right)$ and $M = A^H \cup \left(\cup_{l=1}^L D^l\right)$. Note that $B \subseteq H$ and $A^N \cup M = D$. We want to show $B \neq \emptyset$, and that q violates SP on B. A simple induction by (b) implies $M \subseteq \Pi_q(B)$, and thus $co(M) \subseteq \Pi_q(B)$. Thus, to show that for all $x \in B$, there is $y \in \Pi_q(B)$ such that $x \succ^{**} y$, it suffices to show that for all $x \in B$, there is $y \in co(M)$ such that $x \succ^{**} y$.

Recall that for every $x \in A^N$, $x \succ^* f(x) \in co(A^N \cup M)$, and for every $x \in B$, $x \succ^{**} g(x) \in co(A^N \cup M)$. Label A^N as $A^N = \{x^1, ..., x^n\}$. By Lemma 8, for every $x \in A^N \backslash \{x^1\}$, there is $y \in co((A^N \backslash \{x^1\}) \cup M)$ such that $x \succ^* y$, and for every $x \in B$, there is $y \in co((A^N \backslash \{x^1\}) \cup M)$ such that $x \succ^{**} y$. A simple induction implies that there is $y \in co(\{x^n\} \cup M)$ with $x^n \succ^* y$, and for every $x \in B$, there is $y \in co(\{x^n\} \cup M)$ such that $x \succ^{**} y$. Note that if $B = \emptyset$, then $A^H = \emptyset$ and for all $l \geq 1$, $E^l = \emptyset$. By (b), for all $l \geq 1$, $F^l = \emptyset$, and thus $M = \emptyset$. It follows that $x^n \succ^* x^n$, contradicting with the asymmetry of $\succ^*$. Hence, $B \neq \emptyset$. Applying Lemma 8 one more time, we conclude that for all $x \in B$, there is $y \in co(M)$ such that $x \succ^{**} y$.

Step 2. Consider the general case where $\sim^*$ is non-trivial. For each $x \in X$, define the set $[x] = \{y \in X : y \sim^* x\}$. Let $X/\sim^* = \{[x]\}_{x \in X}$ be the collection of all such sets. Since $\sim^*$ is reflexive, symmetric and transitive, $X/\sim^*$ forms a partition of X. We show that $X/\sim^*$ can be viewed as a mixture space in the following sense. Define a mixture operator such that for all $x, y \in X$ and $\alpha \in [0,1]$, $\alpha[x] + (1-\alpha)[y] := [\alpha x + (1-\alpha)y]$. Since $\succsim^*$ is an I-preorder, $\sim^*$ is also an I-preorder. Hence, for all $\hat{x}, \bar{x} \in [x]$ and $\hat{y}, \bar{y} \in [y]$, $[\alpha\hat{x} + (1-\alpha)\hat{y}] = [\alpha\bar{x} + (1-\alpha)\bar{y}]$. Thus, $\alpha[x] + (1-\alpha)[y]$ is well-defined. Moreover, for all $[x], [y], [z] \in X/\sim^*$ and $\alpha, \beta, \gamma \in [0,1]$, we can show that $\alpha[x] + (1-\alpha)[y] = (1-\alpha)[y] + \alpha[x]$ and $\alpha(\beta[x] + (1-\beta)[y]) + (1-\alpha)[z] = \alpha\beta[x] + (1-\alpha\beta)(\gamma[y] + (1-\gamma)[z])$, where γ satisfies $(1-\alpha\beta)\gamma = \alpha(1-\beta)$ and $(1-\alpha\beta)(1-\gamma) = 1-\alpha$. Hence, $X/\sim^*$ forms a well-defined mixture space under the mixture operator defined above. The result established in Step 1 can then be applied to order pairs defined on $X/\sim^*$.

Slightly abusing notation, define binary relations $\trianglerighteq$, $\succ$, and $\succ^*$ on $X/\sim^*$ such that (i) $[x] \trianglerighteq [y]$ if $\hat{x} \trianglerighteq \hat{y}$ for some $\hat{x} \in [x]$ and $\hat{y} \in [y]$, (ii) $[x] \succ [y]$ if $\hat{x} \succ \hat{y}$ for some $\hat{x} \in [x]$ and $\hat{y} \in [y]$, and (iii) $[x] \succ^* [y]$ if $\hat{x} \succ^* \hat{y}$ for some $\hat{x} \in [x]$ and $\hat{y} \in [y]$. Let $r = (\trianglerighteq, \succ \cup \succ^*)$ be an order pair on $X/\sim^*$. We show that $\succ^*$ is an I-partial order on $X/\sim^*$. Clearly, $\succ^*$ is a partial order on $X/\sim^*$. To see that $\succ^*$ satisfies independence, note that for all $x, y, z \in X$ and $\alpha \in (0,1)$, we have

$$
\begin{aligned}
[x] \succ^* [y] \Leftrightarrow x \succ^* y \Leftrightarrow \alpha x + (1-\alpha)z \succ^* \alpha y + (1-\alpha)z \\
\Leftrightarrow [\alpha x + (1-\alpha)z] \succ^* [\alpha y + (1-\alpha)z] \\
\Leftrightarrow \alpha[x] + (1-\alpha)[z] \succ^* \alpha[y] + (1-\alpha)[z].
\end{aligned}
$$

To proceed, consider a finite menu A such that $p = (\trianglerighteq \cup \sim^*, \succ \cup \succ^*)$ (as an order pair on X) violates SP on A. This implies that for all $x \in A$, there exists $y_x \in \Pi_p(A)$

such that $x \succ \cup \succ^* y_x$. Define $\mathcal{A} = \{[x] : x \in A\}$, which is a nonempty and finite subset of $X/\sim^*$. For any $y \in X$, we have $y \in \Pi_p(A)$ if and only if $[y] \in \Pi_r(\mathcal{A})$. It follows that for each $x \in A$, $[y_x] \in \Pi_r(\mathcal{A})$ and $[x] \, (\succ \cup \succ^*) \, [y_x]$. Therefore, r violates SP on $\mathcal{A}$. Define

$$\mathcal{H} = \{[x] \in X/\sim^*: [x] \trianglerighteq [y] \text{ for some } [y] \in X/\sim^* \text{ with } [y] \neq [x]\}$$
$$\cup \{[x] \in X/\sim^*: [x] \succ [y] \text{ for some } [y] \in X/\sim^*\}.$$

Let $\hat{r} = (\trianglerighteq, \succ \cup (\succ \diamond \succ^*) \cup (\trianglerighteq \diamond \succ^*))$ be an order pair on $X/\sim^*$. By our result established in Step 1, $\hat{r}$ violates SP on some nonempty and finite $\mathcal{B} \subseteq \mathcal{H}$.

Let H and $\succ^{**}$ be given by equation (5). Consider the order pair $q = (\trianglerighteq \cup \sim^*, \succ^{**})$ on X. We construct a finite menu $B \subseteq H$ such that q violates SP on B. Since $\hat{r}$ violates SP on $\mathcal{B}$, for each $[x] \in \mathcal{B}$, there exists $\Theta([x]) \in \Pi_{\hat{r}}(\mathcal{B})$ such that either $[x] \succ \Theta([x])$, or $[x] \succ \diamond \succ^* \Theta([x])$, or $[x] \trianglerighteq \diamond \succ^* \Theta([x])$. Select for each $[x] \in \mathcal{B}$ some $f([x]) \in [x]$ and $g([x]) \in \Theta[x]$ such that (i) if $[x] \succ \Theta([x])$, then $f([x]) \succ g([x])$, (ii) if $[x] \not\succ \Theta([x])$ and $[x] \succ \diamond \succ^* \Theta([x])$, then $f([x]) \succ \diamond \succ^* g([x])$, and (iii) otherwise (in which case it holds that $[x] \trianglerighteq \diamond \succ^* \Theta([x])$), $f[x] \trianglerighteq \diamond \succ^* g[x]$ with $f([x]) \trianglerighteq y$ for some $y \neq f([x])$. Note that for (iii), if $[x] \not\succ^* \Theta([x])$, then we can find some $[y] \neq [x]$ such that $[x] \trianglerighteq [y] \succ^* \Theta([x])$. In this case, we can find $f([x]) \in [x], \hat{y} \in [y]$ and $g([x]) \in \Theta([x])$ such that $f([x]) \neq \hat{y}$ and $f([x]) \trianglerighteq \hat{y} \succ^* g([x])$, ensuring that (iii) holds. If $[x] \succ^* \Theta([x])$, then by $\mathcal{B} \subseteq \mathcal{H}$, we can find $f([x]) \in [x]$ such that $f([x]) \trianglerighteq y$ for some $y \neq f([x])$, and we have $f([x]) \succ^* g([x])$ for any $g([x]) \in \Theta([x])$. This also ensures that (iii) holds. It follows that $B = \{f([x])\}_{[x] \in \mathcal{B}}$ is a subset of H. Since for all $y \in X$, $y \in \Pi_q(B)$ if and only if $[y] \in \Pi_{\hat{r}}(\mathcal{B})$, we conclude that for all $x \in B$, $x \succ^{**} g([x]) \in \Pi_q(B)$. That is, q violates SP on B.

Step 3. Since we already show that if p does not admit a convex completion, then q violates SP on some finite menu contained in H, it remains to show that if p admits a convex completion, then q satisfies SP on every finite menu contained in H. Note that if p admits a convex completion $\succsim^c$, then $\succsim^c$ also extends q. It follows from Theorem 1 that q satisfies SP on every finite menu, including those contained in H. □

Proof of Proposition 1. It suffices to show that for some finite menu $A \subseteq H$, if for all $x \in A$, $\{y \in X : x \succ \diamond \succ^* y \text{ or } x \succ^* y\} \cap co(A) \neq \emptyset$, then there is menu $B \subseteq A$ that violates condition (6), i.e., for all $x \in B$, $\{y \in X : x \succ \diamond \succ^* y\} \cap co(B) \neq \emptyset$. To see this, consider some $\hat{x} \in A$ such that $\hat{x} \succ^* \hat{y}$ for some $\hat{y} \in co(A)$. By Lemma 8, for all $x \in A \backslash \{\hat{x}\}$, we have $\{y \in X : x \succ \diamond \succ^* y \text{ or } x \succ^* y\} \cap co(A \backslash \{\hat{x}\}) \neq \emptyset$. Inductively, we can shrink the menu by consecutively removing such $\hat{x}$ from it and obtain a nonempty subset $B \subseteq A$ such that for all $x \in B$, $\{y \in X : x \succ^* y\} \cap co(B) = \emptyset$.[30] It follows from

[30] We cannot end up with removing all alternatives in A since for any singleton menu $\{z\}$, $z \not\succ^* z$.

the induction that for all $x \in B$,

$$\{y \in X : x \succ \diamond \succ^* y\} \cap co(B) = \{y \in X : x \succ \diamond \succ^* y \text{ or } x \succ^* y\} \cap co(B) \neq \emptyset.$$

That is, B violates condition (6), and we are done. $\square$

Proof of Theorem 5. The equivalence between (i) and (iii) and the implication from (iii) to (ii) are evident. It suffices to prove that (ii) implies (iii). Define a symmetric binary relation $\sim_{\mathcal{S}}$ on X such that $x \sim_{\mathcal{S}} y$ if and only if there exists $N^i \in \mathcal{S}$ and bijection $\phi : N^i \to N^i$ such that $y = \phi(x)$. To show that $\mathcal{O}_{\mathcal{S}}$ can be rationalized by a strictly increasing, convex and $\mathcal{S}$-symmetric preference, it suffices to show that the order pair $p = (\trianglerighteq_{\mathcal{O}_{\mathcal{S}}} \cup \sim_{\mathcal{S}}, >_n)$ satisfies SP on every finite menu. Suppose to the contrary that p violates SP on some finite menu A. Define finite menu B such that $x \in B$ if and only if there exists $y \in A$, $N^i \in \mathcal{S}$ and bijection $\phi : N^i \to N^i$ with $x = \phi(y)$. It follows from the symmetry of the data set $\mathcal{O}_{\mathcal{S}}$ that p violates SP on B. The $\mathcal{S}$-symmetry of B as well as the $\mathcal{S}$-symmetry of the data set $\mathcal{O}_{\mathcal{S}}$ ensures that $\Pi_p(A) = \Pi_q(A)$, where $q = (\trianglerighteq_{\mathcal{O}_{\mathcal{S}}}, >_n)$. Hence, q violates SP on B. This contradicts with the fact that $\mathcal{O}_{\mathcal{S}}$ can be rationalized by a strictly increasing and convex preference. $\square$

Proof of Theorem 7. It remains to show that (iii) implies (ii). Since $\mathcal{U}$ is finite, let $n = |\mathcal{U}|$. We can transfer each F^t and G^t to an n-dimensional vector in $\mathbb{R}^n_{++}$ by defining $x^t = (\int_L u(x)dF^t(x))_{u\in\mathcal{U}} + a$ and $y^t = (\int_L u(x)dG^t(x))_{u\in\mathcal{U}} + a$ for each $t \in T$, where $a \in \mathbb{R}^n_{++}$ is a constant vector to ensure that all x^t and y^t are in $\mathbb{R}^n_{++}$. Statement (iii) then implies that for all $A \subseteq \{x^t\}_{t\in T}$, there is $x \in A$ such that for all $r \in T$ with $x^r = x$, we have $\{y \in \mathbb{R}^n_{++} : y^r \geq_n y\} \cap co(A) \neq \emptyset$, where $y^r \geq_n y$ means that $y^r_i \geq y_i$ for each $i \in \{1, ..., n\}$. By Corollary 2 of Hu et al. (2022), there exists a continuous, strictly increasing, and convex preference $\trianglerighteq$ on $\mathbb{R}^n_{++}$ such that for all $t \in T$, $x^t \triangleright y^t$. This preference induces a continuous and convex preference over X that extends $(\sim^*, \succ^*)$. $\square$

Proof of Theorem 8. It suffices to prove the first statement of the theorem. Let p be consistent and pre-convex on A. Suppose, to the contrary, that p violates SP on a finite menu $B \subseteq A$. By consistency, there is a preference $\succsim^c$ that extends p, i.e., $\trianglerighteq \subseteq \succsim^c$ and $\succ \subseteq \succ^c$. Since p violates SP on B, by Lemma 1, there is a finite sequence of mutually disjoint finite menus $(D^k)^n_{k=0}$ such that (i) $D^0 = B$, (ii) for all $x \in B$, there exists $y \in co(\cup^n_{k=0} D^k)$ such that $x \succ y$, and (iii) for all $k \in \{0, ..., n-1\}$ and $x \in D^{k+1}$, there is $y \in co(\cup^k_{l=0} D^l)$ such that $x \neq y$ and $x \trianglerighteq y$. Let x^* be the $\succsim^c$-worst alternative in B. Since $x^* \succ y^1$ for some $y^1 \in co(\cup^n_{k=0} D_k)$, by the convexity of the set $\{y : x \not\succ y\}$, there is $z^1 \in \cup^n_{k=0} D^k$ such that $x \succ z^1$. If $z^1 \in D^0 = B$, then we have a contradiction since x is the $\succsim^c$-worst alternative in B. Since $z^1 \in D^k$ for some $k \geq 1$, we have $z^1 \trianglerighteq y^2$ for some $y^2 \in co(\cup^{k-1}_{l=0} D^t)$ with $z^1 \neq y^2$. By the convexity of the set $\{y : z^1 \not\trianglerighteq y\}$, we have $z^1 \trianglerighteq z^2$

for some $z^2 \in \cup_{l=0}^{k-1} D^l$. By a simple induction, we have $x \succ z^1 \trianglerighteq z^2 \trianglerighteq ... \trianglerighteq z^m$ for some $z^m \in D_0 = B$. This contradicts with the fact that x is the $\succsim^c$-worst alternative in B. $\square$

Proof of Theorem 9. We only show the equivalence among statements (i)-(iii). The equivalence among statements (iv)-(vi) can be shown similarly. The equivalence between (i) and (iii) follows from Theorem 2 of Nishimura et al. (2017). Since (ii) implies (i), it suffices to prove that (i) implies (ii). Let $\mathcal{O} = \{(x^t, B^t)\}_{t \in T}$. By statement (i), the order pair $p = (\trianglerighteq_{\mathcal{E}}, \succ_{\mathcal{E}})$ is consistent. By Theorem 3, a sufficient and necessary condition for (ii) to hold is that p satisfies SP on all finite menus contained in $\{x^t\}_{t \in T}$. Since p is pre-convex, by Theorem 8, we are done. $\square$

Proof of Theorem 11. Consider an order pair $p = (\trianglerighteq, \succ)$ and assume that it admits a convex completion. Consider any one of its convex completions $\succsim^*$. First note that for any finite menu A and $y \in A$, if y is the only alternative in A such that $\Pi_p(A) \cap y_p^{\downarrow\downarrow} = \emptyset$, then we must have for all $z \in \Pi_p(A)$, $z \succsim^* y$, and for all $z \in A \backslash \{y\}$, $z \succ^* y$. Thus, for all $z \in \Pi_p(A \backslash \{y\})$, we have $z \succ^* y$. This proves the "if" part of (i) and (ii).

To see the "only if" part of (i), note that if x is identified as weakly preferred to y, then $q = (\trianglerighteq, \succ \cup \{(y, x)\})$ must violate SP on some finite menu A. Since p satisfies SP on menu A, to fail SP for q on A, x must be in $\Pi_p(A)$, and y must be the only alternative in A such that $y_p^{\downarrow\downarrow} \cap \Pi_p(A) = \emptyset$.

To see the "only if" part of (ii), note that if x is identified as strictly preferred to y, then $r = (\trianglerighteq \cup \{(y, x)\}, \succ)$ must violate SP on some finite menu A. Since p satisfies SP on A, if $y \in A$ or $x \notin \Pi_p(A)$, then $\Pi_p(A) = \Pi_r(A)$, and we have a contradiction. Thus, we have $y \notin A$ and $x \in \Pi_p(A)$. Consider menu $A \cup \{y\}$, and we have $\Pi_p(A \cup \{y\}) = \Pi_r(A)$. Since r violates SP on A, for each $z \in A$, we have $\Pi_r(A) \cap z_r^{\downarrow\downarrow} = \Pi_p(A \cup \{y\}) \cap z_p^{\downarrow\downarrow} \neq \emptyset$. Since p satisfies SP on $A \cup \{y\}$, it has to be that y is the only alternative in $A \cup \{y\}$ such that $\Pi_p(A \cup \{y\}) \cap y_p^{\downarrow\downarrow} = \emptyset$. $\square$

Proof of Theorem 12. It remains to prove the sufficiency. Let order pair $p = (\trianglerighteq, \succ)$ satisfy TSP on every finite menu. We say that p satisfies the strong separating property (SSP) on finite menu A if there exists $x \in A$ such that $x \in ex(\Pi_p(A))$ and $x_p^{\downarrow\downarrow} \cap \Pi_p(A) = \emptyset$.

We first show that p satisfies SSP on every finite menu. Suppose to the contrary that p violates SSP on a finite menu A. Two facts regarding extreme points are worth-noting. First, if x is an extreme point of a convex set C, then for all convex set $D \subseteq C$ with $x \in D$, x remains to be an extreme point of D. Second, if $x \in A$ is not an extreme point of $\Lambda_p(A)$, then we can find two distinct points $z, w \in \Gamma_p(A)$ and $t \in (0,1)$ such that $x = tz + (1-t)w$. By the definition of $\Gamma_p(A)$, there exists two finite sequences $(z^k)_{k=1}^n$ and $(w^k)_{k=1}^n$ such that $z^1, w^1 \in A$, $z^n = z$, $w^n = w$, and for all $k \in \{1, ..., n-1\}$, $z^{k+1} \trianglerighteq z^k$ and $w^{k+1} \trianglerighteq w^k$. By the two facts, following a similar proof to that of Lemma 1, we can find a finite sequence of finite menus $\{A^k\}_{k=0}^n$ such that (i) $A^0 = A$, (ii) for all $l, l+1 \in \{0, ..., n\}$

and $x \in A^{l+1}$, $x \notin co(\cup_{k=0}^{l} A^l)$ and there exists $\theta(x) \in co(\cup_{k=0}^{l} A^k)$ with $x \trianglerighteq \theta(x)$, and (iii) for every $x \in A \cap ex(co(\cup_{k=0}^{n} A^k))$, there exists $z \in co(\cup_{k=0}^{n} A^k)$ such that $x \succ z$. Let $\bar{A}$ consists of all alternatives in A and every $\theta(x)$ that is not in $ex(co(\cup_{k=0}^{n} A^k))$. A simple induction implies that $\cup_{k=0}^{n} A^k \subseteq \Gamma_p(\bar{A})$. It then follows from the construction of $\bar{A}$ that for all $x \in \bar{A} \cap ex(\Lambda_p(\bar{A}))$, we have $x \in A \cap ex(co(\cup_{k=0}^{n} A^k))$. For such an alternative x, by (iii) and the fact that $co(\cup_{k=0}^{n} A^k) \subseteq \Lambda_p(\bar{A})$, we can find $z \in \Lambda_p(\bar{A})$ such that $x \succ z$. Therefore, p violates STP on $\bar{A}$, a contradiction.

The reminder of the proof is based on the fact that p satisfies SSP on every finite menu. For a given finite menu A, we can construct a strictly decreasing sequence $(A^k)_{k=0}^{n}$ of submenus of A such that $A^0 = A$, $A^n = \emptyset$, and for all $0 \leq k, k+1 \leq n$,

$$A^{k+1} = A^k \backslash \{x \in A^k : x \in ex(\Pi_p(A^k)) \text{ and } x_p^{\downarrow\downarrow} \cap \Pi_p(A^k) = \emptyset\}.$$

Following a proof similar to that of Theorem 2, we can show that if p satisfies SSP on every menu in A, then there is a preference $\succsim^*$ on X such that (i) for all $x, y \in X$, $x \trianglerighteq y$ implies $x \succsim^* y$, (ii) for all $x \in A$ and $y \in X$, $x \succ y$ implies $x \succ^* y$, and (iii) for all $x \in A$, x is an extreme point of the set $\{y : y \succsim^* x\}$. The remaining proof is similar to that of Theorem 1. Let $\mathcal{R}$ be the set of all order pairs that extend p and satisfy SSP on every finite menu. Endowed with the extension order $\subseteq$, $(\mathcal{R}, \subseteq)$ is a partially ordered set. We can then show that $(\mathcal{R}, \subseteq)$ has a maximal element $p^* = (\trianglerighteq^*, \succ^*)$ such that $\trianglerighteq^*$ is a convex preference, and $\succ^* = \triangleright^*$.[31] To see that $\trianglerighteq^*$ is strictly convex, suppose to the contrary that for some distinct $x, y \in X$, $x \trianglerighteq^* y \trianglerighteq^* tx + (1-t)y$ for some $t \in (0, 1)$. It follows that $tx + (1-t)y$ is not an extreme point of its upper counter set $\{z : z \trianglerighteq^* tx + (1-t)y\}$, contradicting to the fact that p^* satisfies SSP on $\{tx + (1-t)y\}$. □

Proof of Proposition 3. (i) ⇒ (iii). Let f, g and N satisfy the primitive conditions stated in statement (iii). Let $\succsim^*$ be a strictly convex preference that extends $(\trianglerighteq, \succ)$. We want to show that there is nonempty $Q \subseteq N$ such that $g(Q) \subseteq f(Q)$. Define $Q \subseteq N$ such that $k \in Q$ if and only if for all $r \in N$, $f(r) \succsim^* f(k)$. For each $k \in Q$, since $f(k) \succsim^* g(k)$, the convexity of $\succsim^*$ implies that for all $y \in co(f(N))$, $y \succsim^* g(k)$. Thus, for all $k \in Q$, $g(k) \in co(f(N))$. By the strict convexity of $\succsim^*$, for all $k \in Q$ and $y \in co(f(N)) \backslash f(Q)$, $y \succ^* f(k)$. Hence, for all $k \in Q$, $g(k) \in f(Q)$, i.e., $g(Q) \subseteq f(Q)$.

(ii) ⇒ (i). It suffices to show that when p violates STP on some finite menu A, then it violates STP on some finite menu $B \subseteq W_p$. Consider such a menu A. Note that p violates

[31] The only difference is to transfer the condition about the extreme points to some "finite" statement. Note that if $x \in A$ is not an extreme point of $\Pi_p(A)$, then we can find distinct $z, w \in \Pi_p(A)$ and $t \in (0, 1)$ such that $x = ty + (1-t)z$. Since for some *finite* integer k, we have $z, w \in \Pi_p^k(A)$, we can show that x is not an extreme point of $\Pi_p^k(A)$. It then follows that the arguments we use to prove Lemmas 1 and 2 can be similarly applied here to show that $(\mathcal{R}, \subseteq)$ has a maximal element.

STP on A is equivalent to that A satisfies the following property.

$$\exists \text{ finite } D \subseteq \Gamma_p(A) : A \subseteq D \text{ and } x \in A \cap ex(co(D)) \text{ implies } x_p^{\downarrow\downarrow} \cap co(D) \neq \emptyset. \quad (10)$$

If $A = \{x\}$, then we must have $\{x\} \subsetneq \Gamma_p(\{x\})$. Thus $x \in W_p$. By letting $B = \{x\}$, we are done. When $|A| \geq 2$ and $x \notin W_p$ for some $x \in A$, let $A^* = A \backslash \{x\}$. Pick D that satisfies condition (10), and let $D^* = D \backslash \{x\}$. Since $x \notin W_p$, we have $D^* \subseteq \Gamma_p(A^*)$. If $x \notin ex(co(D))$, then $co(D) = co(D^*)$, and A^* satisfies condition (10). If $x \in ex(co(D))$, consider any $y \in A^* \cap ex(co(D^*))$. If $y \in ex(co(D))$, then since $x_p^{\downarrow\downarrow} \cap co(D) \neq \emptyset$ and $y_p^{\downarrow\downarrow} \cap co(D) \neq \emptyset$, by Lemma 8, we have $y_p^{\downarrow\downarrow} \cap co(D^*) \neq \emptyset$. If $y \notin ex(co(D))$, we can find some $z \in co(D^*)$ and $t \in (0,1)$ such that $y = tz + (1-t)x$. It follows from Lemma 8 and the independence of $\succ$ that $y_p^{\downarrow\downarrow} \cap co(D^*) \neq \emptyset$. By induction, when $|A| \geq 2$, we can find some finite menu $B \subseteq A$ with $B \subseteq W_p$ that satisfies condition (10), i.e., p violates STP on B.

(iii) $\Rightarrow$ (ii). Assume to the contrary that there exists a finite menu $A \subseteq W_p$ such that for all $x \in A \cap ex(\Lambda_x(A))$, $x_p^{\downarrow\downarrow} \cap \Lambda_x(A) \neq \emptyset$. Consider a finite menu D with $A \subseteq D \subseteq \Gamma_x(A)$ such that for all $x \in A \cap ex(co(D))$, $x_p^{\downarrow\downarrow} \cap co(D) \neq \emptyset$ and for all $y \in D \backslash A$, there is a sequence $(x^k)_{k=1}^m$ with $m \geq 2$ such that $x^1 \in A$, $x^n = y$, and for all $k \in \{1, ..., m-1\}$, $x^{k+1} \trianglerighteq x^k$. Let $B = ex(co(D))$ and we have $co(B) = co(D)$. Let $B^* = \{x \in B : x_p^{\downarrow\downarrow} \cap co(B) = \emptyset\}$. By Lemma 8, we have $B^* \neq \emptyset$. It follows that $A \cap B^* = \emptyset$. Enumerate B^* as $B^* = \{x^1, ..., x^n\}$. Let $N = \{1, ..., n\}$ and define $f : N \to X$ such that for all $k \in N$, $f(k) = x^k$. For each $k \in N$, find a finite sequence of alternatives $(x^{k,i})_{i=1}^{l_k}$ with $l_k \geq 2$ such that $x^{k,1} \in A$, $x^{k,l_k} = x^k$, and for all $i \in \{1, ..., l_k - 1\}$, $x^{k,i+1} \trianglerighteq x^{k,i}$. We further require that for each $k \in N$, the sequence $(x^{k,i})_{i=1}^{l_k}$ is *minimal*, in the sense that there exists no other sequence of alternatives that has the same property but is strictly shorter (i.e., with a length less than l_k). For each $k \in N$, let $g(k) = x^{k,l_k-1}$. The minimality of the sequence $(x^{k,i})_{i=1}^{l_k}$ implies $f(k) \trianglerighteq g(k)$ and $f(k) \neq g(k)$, i.e., condition (iii) holds.

We verify that f, g and N satisfy other conditions stated in statement (iii). Since for all $k \in N$, $f(k) \trianglerighteq g(k)$ and $f(k) \neq g(k)$, we have $g(N) \subseteq W_p$, i.e., condition (i) holds. Condition (ii) holds since $B = ex(co(B))$ and $B^* \subseteq B$. By the definition of B^*, condition (iv) holds. Finally, note that for all $k \in N$, $g(k) \in co(B)$. Either $g(k) \in co(B^*)$ or there is $t \in (0,1]$, $z \in co(B \backslash B^*)$ and $w \in co(B^*)$ such that $g(k) = tz + (1-t)w$. For the latter case, by Lemma 8, there is $w^* \in co(B^*)$ with $g(k) \succ w^*$. Thus, condition (v) holds.

To derive a contradiction, suppose for some nonempty $Q \subseteq N$, $g(Q) \subseteq f(Q)$. Pick $k \in Q$ such that for all $k' \in Q$, $l_k \leq l_{k'}$. Since $x^{k,l_k-1} \in f(Q)$, there is $k' \in Q$ such that $x^{k'} = x^{k,l_k-1}$. The minimality of $(x^{k',i})_{i=1}^{l_{k'}}$ implies $l_{k'} \leq l_k - 1$, contradicting with $l_k \leq l_{k'}$. □

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