Resilience *vs.* Fragility: Global Properties of Economies with Endogenous Production Networks

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Abstract

We characterize equilibria in a canonical class of economies with endogenous production networks. We provide an exact formula for equilibrium prices: the log-price vector is a min-max over networks of an affine function of log-productivities. Economically, the two operators capture the forces of resilience and fragility: the "min" reflects gains from switching suppliers or technologies as prices move, while the "max" reflects amplification driven by rigid input requirements. Standard specifications, such as endogenous Cobb-Douglas and constant elasticity of substitution technologies, hardwire resilience or fragility, ruling out qualitative economic phenomena. We show that a class of endogenous Leontief technologies remains tractable yet sufficiently flexible to generate any feasible equilibrium outcomes. In further applications, we: (i) generalize Hulten's theorem and find that Domar weights are *in*sufficient to characterize the propagation of even small shocks, (ii) characterize the effects of microeconomic volatility on GDP, finding that standard second-order approaches necessarily overstate the adverse effects of volatility, and (iii) provide general conditions for when microeconomic shocks can generate macroeconomic fluctuations.

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1 Introduction

Modern production is characterized by the presence of long and complex supply chains. There are two broad perspectives on the economic implications of these interconnections. On the one hand, there is the perspective that rigid supply chains involved in networked production make the economy *fragile* (see *e.g.,* Elliott, Golub, and Leduc, 2022; Acemoglu and Tahbaz-Salehi, 2025), which we define as the property that (in logarithms) increases in volatility in microeconomic productivity raise prices and depress GDP on average. On the other hand, there is an influential and competing perspective that the ability to distribute production and change suppliers (see *e.g.,* Dew-Becker, 2023; Kopytov, Mishra, Nimark, and Taschereau-Dumouchel, 2024) may make the economy *resilient* or anti-fragile (Taleb, 2013).

At a theoretical level, analyzing the question of when the economy is resilient and when it is fragile requires going beyond canonical settings in which firms' prices are log-linear in their input prices, which obtains under log-linearizations of general models (Hulten, 1978; Liu, 2019; Carvalho, Nirei, Saito, and Tahbaz-Salehi, 2021) or equivalently when all firms operate Cobb-Douglas production functions (Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi, 2012). This is because, in such economies, changes in the microeconomic volatility of log-productivity have no effect on expected log-prices and log-GDP.

However, outside of the log-linear case, there is no known general solution for equilibrium prices and output. To overcome this challenge, the literature has proceeded in three directions, each yielding valuable insights: (i) specializing to specific production technologies (Oberfield, 2018; Acemoglu and Azar, 2020; Elliott et al., 2022; Kopytov et al., 2024), (ii) considering higher-order approximations (Baqaee and Farhi, 2019), or (iii) studying the asymptotic effects of shocks (Dew-Becker, 2023). Thus, while our understanding of endogenous production networks has advanced enormously, we lack a general theory that is valid outside of important but special cases (*cf.* (i)) or for shocks that are neither very small (*cf.* (ii)) nor very large (*cf.* (iii)).

For this reason, we study the global properties of the canonical static single-factor production network model (as in Acemoglu et al., 2012; Carvalho and Tahbaz-Salehi, 2019). We maintain the core assumption at the heart of input-output analysis (Leontief, 1941): production is constant returns to scale. We otherwise make no parametric assumptions on the nature of production technology at the microeconomic level.

Our main result provides globally valid and primitive formulae for equilibrium prices and output that hold for any such economy. We exploit these formulae to understand conditions on technology at the microeconomic level such that resilience and/or fragility emerge as endogenous macroeconomic properties. In proving these results, we also derive a number of new results in producer theory that may be of independent interest. We further apply the general results to provide a generalization of Hulten's (1978) theorem compatible with non-differentiable production technologies—which arise naturally from endogeneity of the production network. Importantly, this result demonstrates that even local Domar weights can be uninformative for the propagation of small shocks, which is the exact opposite implication of the standard result of Hulten (1978). That is, in resilient economies with fast substitution, there is no shortcut in understanding network propagation: an analyst must specify technology and cannot merely lift Domar weights from the data. Moreover, with endogenous technology, we find that standard second-order approaches to studying the effects of shocks to microeconomic volatility always *overstate* the adverse consequences of increased risk as they omit the effects of rapid substitution. Finally, we provide robust bounds that describe when microeconomic productivity shocks can generate sizable GDP fluctuations, without making any additional assumptions on production technology.

Model. The model follows the canonical static single-factor production network setting (see Carvalho and Tahbaz-Salehi, 2019, for a review). There are many sectors, each perfectly competitive, that use as inputs the outputs of other sectors and a single primary factor (such as labor). A representative household inelastically supplies the primary factor. Within this context, we allow for an arbitrary CRS production function. Thus, our analysis allows us to nest: (i) the Cobb-Douglas benchmark (Acemoglu et al., 2012), (ii) nested constant elasticity of substitution (CES) technology (Baqaee and Farhi, 2019; Dew-Becker, 2023), (iii) heterogeneous CES technology (Hanoch, 1971; Chodorow-Reich, Gabaix, and Viviano, 2026), (iv) Cobb-Douglas production technology with endogenous supplier choice at the extensive and intensive margins (Acemoglu and Azar, 2020; Kopytov et al., 2024), (v) Leontief production technology with and without endogenous supplier choice (Acemoglu and Tahbaz-Salehi, 2025), and (vi) task-based production functions (Acemoglu and Restrepo, 2018).

Main Characterization Results. Our main results provide globally valid and primitive formulae for equilibrium prices and GDP for these economies. Our formula for equilibrium prices asserts that the equilibrium log-price vector is the "min-max" over pairs of input-output networks of the Leontief inverse of the maximizing network applied to the vector of log-productivities plus a *production wedge*. This wedge depends on the primitive production functions evaluated at the minimizing network and the relative entropies between the rows of the minimizing and maximizing networks. Intuitively, equilibrium prices are always the "min-max" of what prices would be in Cobb-Douglas economies with a certain kind of productivity distortion. The resulting formula for log-GDP is simply the inner product of the household's expenditure shares and this equilibrium log-price vector.

To derive these results, we first express the fixed-point equation for equilibrium prices that holds in such economies in terms of log-prices. In this system of equations, each sector's log-price is affine in its log-productivity and the log of its unit cost function. Next, we show that any unit cost function that arises from a CRS production function also results from a seemingly more specific endogenous Leontief production function. Accordingly, the cost function is simply the minimum over vectors of Leontief input weights of the inner product of these input weights and input prices, divided by the *as if* productivity of operating that Leontief technology. From this, we show that the log-cost function is simply a "min-max" over pairs of vectors of input weights of an *affine* function of the input log-prices. Finally, we show that the operations of finding the fixed point and computing the "min-max" commute. That is, if we naively ignore the "min-max," compute the fixed point, and only then apply the "min-max" to the solution, then we obtain the true solution to the fixed point system. This allows us to obtain the primitive formula because, if we ignore the "min-max," then the fixed point system is linear and can be solved analytically.

Resilience *vs.* Fragility. We then apply this general representation to understand when the economy is resilient (fragile), *i.e.,* when equilibrium log-prices are concave (convex) functions of the log-productivity vector. The general representation theorem makes clear that neither property should generally be expected globally: the "min" will serve to induce concavity and the "max" will lead to convexity. Thus, the general representation suggests that prices will generally be concave in some regions and convex in others, depending on whether the force of the "min" or the "max" dominates.

Indeed, in a leading example, we show how switching between technologies can naturally give rise to an "S-shaped" response of average log-GDP to increases in the volatility of microeconomic log-productivity: small levels of volatility lower GDP on average, intermediate levels raise it, and high levels lower it again. Intuitively, if production technology is always locally rigid (*e.g.,* Leontief), then small levels of volatility are always harmful. However, the ability to switch between different technologies with different input intensities creates option value and a love of volatility. Ultimately, though, if production technology is always locally rigid, then very high volatility is asymptotically harmful.

This notwithstanding, many standard specifications of production functions ensure that the economy is globally resilient or fragile. This occurs when each sector's log-cost function is always a concave or convex function of log-prices, respectively. When this occurs, such specifications necessarily rule out qualitative phenomena: increases in productivity volatility are always either good or bad on average. Indeed, for this reason, second-order approximations in logarithms (as in Baqaee and Farhi, 2019) or a focus on asymptotics (as in Dew-Becker, 2023) do not allow us to detect the presence of "S-shaped" responses.

To understand when assumptions on technology can rule out these qualitative phenomena, we characterize the property that a log-cost function is globally concave or convex. As a concrete example, a CES production function with elasticity of substitution greater (lesser) than unity has a globally concave (convex) log-cost function. More generally, we show that any concave log-cost function arises from an endogenous Cobb-Douglas production function, which has become a particularly popular production function in the study of endogenous production networks (see *e.g.,* Acemoglu and Azar, 2020; Kopytov et al., 2024). This shows both that endogenous Cobb-Douglas technology is more general than it may appear: it allows the analyst to capture *any* concave log-cost function. However, by the same token, by ruling out convexity, such a specification ensures global resilience. On the other side, we show that any convex log-cost function arises from a particular form of task-based production (see *e.g.,* Acemoglu and Restrepo, 2018) where a set of tasks, each of which has Cobb-Douglas technology, must be completed to produce output.

Further Applications: Hulten's (1978) Theorem, The Effects of Volatility, and Robust Variance Bounds. We next apply our general representation to derive the effects of small shocks of productivity on equilibrium prices and output. We show that equilibrium prices are always directionally differentiable, but not necessarily Gateaux differentiable, *i.e.,* the derivative depends on the direction of shock considered. The failure of Gateaux differentiability is economically important because it corresponds to the presence of technology switching, the core phenomenon that the endogenous production networks literature sets out to study: if a sector is ever indifferent at equilibrium prices between producing using different input mixes, then different directions of shocks will lead to different input mixes and different shock propagation. This is likely to occur when a sector can use multiple different production technologies: as relative prices change, there will be a critical point at which they become indifferent between a given pair of technologies, inducing non-differentiability. While such indifferences are non-generic (*i.e.,* they hold at a zero measure set of prices), ignoring the non-differentiabilities they induce can lead to inaccurate quantitative guidance, as we show in a leading application. Intuitively, when such a switch happens, the correct coefficients in any order of Taylor expansion jump and the previously accurate approximation suddenly breaks down.

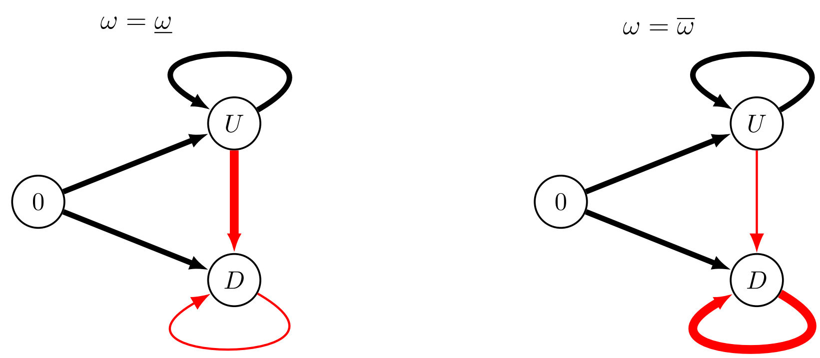
We use this formula for the derivative of prices to derive a corresponding formula for the derivative of GDP, providing a generalization of Hulten's theorem: the effect on GDP is the *maximum* over the set of Domar weights (the vector of sales shares of nominal GDP across sectors) that may prevail at equilibrium of the inner product of these Domar weights and the shock direction expressed in log-TFP units (Proposition 7). The presence of the maximum generalizes Hulten's theorem beyond the differentiable case and allows economically for the presence of technology switching. This provides nuance to the typical interpretation of Hulten's theorem as implying that, for small shocks, input-output structure can be ignored. This is because, even for small shocks, it is important to account for the possibility that changes in input mix lead to rapid changes in Domar weights and the direction of shock propagation.

Next, we study the effects of increasing the volatility of microeconomic productivity on average GDP. We call an economy locally resilient (resp. fragile) if increases in volatility increase (resp. decrease) average GDP. We provide a general formula for the local resilience of an economy, finding it is the sum of two terms: the average derivative of the Domar weights in productivity—which may be positive or negative—and a term reflecting the average jump in Domar weights from technology switching—which is always positive. The first term reflects an *exact* version of the Baqaee and Farhi (2019) logic: derivatives of Domar weights describe the second-order effects of productivity shocks. The second term reveals an additional force: when technology switches can occur, there is a form of option value in the economy, which always pushes toward macroeconomic resilience. Moreover, the presence of this switching term further underscores the importance of technology switching; even if these switches occur at a zero-measure set of points, that does not mean that they can be neglected.

Finally, we apply our generalization of Hulten's theorem to derive robust upper and lower bounds on the variance of log-GDP that hold without restricting technology. This allows us to understand when microeconomic fluctuations in productivity can generate sizable fluctuations in aggregate output without making strong and untestable assumptions on production technology. We define, for each sector, the smallest and largest Domar weight that arises in equilibrium across all productivity realizations. The squared largest (across sectors) smallest (across productivity realizations) Domar weight multiplied by the microeconomic volatility of log-productivity provides a lower bound, while the Herfindahl index of the biggest (across productivity realizations) Domar weights multiplied by the microeconomic volatility of log-productivity provides an upper bound. This result highlights that microeconomic fluctuations can generate large aggregate fluctuations when at least one sector maintains a non-negligible sales share of GDP even when it is very unproductive. Intuitively, this corresponds to the presence of sectors that cannot be substituted even when their products are very expensive.

Additional Technical Literature. Having already summarized how our paper contributes to the macroeconomic literature on production networks, we discuss how the methods we use relate to existing technical work that characterizes fixed points. Most relatedly, Cerreia-Vioglio, Corrao, and Lanzani (2025) characterize the properties of the limit of a general family of fixed point problems arising from operators studied in decision theoretic analyses (see *e.g.,* Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio, 2011; Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini, 2014; Maccheroni, Marinacci, and Rustichini, 2006). The fixed point problem that characterizes equilibrium prices in production networks lies within this class. In this paper, we show that the additional structure of cost minimization imposes restrictions that allow us to characterize the fixed point globally, away from the specific limits that have previously been characterized.

Figure 1: Illustrating the Potential Network Configurations



Notes: The left network corresponds to D using U intensively ($\omega = \underline{\omega}$). The right network corresponds to D using D intensively ($\omega = \overline{\omega}$). The thickness of arrows denotes the strength of the connection and the direction denotes input usage.

2 Why Global Properties Are Important: An Example

We begin with a simple example to motivate the importance of a global approach. We do so by highlighting the potentially inaccurate conclusions that arise from standard approximations.

Setting. Suppose that there are two sectors in the economy, an upstream sector U and a downstream sector D. The upstream sector uses labor and upstream inputs to produce output:

$$Y_U = 2Z_U Q_{U,0}^{\frac{1}{2}} Q_{U,U}^{\frac{1}{2}} \tag{1}$$

where Z_U is the productivity of sector U, $Q_{U,0}$ is the input of a primary factor, and $Q_{U,U}$ is sector U's usage of goods produced in sector U. The downstream sector uses labor, upstream inputs, and downstream inputs to produce output and can do so using two different production technologies indexed by $\omega \in \{\underline{\omega}, \overline{\omega}\} \subset (0, 1)$:

$$Y_D^{\omega} = 2Z_D Q_{D,0}^{\frac{1}{2}} \left(\min \left\{ \frac{Q_{D,D}}{\omega}, \frac{Q_{D,U}}{1-\omega} \right\} \right)^{\frac{1}{2}} \tag{2}$$

where Z_D is the productivity of sector D, $Q_{D,0}$ is the input of a primary factor, $Q_{D,D}$ is D's usage of goods produced downstream, and $Q_{D,U}$ is D's usage of upstream inputs. Intuitively, upstream and downstream goods are always perfect complements in the production of downstream goods, but the downstream sector can choose between two distinct production technologies. We illustrate the potential network configurations that can arise in Figure 1.

To close the model, we assume that the primary factor has an aggregate supply of one and has a price that we normalize to one. Finally, a household has Cobb-Douglas preferences:

$$U = 2C_D^{\frac{1}{2}}C_U^{\frac{1}{2}} \tag{3}$$

where C_i is final consumption from sector $i \in \{D, U\}$. We will examine the outcome of a competitive equilibrium in this simple setting, in which both sectors minimize costs and make no profits, the household maximizes its utility, and all markets clear.

Solving for Equilibrium Outcomes. By solving the cost-minimization problem of the upstream sector and applying the zero-profit condition, the price of the upstream sector follows $P_U = 1/Z_U^2$. We now perform the same steps for the downstream sector and find that:

$$P_D = \mathbb{I}[Z_D \leq Z_U]\frac{\underline{\omega} + \sqrt{\underline{\omega}^2 + 4(1-\underline{\omega})\frac{Z_D^2}{Z_U^2}}}{2Z_D^2} + \mathbb{I}[Z_D > Z_U]\frac{\overline{\omega} + \sqrt{\overline{\omega}^2 + 4(1-\overline{\omega})\frac{Z_D^2}{Z_U^2}}}{2Z_D^2} \tag{4}$$

Intuitively, the cost of producing one unit of the composite intermediate input is just the weighted sum of input prices, and the sector will always choose the cheapest technology ω and D is cheaper than U in equilibrium iff and only if $Z_D \leq Z_U$.

By solving the household's utility maximization problem, we obtain that log-GDP follows:

$$\ln \mathrm{GDP} = \frac{1}{2}\ln 2 + \ln Z_U + \ln Z_D - \frac{1}{2}\begin{cases} \ln\left(\underline{\omega} + \sqrt{\underline{\omega}^2 + 4(1-\underline{\omega})\frac{Z_D^2}{Z_U^2}}\right), Z_D \leq Z_U, \\ \ln\left(\overline{\omega} + \sqrt{\overline{\omega}^2 + 4(1-\overline{\omega})\frac{Z_D^2}{Z_U^2}}\right), Z_D > Z_U. \end{cases} \tag{5}$$

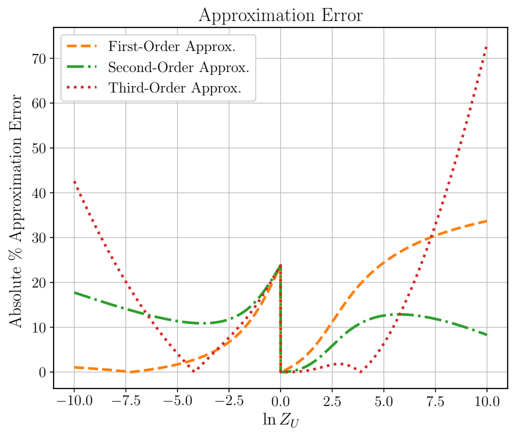
We now proceed to study the global properties of equilibrium prices and GDP. The standard approach to studying the properties of this economy would be to take a first-order approximation (Hulten, 1978) or a second-order approximation (Baqaee and Farhi, 2019) of $\ln \mathrm{GDP}$ in $\ln Z = (\ln Z_U, \ln Z_D)$. At all points $\ln Z_0$ (except those such that $Z_U = Z_D$), we have that:

$$\ln \mathrm{GDP} \approx \overline{\ln \mathrm{GDP}} + D^T \Delta \ln Z + \frac{1}{2}(\Delta \ln Z)^T H \Delta \ln Z \tag{6}$$

where $\Delta \ln Z = \ln Z - \ln Z_0$, $\overline{\ln \mathrm{GDP}}$ is a constant, and D is the gradient and H is the Hessian of $\ln \mathrm{GDP}$ evaluated at the initial $\ln Z_0$.

Approximations Can Give Inaccurate Quantitative Guidance. We first examine the accuracy of these approximations, considering approximations of $\ln \mathrm{GDP}$ in $\ln Z_U$ at a value that is positive but very close to zero, holding fixed $\ln Z_D = 0$. In Figure 2, we plot the resulting approximations (Panel (a)) and percentage absolute approximation errors (Panel (b)). For positive shocks, the approximations are highly accurate. Even a 100% positive shock gives rise to only a small error under the first-order approximation and an utterly negligible error under

Figure 2: Local Approximations Can Provide Inaccurate Quantitative Guidance



Notes: This figure plots the percent absolute approximation error, *i.e.*, $|(\text{True} - \text{Approx.})/\text{True}|$, for first- (orange), second- (green), and third-order (red) Taylor approximations to $\ln \text{GDP}$ in $\ln Z_U$ around the point $\ln Z_U = 1 \times 10^{-4}$. Parameters: $\underline{\omega} = 0.2$, $\overline{\omega} = 0.8$, $\ln Z_D = 0$.

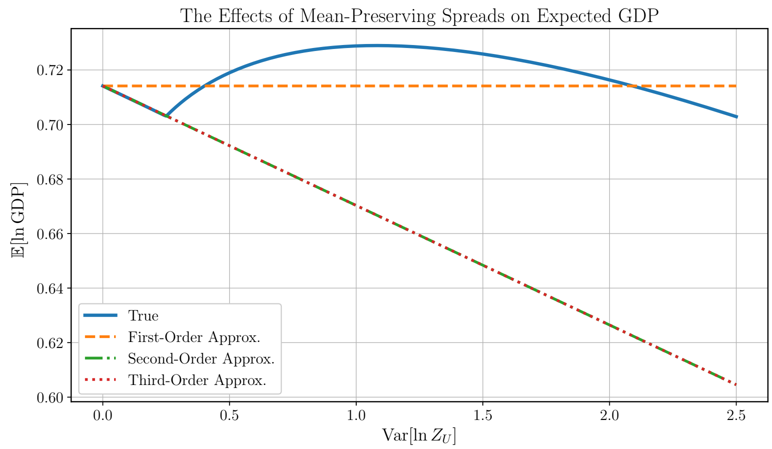
the second-order approximation. For negative shocks, however, the approximations break down, even locally. In particular, negative shocks lead to an approximation error of about 20% and the first-order approximation performs substantially better than the second-order approximation.

One natural reaction to these results is to posit that second-order approximations are not enough and higher-order series expansions must be considered. However, as can be seen in Figure 2, this does not solve the problem that the local approximation breaks down for small negative shocks, and it can lead to larger approximation errors for non-negligible positive shocks. Indeed, the issue could not be remedied by any series expansion. The problem is that the Taylor approximation *functional* that returns an n-th order series approximation at a given value of $\ln Z$ is discontinuous at the kink at which $\ln Z_D = \ln Z_U$ and the downstream sector switches technology. For this reason, a global approach is necessary to provide accurate guidance.

Approximations Can Rule Out Qualitative Phenomena. More importantly, putting aside the issue of accuracy, we also argue that first-order and second-order approximations rule out qualitatively interesting and potentially important economic phenomena. One of the most important motivations in the modern production network literature is understanding how microeconomic shocks at the sector-level may affect the first and second moments of $\ln \text{GDP}$. Consider now that $\ln Z$ is an IID random vector with mean 0 and variance-covariance matrix $\sigma^2 I$. Under the second-order approximation of Baqaee and Farhi (2019) in an arbitrary economy with n sectors, we have that:

$$\mathbb{E}[\ln \text{GDP}] = \overline{\ln \text{GDP}} + \frac{1}{2}\sigma^2 \sum_{i=1}^{n} H_{ii} \quad \text{Var}[\ln \text{GDP}] = 2\sigma^4 \sum_{i=1}^{n} H_{ii}^2 + \sigma^2 \sum_{i=1}^{n} D_i^2. \tag{7}$$

Figure 3: Local Approximations Can Rule Out Qualitative Phenomena



Notes: This figure plots expected ln GDP when $\ln Z_U = k \pm x$ (each with probability 0.5) against $\text{Var}[\ln Z_U] = x^2$ under the global solution (blue) and under first- and second-order Taylor approximations around the point $\ln Z_U = k$. Parameters: $\underline{\omega} = 0.2$, $\overline{\omega} = 0.8$, $\ln Z_D = 0$, $k = 0.5$.

How does increasing the variance of microeconomic shocks affect the mean and variance of GDP growth? Thus, second-order methods imply that as we increase the variance of shocks, the effect on output must be (i) unambiguously positive or negative, and (ii) convex.

In our example economy, suppose that $\ln Z_U = k \pm x$, each with probability 0.5, while $\ln Z_D = 0$. Figure 3 plots the effects of increasing the volatility of upstream productivity on expected output. The global solution reveals an S-shaped response in which initial increases in variance lower expected output, intermediate levels of variance increase aggregate output, and high levels of variance decrease output again. Intuitively, at low levels of variance, the economy mostly operates one Leontief technology downstream, and so variance in input prices is harmful. However, at intermediate levels of variance, the downstream sector can switch across technologies and the option value of switching technologies makes variance beneficial. Ultimately though, when variance becomes very large, the fact that they always operate a Leontief technology (even if they can choose the best one) makes variance lower average output. These qualitative patterns cannot be captured by first-, second-, and even third-order approximations.

Summary. Of course, one might think that we are presenting a pathological example, and no reasonable model would display kinks like this. However, we will later show that such kinks are a *general property* of models with endogenous production networks—including many leading examples in the literature. Intuitively, when relative prices cross critical values, firms may switch across technologies or input suppliers, leading input demands to jump and generating a kinked cost function. Thus, to capture the force of technology switching, it is insufficient to study approximations; understanding the global properties of such economies is required.

3 Model

We consider a canonical static model of a production network with $n \in \mathbb{N}$ sectors, following Acemoglu et al. (2012), Baqaee and Farhi (2019), Acemoglu and Azar (2020), Dew-Becker (2023), and Kopytov et al. (2024). The only novelty of our setting is its generality.

3.1 Primitives

Supply. Each sector $i \in [n] = \{1, \ldots, n\}$ produces a single output by combining a primary factor in amounts Q_{i0} and intermediate goods in amounts $Q_i = (Q_{ij})_{j=1}^n$ according to a production function $F_i : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$:

$$F_i(Q_{i0}, Q_i) = Z_i \tilde{G}_i(Q_{i0}, Q_i) \tag{8}$$

where $Z_i \in \mathbb{R}_{++}$ is the Hicks-neutral productivity and $\tilde{G}_i : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$ is the underlying production technology.

The goal of our analysis is to study the *robust* properties of equilibrium outcomes in production networks, *i.e.,* those properties that obtain without functional form restrictions on the nature of the production network. Thus, we impose minimal regularity assumptions, a normalization that allows us to interpret the Z_i as Hicks-neutral productivity shocks, and the core economic assumption at the heart of input-output analysis: production is CRS.

Assumption 1. *For all sectors $i \in [n]$, the production function $\tilde{G}_i : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$ is positive homogeneous, non-decreasing, continuous, and such that the optimal expenditure share on the primary factor is bounded away from zero.*[1]

In order to simplify notation and exposition for the main text, we express production technologies in the form:

$$\tilde{G}_i(Q_{i0}, Q_i) = \left(\frac{Q_{i0}}{1-\beta}\right)^{1-\beta} \left(\frac{G_i(Q_{i1}, \ldots, Q_{in})}{\beta}\right)^{\beta} \tag{9}$$

where $\beta \in (0, 1)$ is the intermediate-input share of costs, and $G_i : \mathbb{R}_+^n \to \mathbb{R}_+$ satisfies all but the last of the requirements of Assumption 1. We show in Lemma 16 in Appendix G that by redefining the set of sectors, *any* economy with production technologies satisfying Assumption 1 can be *exactly* represented by technologies of this seemingly more restrictive form. Moreover, so

[1]First, it is equivalent to assume that each G_i is only upper semicontinuous since we restrict to intermediate production functions that are positive homogeneous (see, e.g., Moore, 1999, Proposition 5.95). Second, our definition of the property that the expenditure share on the primary factor is bounded away from zero is that the production function of sector i can be written for some $\bar{\beta} > 0$ as $\tilde{G}_i(Q_0, Q) = Z_i(Q_0/(1-\bar{\beta}))^{1-\bar{\beta}}(\hat{G}_i(Q_0, Q)/\bar{\beta})^{\bar{\beta}}$, where $\hat{G}_i$ satisfies the other criteria of Assumption 1.

that Z_i has an unambiguous meaning, we normalize G_i such that $\max_{Q_i \in \Delta} G_i(Q_{i1}, \ldots, Q_{in}) = 1$, where $\Delta = \left\{ w \in \mathbb{R}^n_+ : \sum_{j=1}^n w_j = 1 \right\}$. We call such a G_i *regular*.

In light of this representation, the key object that determines the network of input-output relationships is the intermediate input production function G_i. The current literature that provides analytical results for the equilibrium properties of production networks has considered various assumptions on G_i: Acemoglu et al. (2012) and a large subsequent literature (see Carvalho and Tahbaz-Salehi, 2019, for a review) assume a Cobb-Douglas production function, Dew-Becker (2023) assumes a constant elasticity of substitution (CES) production function, Kopytov et al. (2024) assume a Cobb-Douglas production function with endogenous weights. Our formulation is sufficiently general to encompass the universe of intermediate input production functions considered in this previous work. As we demonstrate in Section 3.4, this allows us to consider models in which the technologies, suppliers, and substitution elasticities of firms are chosen endogenously.

Demand. As the focus of our analysis is on the production network, we follow Acemoglu et al. (2012), Dew-Becker (2023), and Kopytov et al. (2024). This is justified because, as is well known and we will shortly see, demand has no role in determining equilibrium prices. Instead, demand matters only for computing GDP given prices. A representative household inelastically supplies a unit of primary factor and consumes a bundle of goods $C \in \mathbb{R}^n_+$ with a Cobb-Douglas utility function:

$$U(C) = \prod_{i=1}^{n} \left(\frac{C_i}{\xi_i} \right)^{\xi_i} \qquad \forall C \in \mathbb{R}^n_+ \tag{10}$$

where each $\xi_i \geq 0$ captures the importance of good i for the final consumer and $\sum_{i=1}^n \xi_i = 1$. Normalizing the wage in the economy to one and denoting the price of good i as P_i, the household faces the standard budget constraint $\sum_{i=1}^n C_i P_i \leq 1$. In general, so long as U is positive homogeneous, our results characterizing GDP extend.[2]

3.2 Equilibrium

Given this structure for production and demand, the competitive equilibrium is defined by requiring that firms and consumers are price takers and firms make zero profits.[3]

Definition 1. *A competitive equilibrium is a vector of prices, $P^* \in \mathbb{R}^n_{++}$, a matrix of unitary input quantities $Q^* \in \mathbb{R}^{n \times (n+1)}_+$, a vector of production levels $Y^* \in \mathbb{R}^n_+$ and a final consumption vector $C^* \in \mathbb{R}^n_+$ such that*

[2]This is because we can treat U as if it were a production function and apply Proposition 1 below. For the later GDP characterization this induces an additional min-max layer over the household's expenditure shares (see Footnote 6).

[3]As technology is constant returns to scale, these requirements are equivalent to profit maximization in the more standard definition of competitive equilibrium.

1. Firms choose a cost-minimizing vector of unitary inputs:

$$(Q_{i0}^*, Q_i^*) \in \underset{(Q_{i0},Q_i)\in\mathbb{R}_+^{n+1}}{\arg\min} \sum_{j=1}^{n} Q_{ij}P_j^* + Q_{i0} \quad s.t. \quad F_i(Q_{i0}, Q_i) \geq 1 \qquad \forall i \in [n] \,. \tag{11}$$

2. Firms price at their marginal cost:

$$P_i^* = \sum_{j=1}^{n} Q_{ij}^* P_j^* + Q_{i0}^* \qquad \forall i \in [n] \,. \tag{12}$$

3. The labor market and all product markets clear:

$$\sum_{i=1}^{n} Y_i^* Q_{i0}^* = 1 \,, \quad \sum_{i=1}^{n} Y_i^* Q_{ij}^* + C_j^* = Y_j^* \qquad \forall j \in [n] \,. \tag{13}$$

4. The household chooses consumption to maximize its utility:

$$C^* \in \underset{C\in\mathbb{R}_+^n}{\arg\max}\, U(C) \quad s.t. \quad \sum_{i=1}^{n} C_i P_i^* \leq 1. \tag{14}$$

We will be concerned with characterizing equilibrium properties of prices and GDP. We call a solution to Equation 12 an equilibrium price vector.

3.3 Preliminaries: The Log-Cost Operator and Equilibrium Prices

Our characterization of the equilibrium leverages the properties of a particular transformation of the sectors' cost functions. By solving each sector's cost minimization problem, we obtain:

$$\min_{\substack{(Q_{i0},Q_i)\in\mathbb{R}_+^{n+1}: \\ F_i(Q_{i0},Q_i)\geq 1}} \sum_{j=1}^{n} Q_{ij}P_j + Q_{i0} = \frac{1}{Z_i} K_i\left(P\right)^{\beta} \qquad \forall i \in [n] \,, \forall P \in \mathbb{R}_{++}^n \tag{15}$$

where $K_i(P)$ is defined as:[4]

$$K_i(P) = \min_{Q_i\in\mathbb{R}_+^n} \sum_{j=1}^{n} Q_{ij}P_j \quad \text{s.t.} \quad G_i(Q_i) \geq 1. \tag{16}$$

We define the associated map from log-prices to log-costs as:

$$T_i(p) = \ln(K_i(\exp(p))) \quad \forall p \in \mathbb{R}^n. \tag{17}$$

[4]See Lemma 9 in the appendix for a proof of Equation 15 and that the minima in Equations 15 and 16 are attained.

The cost operator $K : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ and the *log-cost operator* $T : \mathbb{R}^n \to \mathbb{R}^n$ are the operators whose i-th row are respectively equal to K_i and T_i.

By defining the inverse log-productivity of sector $i \in [n]$ as $x_i = -\frac{1}{1-\beta} \ln Z_i$, we can rewrite the fixed-point equation that defines equilibrium price vectors Equation 12 as:

$$p_i = (1-\beta)x_i + \beta T_i(p) \qquad \forall i \in [n]. \tag{18}$$

It is standard to show that the operator defining this fixed point equation is a contraction,[5] and therefore that there exists a unique solution to this equation; we denote this unique equilibrium price vector by $\Pi(x)$. This represents a vector of equilibrium log-prices at a given vector of log-productivity shocks $x \in \mathbb{R}^n$.

We denote the equilibrium aggregate log-output at such a vector of log-prices by

$$\Upsilon(x) = \max_{C \in \mathbb{R}^n_+} \ln U(C) \quad \text{s.t.} \quad \sum_{i=1}^n C_i \exp(\Pi_i(x)) \leq 1. \tag{19}$$

It is standard to show that:[6]

$$\Upsilon(x) = -\sum_{i=1}^n \xi_i \Pi_i(x). \tag{20}$$

Because of this observation, it suffices to characterize equilibrium prices. While equilibrium prices and *aggregate* output are unique, there is not necessarily a unique equilibrium matrix of *input* quantities Q. We will characterize this stronger uniqueness property in Proposition 6.

3.4 Example Economies

Before embarking on our analysis, we give four classes of example economies that fall within our general framework and to which we will often turn to exemplify our general results.

Example 1 (Nested CES)**.** A standard non-linear specification of production in which the input-output network depends endogenously on equilibrium prices is the nested CES structure:

$$G_i(Q_i) = \left(\sum_{j=1}^n \alpha_{ij}^{\frac{1}{\sigma_i}} Q_{ij}^{\frac{\sigma_i - 1}{\sigma_i}} \right)^{\frac{\sigma_i}{\sigma_i - 1}} \tag{21}$$

for some vector of weights $\alpha_i \in \Delta$ and elasticity of substitution $\sigma_i \in \bar{\mathbb{R}}_+$. As is well-known, $\sigma_i =$

[5] As T is monotone and translation invariant (see Proposition 1), Blackwell's sufficient conditions imply that the fixed point operator is a contraction

[6] With more general preferences, the GDP formula would instead be a non-linear aggregate of prices. Under positive homogeneity of U, Proposition 1 can be applied directly to provide a general formula for GDP given prices.

0 corresponds to the Leontief production function $G_i(Q_i) = \min\left\{\frac{Q_{ij}}{\alpha_{ij}}\right\}$, $\sigma_i = \infty$ corresponds to the perfect substitutes production function $G_i(Q_i) = \sum_{i=1}^n Q_{ij}$, and $\sigma_i = 1$ corresponds to a Cobb-Douglas production function $G_i(Q_i) = \prod_{i=1}^n (Q_{ij}/\alpha_{ij})^{\alpha_{ij}}$. For the nested CES structure, the log-cost operator is given in general by:

$$T_i(p) = \frac{1}{1-\sigma_i} \ln\left(\sum_{j=1}^n \alpha_{ij} \exp\left((1-\sigma_i)p_j\right)\right). \tag{22}$$

Existing work has characterized the second-order local implications of shocks with nested CES (Baqaee and Farhi, 2019) and the asymptotic effects with large shocks (Dew-Becker, 2023). However, the global equilibrium properties of these economies have not been characterized. △

Example 2 (Endogenous Cobb-Douglas)**.** Recent work that explores the endogenous determination of supply linkages has proposed models in which each sector $i \in [n]$ chooses its Cobb-Douglas input shares to economize on costs at the opportunity cost of perhaps suffering reduced productivity:

$$G_i(Q_i) = \max_{\alpha_i \in C_i} A_i(\alpha_i) \prod_{j=1}^n \left(\frac{Q_{ij}}{\alpha_{ij}}\right)^{\alpha_{ij}} \tag{23}$$

for some upper semicontinuous and log-concave productivity shifter $A_i : C_i \to (0, 1]$ where $C_i \subseteq \Delta$ is convex and $\max_{\alpha_i \in C_i} A_i(\alpha_i) = 1$. The interpretation of $A_i(\alpha_i)$ is that it represents the productivity of the sector when it operates using some feasible input shares $\alpha_i \in C_i$. The closer $A_i(\alpha_i)$ is to 1, the cheaper it is (in terms of lost productivity) for sector i to use technology α_i. The technologies $\alpha_i \notin C_i$ are the infeasible ones for sector i.

For the endogenous Cobb-Douglas structure, the log-cost operator is given by:

$$T_i(p) = \min_{\alpha_i \in C_i} \left\{\sum_{j=1}^n \alpha_{ij} p_{ij} + \ln\left(\frac{1}{A_i(\alpha_i)}\right)\right\} \tag{24}$$

Influential papers in the literature have studied particular cases of this class. Acemoglu and Azar (2020) consider as a leading application a specification in which productivity depends only on the identity of the suppliers chosen by each sector, but their input shares are fixed. Kopytov et al. (2024) study a specification in which $A_i(\alpha_i)$ is smooth and strictly log-concave. △

Example 3 (Task-Based Production)**.** Recent work that studies the substitutability of different factors of production has studied task-based models in which output is produced by completing a set of different tasks and each task has its own production function (see *e.g.,* Acemoglu and Restrepo, 2018). Inspired by this, we consider an economy in which the set of tasks corresponds to the nonempty and convex set $C_i \subseteq \Delta$. Output is only produced if each task in this set is completed. That is, each sector has a Leontief production function over tasks. The output of each task is itself given by a Cobb-Douglas production with weights $\alpha_i \in C_i$ and some lower semicontinuous and entropy log-convex (*i.e.*, $w_i \mapsto \ln A_i(w_i) - \sum_{j=1}^n w_{ij} \ln w_{ij}$ is convex) productivity shifter $A_i : C_i \to [1, \infty)$ with $\min_{\alpha_i \in C_i} A_i(\alpha_i) = 1$. That is, the production function is given by:

$$G_i(Q_i) = \min_{\alpha_i \in C_i} A_i(\alpha_i) \prod_{j=1}^n \left(\frac{Q_{ij}}{\alpha_{ij}} \right)^{\alpha_{ij}}. \tag{25}$$

The interpretation of this production function becomes clear when we plug it into the unitary cost-minimization problem defined in Equation 16:

$$K_i(P) = \min_{Q_i \in \mathbb{R}_+^n} \sum_{j=1}^n Q_{ij} P_j \quad \text{s.t.} \quad A_i(\alpha_i) \prod_{j=1}^n \left(\frac{Q_{ij}}{\alpha_{ij}} \right)^{\alpha_{ij}} \geq 1 \quad \forall \alpha_i \in C_i. \tag{26}$$

That is, the firm must acquire a vector of inputs Q_i that produces a unit of all the tasks in C_i. With this, the tasks such that $A_i(\alpha_i)$ is high are those for which the constraint is less binding, while the tasks such that $A_i(\alpha_i)$ is close to 1 are bottlenecks. In this case, the log-cost operator is given by that of Equation 24, with the min replaced by a max. $\triangle$

Example 4 (Endogenous CES)**.** Our general framework allows us to study a new class of nested CES economies in which input weights and elasticities are chosen endogenously:

$$G_i(Q_i) = \max_{\alpha_i \in \Delta, \sigma_i \in [0, \bar{\sigma}_i]} A_i(\alpha_i, \sigma_i) \left(\sum_{j=1}^n \alpha_{ij}^{\frac{1}{\sigma_i}} Q_{ij}^{\frac{\sigma_i - 1}{\sigma_i}} \right)^{\frac{\sigma_i}{\sigma_i - 1}} \tag{27}$$

for some $\bar{\sigma}_i \in \overline{\mathbb{R}}_+$ indexing the maximum possible degree of substitution and an upper-semicontinuous $A_i : \Delta \times \overline{\mathbb{R}}_+ \to [0, 1]$ such that $\max_{\alpha_i \in \Delta, \sigma_i \in [0, \bar{\sigma}_i]} A_i(\alpha_i, \sigma_i) = 1$, representing the productivity of producing using different input weights and with different elasticities of substitution. In this case, the log-cost operator is given by:

$$T_i(p) = \min_{\alpha_i \in \Delta, \sigma_i \in [0, \bar{\sigma}_i]} \left\{ \frac{1}{1 - \sigma_i} \ln \left(\sum_{j=1}^n \alpha_{ij} \exp((1 - \sigma_i) p_j) \right) + \ln \left(\frac{1}{A_i(\alpha_i, \sigma_i)} \right) \right\}$$

A natural sub-class of the endogenous CES family emerges if we assume that sector $i \in [n]$ is endowed with some initial input weights α_i^0 and elasticity of substitution σ_i^0 and it can suffer an "adjustment cost" in productivity-equivalent units that is given by:

$$A_i(\alpha_i, \sigma_i) = \exp\left(-D(\alpha_i \| \alpha_i^0) - d(\sigma_i \| \sigma_i^0) \right) \tag{28}$$

where $D : \Delta^2 \to \overline{\mathbb{R}}_+$ is a divergence between α_i and α_i^0 and $d : \overline{\mathbb{R}}_+^2 \to \overline{\mathbb{R}}_+$ is a divergence between

σ_i and σ_i^0. This specification captures the intuitive idea that if a sector tries to produce using a technology that is different from where it started, then there will be an opportunity cost. $\triangle$

4 A General Formula for Equilibrium Prices and Output

In this section, we derive a general formula for equilibrium prices and output in terms of the primitive, non-parametric production structure. In so doing, we provide a primitive formula for the global solution for equilibrium prices in a general production network economy. A key step in proving this result is showing that an arbitrary CRS economy can be represented "as if" it were an economy in which each sector can choose among a set of Leontief technologies with different input weights and productivities (as in the activity analysis of Koopmans, 1951).

4.1 The Sufficiency of Endogenous Leontief Technology

We begin our analysis by providing a characterization of the structure of log-cost functions. To do this, we define the relative entropy $R : \Delta \times \Delta \to \overline{\mathbb{R}}_+$:

$$R\left(w \| \tilde{w}\right) = \begin{cases} \sum_{j=1}^{n} w_j \ln\left(\frac{w_j}{\tilde{w}_j}\right) & \text{if} \quad w \ll \tilde{w}, \\ \infty & \text{otherwise,} \end{cases} \tag{29}$$

where $w \ll \tilde{w}$ means that w is absolutely continuous with respect to $\tilde{w}$. The following result reveals that we can work directly with log-costs T rather than production functions G so long as T has a specific structure that is consistent with cost minimization:

Proposition 1. *The following are equivalent:*

1. *There exists a regular G that induces T.*
2. *T is monotone, translation invariant, normalized, and $K_T := \exp \circ T \circ \ln$ is concave.*

Moreover, G and T are related through:

$$T(p) = \min_{\tilde{w} \in \Delta} \max_{w \in \Delta} \left\{ \sum_{j=1}^{n} w_j p_j - R(w || \tilde{w}) + \ln\left(\frac{1}{G(\tilde{w})}\right) \right\} \tag{30}$$

This result follows from the constant returns to scale assumption on production functions. We derive this representation by first showing that the cost function K corresponding to any production function G can always be written as:

$$K(P) = \min_{\tilde{w} \in \Delta} \frac{\sum_{j=1}^{n} \tilde{w}_j P_j}{G(\tilde{w})}, \tag{31}$$

More evocatively, we observe that this is the same cost function as the one that would obtain if the production function were given by the following endogenous Leontief production function:

$$G^*(Q) = \max_{\tilde{w} \in \Delta} \left\{ G(\tilde{w}) \min_{j \in [n]} \frac{Q_j}{\tilde{w}_j} \right\}$$

That is, it is *as if* any production function corresponds to choosing a Leontief technology with input weights $\tilde{w} \in \Delta$ and corresponding endogenous productivity $G(\tilde{w})$. Intuitively, the resulting technology arises from the trade-off between choosing cheap inputs and selecting an input mix at which productivity is high.

In turn, this endogenous Leontief representation can be directly connected with the models of endogenous production technologies recently analyzed in the literature (Acemoglu and Azar, 2020; Kopytov et al., 2024). These papers consider CRS production functions that can be obtained as upper envelopes over technologies of smooth production functions, usually Cobb-Douglas, with technology-dependent productivity shifters. This result shows that there is no loss of generality in making this kind of assumption, so long as the base production technology is taken as Leontief.[7]

In the final part of the result, we derive the representation of T that will be essential for our subsequent analysis of equilibrium prices. This expression rewrites Equation 31 by first transforming it into the space of log prices:

$$T(p) = \min_{\tilde{w} \in \Delta} \left\{ \ln \left(\sum_{j=1}^{n} \tilde{w}_j \exp(p_j) \right) + \ln \left(\frac{1}{G(\tilde{w})} \right) \right\} \tag{32}$$

and then leveraging the Donsker-Varadhan formula to express the first term as:

$$\ln \left(\sum_{j=1}^{n} \tilde{w}_j \exp(p_j) \right) = \max_{w \in \Delta} \left\{ \sum_{j=1}^{n} w_j p_j - R(w || \tilde{w}) \right\} \tag{33}$$

which then directly yields Equation 30. The economic interpretation of this formula is that the Leontief technology with weights $\tilde{w}$ is the worst possible Cobb-Douglas technology with a productivity gain that depends on the relative entropy between the chosen Cobb-Douglas weights w and the Leontief weights $\tilde{w}$. The key mathematical upshot of this representation is that T is the min-max of an *affine* function of log-prices. This is the property that will allow us to characterize equilibrium prices in the sequel.

[7]This may be reminiscent of Houthakker (1955), who shows that aggregation of Leontief technologies with fixed inputs and heteogeneous input-specific productivity can yield an aggregate Cobb-Douglas technology, we emphasize that our result is microeconomic in nature and has nothing to do with aggregation.

4.2 Formulae for Equilibrium Prices and Output

We now derive our main formula for equilibrium log-prices. Let $\mathcal{W} = \left\{W \in \mathbb{R}_+^{n\times n} : \forall i \in [n], w_i \in \Delta\right\}$ be the set of $n \times n$ row-stochastic matrices, which can be interpreted as the set of all possible input-output matrices. Given an input-output matrix $W \in \mathcal{W}$, we define the Leontief-inverse:

$$\Omega_W = (1-\beta)(I-\beta W)^{-1} \tag{34}$$

It is well known (see *e.g.,* Acemoglu et al., 2012) that in a Cobb-Douglas economy with fixed input shares W, the equilibrium price vector is given by $\Pi(x) = \Omega_W x$. That is, Ω_W uniquely pins down how log-productivity shocks propagate in equilibrium.

To generalize this, we also define the production-wedge operator $\tau^G : \mathcal{W}^2 \to \overline{\mathbb{R}}^n$ as:

$$\tau_i^G(W\|\tilde{W}) = \frac{\beta}{1-\beta}\left[\ln\left(\frac{1}{G_i(\tilde{w}_i)}\right) - R\left(w_i\|\tilde{w}_i\right)\right] \qquad \forall i \in [n] \tag{35}$$

which is simply a rescaling of the additive productivity distortion term in the affine min-max representation of T_i (see Equation 30).

With these objects in hand, we state our formula for equilibrium prices:

Theorem 1 (Equilibrium Prices)**.** *The unique equilibrium log-prices are given by*

$$\Pi(x) = \min_{\tilde{W}\in\mathcal{W}} \max_{W\in\mathcal{W}} \left\{\Omega_W\left(x + \tau^G(W\|\tilde{W})\right)\right\} \qquad \forall x \in \mathbb{R}^n \tag{36}$$

Before unpacking the intuition behind this formula, we remark that the right-hand side of Equation 36 is entirely *primitive*: it depends only on the production functions and not on any equilibrium objects. To interpret the formula, fix $\tilde{W}$ and W that respectively are minimizers and maximizers in Equation 36. We then have that $\Pi(x) = \Omega_W(x + \tau^G(W\|\tilde{W}))$. That is, it is as if the initial adverse productivity shock x is further increased by the production wedge $\tau^G(W\|\tilde{W})$, and then this composite productivity shock propagates according to the Leontief-inverse of the input-output matrix W. This is the formula for equilibrium prices in a Cobb-Douglas economy with input-output matrix W and inverse log-productivity vector $x + \tau^G(W\|\tilde{W})$.

Turning to intuition, given that the economy we are studying is non-linear, the fact that the solution for equilibrium prices inherits the linear form corresponding to a distorted Cobb-Douglas form is perhaps surprising. We can use the endogenous Leontief representation of Proposition 1 to understand where this comes from. In particular, the fixed point equation for equilibrium prices becomes:

$$p_i = (1-\beta)x_i + \beta \min_{\tilde{w}_i\in\Delta} \max_{w_i\in\Delta} \left\{\sum_{j=1}^n w_{ij}p_j - R(w_i||\tilde{w}_i) + \ln\left(\frac{1}{G_i(\tilde{w}_i)}\right)\right\}, \quad \forall i \in [n] \tag{37}$$

Thus, stacking these equations and using the definition of the production-wedge operator, we have that:

$$p = (1-\beta)x + \beta \min_{\tilde{W}\in\mathcal{W}} \max_{W\in\mathcal{W}} \left\{ Wp + \frac{1-\beta}{\beta}\tau^G(W||\tilde{W}) \right\} \tag{38}$$

To understand why we can view row-by-row min-maximization over vectors as vectorial min-maximization over matrices, observe that each sector chooses its own technology, taking the prices as given. We then proceed by doing something seemingly naive. Suppose that we fixed a given pair of input-output matrices $(\tilde{W}, W)$. The resulting fixed point equation is simply a linear system in p and can be exactly inverted to yield:

$$p = \Omega_W(x + \tau^G(W||\tilde{W})) \tag{39}$$

The key step of the proof is then to show that we can apply the min-max structure to this solution and obtain the solution that we would have obtained if we applied the min-max and then computed the fixed point. That is, the operations of taking the fixed point and the min-max can be performed in either order and the answer is the same.

By way of analogy, this ability to interchange optimality and fixed points appears regularly in economics via the optimality principle of Bellman: the value of a dynamic optimization problem (under standard conditions) is the same as the fixed point of a Bellman operator. Though the details and argument differ, here we are establishing a similar conclusion that applies when computing equilibrium prices. As a by-product, this result guarantees the existence of a pair of input-output networks W and $\tilde{W}$ that respectively and *simultaneously* maximize and minimize the price of *every* sector.[8]

To understand what the production wedge looks like, we observe that if G_i is a Cobb-Douglas production function with *fixed* weights $\alpha_i \in \Delta$, then the wedge is proportional to the difference in the relative entropy between α_i and $\tilde{w}_i$ and the relative entropy between w_i and $\tilde{w}_i$:

$$\tau_i^G(W\|\tilde{W}) = \frac{\beta}{1-\beta}\left[R\left(\alpha_i\|\tilde{w}_i\right) - R\left(w_i\|\tilde{w}_i\right)\right] \tag{40}$$

Given this form, solving the min-max problem in Equation 36 yields the standard Cobb-Douglas solution that $\Pi(x) = \Omega_W x$, where $W = (\alpha_i)_{i=1}^n$ is the matrix of Cobb-Douglas weights. Similarly, if G_i is a CES production function with weights $\alpha_i \in \Delta$ and $\sigma_i \in \overline{\mathbb{R}}_+$, then the wedge is proportional to the Renyi divergence with parameter $\frac{1}{\sigma_i}$ between α_i and $\tilde{w}_i$ and the relative

[8] In dynamic programming, this amounts to showing that, in stationary environments, Markov policies are enough to find the unconstrained optimum.

entropy between w_i and $\tilde{w}_i$:

$$\tau_i^G(W\|\tilde{W}) = \frac{\beta}{1-\beta}\left[D_{\sigma_i^{-1}}\left(\alpha_i\|\tilde{w}_i\right) - R\left(w_i\|\tilde{w}_i\right)\right] \tag{41}$$

where D_t is the Renyi divergence with parameter $t \in \overline{\mathbb{R}}$. In this case, our result provides the first formula for equilibrium prices under the nested CES structure.

From Prices to GDP. With the prices derived, equilibrium GDP follows simply by solving the household's utility maximization problem given the equilibrium prices. To illustrate how this can be applied, suppose that the household has a Cobb-Douglas utility function:

$$U\left(C\right) = \prod_{i=1}^{n}\left(\frac{C_i}{\xi_i}\right)^{\xi_i} \qquad \forall C \in \mathbb{R}_+^n \tag{42}$$

where each $\xi_i \geq 0$ captures the importance of good i for the final consumer and $\sum_{i=1}^n \xi_i = 1$. It is standard to show that:

$$\Upsilon(x) = -\sum_{i=1}^{n}\xi_i\Pi_i(x). \tag{43}$$

In this case, it follows immediately from Theorem 1 that the corresponding equilibrium log-output is given by:

$$\Upsilon(x) = \max_{\tilde{W}\in\mathcal{W}}\min_{W\in\mathcal{W}}\left\{-\xi'\Omega_W\left(x+\tau^G(W\|\tilde{W})\right)\right\} \tag{44}$$

Intuitively, as we are solving all n min-max problems simultaneously, the expenditure weights pass through the min-max structure.

Illustrating The Formula in a Two-Sector Economy. To understand the economic interpretation of the minimum and the maximum, it is illustrative to specialize to a setting with one upstream sector and one downstream sector, like the one analyzed in Section 2. As before, assume that the production function of the upstream sector is given by $Y_U = 2Z_UQ_{U,0}^{\frac{1}{2}}Q_{U,U}^{\frac{1}{2}}$. Now, instead of the double-Leontief form of the initial example, we generalize the downstream production function to take the form $Y_D = 2Z_DQ_{D,0}^{\frac{1}{2}}G_D(Q_{D,D},Q_{D,U})^{\frac{1}{2}}$, where G_D is regular. For simplicity, we assume that the household consumes only downstream output and set $Z_D = 1$.

We now use Theorem 1 to understand the propagation of upstream productivity shocks into real GDP. Recalling that $x_U = -2\ln Z_U$, we therefore have that the unique equilibrium log prices are given by:

$$\begin{pmatrix} p_U \\ p_D \end{pmatrix} = \min_{\tilde{W}\in\mathcal{W}}\max_{W\in\mathcal{W}}\left\{\begin{pmatrix} \frac{2-w_{DD}}{3-w_{UU}-w_{DD}} & \frac{1-w_{UU}}{3-w_{UU}-w_{DD}} \\ \frac{1-w_{DD}}{3-w_{UU}-w_{DD}} & \frac{2-w_{UU}}{3-w_{UU}-w_{DD}} \end{pmatrix}\begin{pmatrix} x_U + \ln\left(\frac{1}{\tilde{w}_{UU}}\right) - R(w_U\|\tilde{w}_U) \\ \ln\left(\frac{1}{G_D(\tilde{w}_D)}\right) - R(w_D\|\tilde{w}_D) \end{pmatrix}\right\} \tag{45}$$

where we have solved for the Leontief inverse. We begin by inspecting the first row of this problem. First, observe that the minimizer can guarantee a value of x_U by setting $\tilde{w}_{UU} = 1$ (as the maximizer must then set $w_{UU} = 1$). Second, observe that the maximizer can also guarantee a value of x_U by setting $w_{UU} = 1$ (as the production wedge cancels to zero at this point). Thus, we have that $p_U = x_U$. We now turn to the second row, which allows us to express log real GDP (as the household only consumes downstream goods) $\Upsilon(x_U) = -p_D$ as:

$$\Upsilon(x_U) = \max_{\tilde{w}_{DD}\in[0,1]} \min_{w_{DD}\in[0,1]} \left\{ -\frac{1}{2-w_{DD}} \Big((1-w_{DD})x_U + \hat{\tau}_D(w_{DD}, \tilde{w}_{DD}) \Big) \right\} \tag{46}$$

where $\hat{\tau}_D(s, \tilde{s}) = -\ln G_D(\tilde{s}, 1-\tilde{s}) - R((s, 1-s)||(\tilde{s}, 1-\tilde{s}))$ is the production wedge of the downstream sector.

To unpack further the form of the solution and the roles of the max and the min, first suppose that the downstream sector operates a Leontief technology $G_D(Q_{D,D}, Q_{D,U}) = \min\{Q_{D,D}/\alpha, Q_{D,U}/(1-\alpha)\}$ for $\alpha \in (0,1)$. In the formula above, it is simple to verify that $\tilde{w}_{DD} = \alpha$ is optimal and hence that:

$$\Upsilon^{\text{Leontief}}(x_U) = \min_{w_{DD}\in[0,1]} \left\{ -\frac{(1-w_{DD})x_U - R((w_{DD}, 1-w_{DD})||(\alpha, 1-\alpha))}{2-w_{DD}} \right\} \tag{47}$$

Thus, log GDP is a minimum of an *affine* function of x_U and is hence concave in upstream productivity shocks. Thus, in Leontief economies, which are intuitively the most fragile to shocks, the force of the minimum dominates.

To consider the polar opposite case, now suppose that the downstream sector operates a perfect substitutes production technology $G_D(Q_{D,D}, Q_{D,U}) = Q_{DD} + Q_{D,U}$. Again, in the GDP formula, one can verify that $w_{DD} = \tilde{w}_{DD}$ and hence that:

$$\Upsilon^{\text{Substitutes}}(x_U) = \max_{\tilde{w}_{DD}\in[0,1]} \left\{ -\frac{(1-\tilde{w}_{DD})x_U}{2-\tilde{w}_{DD}} \right\} \tag{48}$$

And so log GDP is a maximum of an affine function of x_U and is therefore convex in upstream productivity shocks. Hence, in economies with perfect substitutes, which are intuitively the most resilient to shocks, the force of the maximum dominates.

In general of course, one cannot resolve the system so that either the min or the max dominates, and the two interact to determine the response of prices and GDP to shocks. In the following section, we proceed to formally define notions of resilience and fragility and we show that it maps formally to the dominance of the forces of the minimum and the maximum in the general representation.

5 Application I: Resilience *vs.* Fragility

We now apply our general representation of equilibrium prices to derive how properties of production functions translate into general equilibrium properties of prices. In the process, we will formalize the mathematical sense in which the general min-max formulation of equilibrium prices encodes the dueling forces of resilience and fragility.

5.1 Defining Resilience and Fragility

To define the notions of resilience and fragility, we ask: when is it true that making shocks to productivity more dispersed has adverse economic consequences? Informally, if the consequences are adverse, we call the economy fragile, and if the consequences are beneficial, we call the economy resilient. While this is far from the only reasonable definition of fragility and resilience, this corresponds to the popular notion advanced by Taleb (2013): "Fragility can be defined as an accelerating sensitivity to a harmful stressor: this response plots as a concave curve and mathematically culminates in more harm than benefit from random events. Antifragility is the opposite, producing a convex response that leads to more benefit than harm." We follow this definition:

Definition 2 (Resilience and Fragility)**.** *Sector i is resilient (fragile) if T_i is concave (convex). The economy is resilient (fragile) if Π is concave (convex).*

To understand the motivation for this definition, assume that the log-prices faced by sector i have an initial distribution π over $\mathbb{R}^n$. The expected unitary log-cost of sector i is then $\mathbb{E}_\pi[T_i]$. We are interested in the effect on the expected log-cost of a mean-preserving spread of the distribution of log-prices, that is, the new distribution π' dominates π in the convex order. We have that sector i is resilient *if and only if* $\mathbb{E}_{\pi'}[T_i] \leq \mathbb{E}_\pi[T_i]$ for any such pair of distributions. Conversely, sector i is fragile *if and only if* $\mathbb{E}_{\pi'}[T_i] \geq \mathbb{E}_\pi[T_i]$ for any such pair of distributions. That is, resilient sectors have lower log-costs in expectation when the volatility of log-prices rises, while the opposite happens for fragile sectors. We can interpret resilience and fragility of the economy analogously by looking at the expected log-price effect of mean-preserving spreads in the distributions of log-shocks.

This notion of resilience and fragility is consistent with the intuitive idea that CES technologies are fragile (resilient) when the elasticity of substitution is smaller (larger) than one:

Example 1 (continuing from p. 13)**.** *A sector $i \in [n]$ with CES production technology with elasticity of substitution σ_i is resilient if and only if $\sigma_i \geq 1$ and fragile if and only if $\sigma_i \leq 1$.*

5.2 Resilience

We now characterize resilience at the sector level and show that equilibrium prices preserve resilience. The next result shows that a sector is resilient if and only if its cost operator is generated by an endogenous Cobb-Douglas production function (as in Example 2).

Proposition 2. *Fix $i \in [n]$ with log-cost function T_i. The following are equivalent:*

(i) Sector i is resilient.

(ii) T_i is generated by an endogenous Cobb-Douglas production function G_i^*:

$$G_i^*(Q_i) = \max_{w_i \in C_i} \left\{ A_i(w_i) \prod_{j=1}^{n} \left(\frac{Q_{ij}}{w_{ij}} \right)^{w_{ij}} \right\} \tag{49}$$

for some nonempty, compact, and convex set $C_i \subseteq \Delta$ and production shifter $A_i : C_i \to [0,1]$ that is upper semi-continuous, log-concave, and such that $\max_{w_i \in C_i} A_i(w_i) = 1$.

Moreover, the productivity shifter A_i is uniquely defined given T_i.

One important implication of this result is that endogenous Cobb-Douglas production functions span the full scope of possibilities under resilience. Thus, insofar as one is interested in resilient technologies, this result implies that the approach of imposing endogenous Cobb-Douglas technology (as done in applications by Acemoglu and Azar, 2020; Kopytov et al., 2024) is not a restriction. However, this formulation rules out the possibility of fragility.

Example 1 (continuing from p. 13)**.** *Interestingly, because CES production functions with elasticity $\sigma_i > 1$ induce a concave log-cost operator, it follows that they are cost-equivalent to some endogenous Cobb-Douglas production function. By inspection of the proof of Proposition 2, it is possible to show that the productivity shifter recovering the CES log-cost operator is $A_i(w_i) = \exp\left(-\frac{1}{1-\sigma_i} R(w_i || \alpha_i)\right)$ and $C_i = \{w_i \in \Delta : w_i \ll \alpha_i\}$, where $\alpha \in \Delta$ are the weights in the original CES production function.*

The next result uses this representation and Theorem 1 to derive the formula for equilibrium prices in economies in which all sectors are resilient:

Corollary 1. *If every sector is resilient, then the economy is resilient and such that*

$$\Pi(x) = \min_{W \in \mathcal{W}_C} \left\{ \Omega_W(x + \tau^A(W)) \right\} \tag{50}$$

where $\mathcal{W}_C = \{W \in \mathcal{W} : w_i \in C_i, \forall i \in [n]\}$ and $\tau_i^A(W) = \frac{\beta}{1-\beta} \ln\left(\frac{1}{A_i(w_i)}\right)$ for all $i \in [n]$.

Thus, in resilient economies, the max in the general min-max representation of Theorem 1 disappears. This justifies our earlier informal interpretation of the min as inducing resilience. In this case, the production wedge also simplifies: G_i is replaced by A_i from the endogenous Cobb-Douglas representation.

5.3 Fragility

We now turn to the opposite question of when sectors and the aggregate economy are fragile. The following result shows that fragility is equivalent to a sector having a task-based production function with an entropy log-convex shifter A_i (as in Example 3). To understand this property, recall that cost operators K are always concave. Thus, while log-cost operators T can be convex, this comes with the additional restriction that A_i is sufficiently convex that its logarithm plus a concave function (i.e., the entropy) remains convex. This is exactly the extra restriction needed to offset the resilience coming from cost minimization.

Proposition 3. *Fix $i \in [n]$ with log-cost function T_i. The following are equivalent:*

(i) Sector i is fragile

(ii) T_i is generated by a task-based production function G_i^*:

$$G_i^*(Q_i) = \min_{w_i \in C_i} \left\{ A_i(w_i) \prod_{j=1}^{n} \left(\frac{Q_{ij}}{w_{ij}} \right)^{w_{ij}} \right\} \tag{51}$$

for some nonempty, compact, and convex set $C_i \subseteq \Delta$ and production shifter $A_i : C_i \to [1, \infty]$ that is lower semi-continuous, entropy log-convex, and such that $\min_{w_i \in C_i} A_i(w_i) = 1$.

Moreover, the productivity shifter A_i is uniquely defined given T_i.

While the endogenous Cobb-Douglas production function that characterizes resilience is familiar in the literature, this task-based mirror is (to the best of our knowledge) new. Moreover, the shifter in the task-based production function is required to be entropy log-convex, a stronger property than log-convexity. This stronger requirement comes from the need to be consistent with the co-occurrence of T_i being convex while K_i is concave. Indeed, it can be shown that this requirement is so strong that convexity of T_i implies differentiability of T_i.[9] Intuitively, under convexity, firms are not able to change their production technique fast enough to generate kinks in T_i. Thus, by the subsequent Proposition 6, if all sectors of the economy are fragile, then there is a unique equilibrium.

[9] Formally, if T_i is a convex log-cost functional, then T_i is continuously Frechet differentiable.

Example 1 (continuing from p. 13). *Similarly to the resilient case, because CES production functions with elasticity $\sigma_i < 1$ induce a convex log-cost operator, it follows that they are cost-equivalent to some task-based production function. By inspection of the proof of Proposition 3, it is possible to show that the productivity shifter recovering the CES log-cost operator is $A_i(w_i) = \exp\left(\frac{1}{1-\sigma_i} R(w_i||\alpha_i)\right)$ and $C_i = \{w_i \in \Delta : w_i \ll \alpha_i\}$.*

The next result leverages this representation and Theorem 1 to provide the equilibrium prices in economies in which all sectors are fragile:

Corollary 2. *If every sector is fragile, then the economy is fragile and such that*

$$\Pi(x) = \max_{W \in \mathcal{W}_C} \left\{ \Omega_W(x + \tau^A(W)) \right\} \tag{52}$$

Thus, just as resilience eliminated the max in the general min-max representation of Theorem 1, fragility eliminates the min in the representation. This completes our justification of the general max as corresponding to fragility and the min as corresponding to resilience. Moreover, as in the resilient case, the production wedge is substantially simpler than the general representation.

5.4 Resilience, Fragility, and the Ability to Abate Adverse Shocks

In this subsection, while not the motivation for our definition, we nevertheless show that our notions of resilience and fragility provide a useful diagnostic for the concept that a sector is (un)able to abate large shocks to upstream suppliers. For this thought experiment, consider a pair of sectors $i, j \in [n]$, and suppose that there is a change in the price of sector j by an amount $\lambda > 0$ such that the input price vector changes to $p \mapsto p + \lambda e_j$. We say that a large such shock to sector j can be *abated* by sector i if:[10]

$$\lim_{\lambda \to \infty} \frac{T_i(p + \lambda e_j) - T_i(p)}{\lambda} = 0 \tag{53}$$

Intuitively, a shock to j can be abated by sector i if, as the shock to j's price becomes large, the ultimate effect on the costs of sector i becomes proportionately insignificant. To exemplify this definition, in the case in which sector i has a Cobb-Douglas production technology with weights $\alpha_i \in \Delta$, i can abate shocks to j if and only if $\alpha_{ij} = 0$.

To describe how resilience and fragility relate to the ability to abate shocks, consider first a resilient sector $i \in [n]$. By Proposition 2, such a sector can be represented as possessing an endogenous Cobb-Douglas production function with productivity shifter $A_i : C_i \to [0, 1]$. For such a production function, we define the set of feasible production technologies $C_i^R =$

[10] We observe that this property is independent of the initial p at which the object inside the limit is evaluated.

$\text{cl}\{w_i \in C_i : A_i(w_i) > 0\}$, *i.e.,* the closure of the set of all of the mixes of inputs such that the sector can generate production using that input mix. Now consider a fragile sector $i \in [n]$. By Proposition 3, such a sector can be represented via a task-based production function with productivity shifter $A_i : C_i \to [1, \infty]$. For such a production function, we define the set of necessary tasks $C_i^F = \{w_i \in C_i : A_i(w_i) < \infty\}$, *i.e.,* the set of tasks which must be completed in order to produce output. With these definitions in hand, we have that:

Proposition 4. *The following statements are true:*

1. *Suppose sector i is resilient. Sector i can abate shocks from sector j if and only if $w_{ij} = 0$ for some $w_i \in C_i^R$.*
2. *Suppose sector i is fragile. Sector i can abate shocks from sector j if and only if $w_{ij} = 0$ for all $w_i \in C_i^F$.*

Thus, our notion of resilience captures the idea that a sector can abate adverse shocks so long as it is *possible* to adopt a technology that substitutes away from an adversely affected supplier. By contrast, our notion of fragility captures the idea that a sector can only abate shocks if *all* of the tasks that it performs do not require the outputs of the shocked supplier. Moreover, if a sector is fragile, it follows immediately that there always exists some sector such that shocks to that sector cannot be abated. Thus, while our definition does not use these concepts in its conception, it has the desirable property that it maps to these potentially natural notions of the ability to abate shocks.

Example 1 (continuing from p. 13)**.** *As one application of this result, consider a sector i with a CES production technology with input weights $\alpha_i \in \Delta$ and elasticity of substitution $\sigma_i \in \overline{\mathbb{R}}_+$. If sector i is resilient (i.e., $\sigma_i > 1$), then it can abate shocks to sector j if and only if $\alpha_{ij} < 1$. That is, it can abate shocks originating from a sector if and only if that sector is not its sole input supplier. If sector i is fragile (i.e., $\sigma_i < 1$), then it can abate shocks to sector j if and only if $\alpha_{ij} = 0$. That is, it can abate shocks from a sector only if it is not exposed to that sector.*

5.5 Alternative Fragility Notions: Big Shocks and Elasticity

So far, we have considered a notion of fragility that corresponds to more dispersed shocks leading to worse outcomes. While this is a standard notion, there are other reasonable natural-language notions of fragility. Two potentially natural additional notions are that: (i) as shocks become bigger, their marginal adverse effect is larger, and (ii) the elasticity of substitution is greater or smaller than unitary. In this section, we show how these notions are related.

Formally, we define resilience and fragility to big shocks as:

Definition 3 (Resilience and Fragility to Big Shocks)**.** *Sector i is resilient (fragile) to big shocks if T_i is concave (convex) at zero. The economy is resilient (fragile) to big shocks if Π is concave (convex) at zero.*

To understand this, observe that T_i is convex at zero if for all $\lambda \in (0,1)$ and $p \in \mathbb{R}^n$:

$$\underbrace{T_i(p)}_{\text{Total Effect}} \geq \underbrace{\frac{T_i(\lambda p)}{\lambda}}_{\text{Average Effect}} \tag{54}$$

That is, the *total effect* of a shock in direction p—or $T_i(p)$—is larger than the *average effect* for all scaled down versions of that shock— or $T_i(\lambda p)/\lambda$. This is a mathematical definition of the intuitive notion that the marginal effect of a shock is increasing in the size of the shock.

Moreover, we define elasticity as:

Definition 4 (Elasticity)**.** *Sector i is elastic (inelastic) if T_i is supermodular (submodular).*

To understand why we use this definition of elasticity, we observe that when T_i is twice continuously differentiable, it collapses to the standard notion of elasticity from producer theory. More technically, given the corresponding cost function K_i, we define the Allen-Uzawa elasticity of substitution as

$$\sigma_{jk}^{K_i}(P) = \frac{K_i(P)\frac{\partial^2 K_i(P)}{\partial P_j \partial P_k}}{\frac{\partial K_i(P)}{\partial P_j}\frac{\partial K_i(P)}{\partial P_k}} = 1 + \frac{\frac{\partial^2 T_i(p)}{\partial p_j \partial p_k}}{\frac{\partial T_i(p)}{\partial p_j}\frac{\partial T_i(p)}{\partial p_k}} \tag{55}$$

where the second equality follows by simple algebra. Here, we can see that supermodularity (submodularity) is equivalent to the elasticity of substitution being globally greater than one or smaller than one—the standard notion of elastic and inelastic. Indeed, for a CES production function, this notion of elasticity corresponds to the standard condition that $\sigma \lesseqgtr 1$. Our definition is more general in that it can be specified without assuming any differentiability property of T_i.

From this, we can see that elasticity encodes a notion of fragility to correlation. To understand this, suppose as before that the log-prices faced by sector i have distribution $\pi \in \mathbb{R}^n$. A standard notion of a random vector with distribution π' being more correlated than another random vector with distribution π is that the distribution π' dominates π in the supermodular stochastic order (see Shaked and Shanthikumar, 2007, for a discussion). That is, for all supermodular $f : \mathbb{R}^n \to \mathbb{R}$, we have that $\mathbb{E}_{\pi'}[f(p)] \geq \mathbb{E}_{\pi}[f(p)]$. Intuitively, if π' is more correlated than π, the expected value of a supermodular function will be larger under π' than π because the arguments are more likely to be *jointly* large and small. Thus, a sector being elastic implies that that sector is fragile to correlation.

The next result shows that these different notions of resilience and fragility can be ranked in order of stringency.

Proposition 5. *The following statements are true:*

(i) If a sector is inelastic (elastic), then it is fragile (resilient)

(ii) If a sector is fragile (resilient), then it is fragile (resilient) to big shocks

Thus, while all sectors being elastic or inelastic is sufficient for the conclusion that the economy is resilient or fragile, it is far from necessary. While these notions are equivalent with CES production technologies, away from this special case our analysis makes clear the distinct roles played by elasticity and resilience/fragility: elasticity determines the extent to which correlation is good or bad, while resilience and fragility determine the extent to which volatility is good or bad. As Theorem 1 and Corollaries 1 and 2 have shown that resilience and fragility characterize equilibrium outcomes, our results indicate that elasticity (while more conventional) is less reflective of the general structure of equilibrium outcomes.

6 Application II: Shock Propagation

Understanding the effects of small shocks is a classic analytical method in macroeconomic analysis. This is typically for the reason that, by focusing on small shocks, non-linearities can be ignored. We will find, however, that the presence of technology switching implies that non-linearities affect even such a local analysis, caveating the standard takeaway from Hulten (1978): local Domar weights are not generally sufficient to determine the effects of even small shocks.

6.1 Shock Propagation in the Small

To understand the effects of small shocks, we denote the starting point from which the economy is shocked by $x \in \mathbb{R}^n$, the direction of a productivity shock by $d \in \mathbb{R}^n$, and the size of the shock by $t \in \mathbb{R}_{++}$. With this, we define the local effect of a shock in direction d of size t starting from x as:

$$\Pi_0(x; d) = \lim_{t \to 0^+} \frac{\Pi(x + td) - \Pi(x)}{t}. \tag{56}$$

We say that Π is *directionally differentiable* at x if this limit exists for every d and that Π is *Gateaux differentiable* at x if this limit is a linear function of d.[11]

We define the optimal technology choice profile across sectors from the endogenous Leontief representation (Proposition 1) as:

$$\Sigma(p) = \left\{ \tilde{W} \in \mathcal{W} : \tilde{w}_i \in \arg\min_{\hat{w}_i \in \Delta} \frac{\sum_{j=1}^n \hat{w}_{ij} \exp(p_j)}{G_i(\hat{w}_i)}, \forall i \in [n] \right\} \qquad \forall p \in \mathbb{R}^n \tag{57}$$

[11] All these properties and derived objects are analogously defined for the log-cost operator T and the real GDP function Υ.

Intuitively, $\Sigma(p)$ corresponds to the set of cost-minimizing technology choices across firms when they face the log-price vector p. We also define the operator $L : \mathcal{W} \times \mathbb{R}^n \to \mathcal{W}$:

$$L_{ij}(\tilde{W}, p) = \frac{\tilde{w}_{ij} \exp(p_j)}{\sum_{k=1}^{n} \tilde{w}_{ik} \exp(p_k)} \qquad \forall i, j \in [n] \tag{58}$$

which returns the Cobb-Douglas expenditure shares given a technology choice matrix $\tilde{W}$. With these definitions in hand, we arrive at the following formula for the directional derivative of equilibrium log-prices:

Theorem 2. *The equilibrium log-price operator* Π *is directionally differentiable with:*

$$\Pi_0(x; d) = \min_{\tilde{W} \in \Sigma(\Pi(x))} \Omega_{L(\tilde{W}, \Pi(x))} d \qquad \forall x, d \in \mathbb{R}^n \tag{59}$$

Intuitively, this formula says that for every direction d, shocks propagate according to the Leontief inverse of a fictitious Cobb-Douglas economy with expenditure shares given by L. Moreover, the result says that among all of the cost-minimizing technologies that prevail at the equilibrium log-prices $\Pi(x)$, the fictitious Cobb-Douglas is the one that minimizes equilibrium prices following the shock.

We emphasize that different directions of shocks can give rise to different technology choices of firms, causing the derivative of prices to jump. Theorem 2 thus implies that the derivative of prices jumps at exactly those points where many technologies are optimal for any sector. The following four equivalent statements characterize the points at which technology switching introduces jumps.

Proposition 6 (When Jumps Feature)**.** *Fix* $x \in \mathbb{R}^n$. *The following statements are equivalent:*

(i) T *is Gateaux differentiable at* $\Pi(x)$

(ii) Π *is Gateaux differentiable at* x

(iii) $\Sigma(\Pi(x))$ *is a singleton*

(iv) There exists a unique competitive equilibrium at x.

Moreover, for almost all $x \in \mathbb{R}^n$, *these four equivalent conditions are true.*

Thus, whenever there are multiple competitive equilibria or T is not Gateaux differentiable at equilibrium, jumps occur and the fundamental theorem of calculus may serve as a poor guide. While this result clarifies that these jump points comprise a zero measure subset of $\mathbb{R}^n$, their presence can nonetheless lead to arbitrarily large errors if the analyst ignores them. As we will see in our later applications, the presence of jumps is an essential consideration in computing the effects of shock volatility on expected GDP.

6.2 The *In*sufficiency of Local Domar Weights

We now use Theorem 2 to derive the directional derivative of equilibrium output $\Upsilon_0(x; d) = \lim_{t \to 0} \frac{\Upsilon(x+td)-\Upsilon(x)}{t}$ and provide a generalization of Hulten's theorem that the change in aggregate labor productivity induced by infinitesimal productivity shocks is given by the inner product of the sales share vector (or Domar weights) with the vector of productivity shocks. To this end, let $\mathcal{E}(x)$ be the set of competitive equilibria at $x \in \mathbb{R}^n$. For any competitive equilibrium E, we define the Domar weight vector as $\lambda_E = (P_i^E Y_i^E)_{i=1}^n$.

Proposition 7 (Generalized Hulten's Theorem)**.** *Equilibrium log output Υ is directionally differentiable with derivative:*

$$\Upsilon_0(x; d) = \max_{E \in \mathcal{E}(x)} \lambda_E' d_Z \qquad \forall x, d \in \mathbb{R}^n \tag{60}$$

where $d_Z = -(1-\beta)d$ is the direction of the shock expressed in units of log-TFP.

This generalizes Hulten's theorem beyond the case in which firms' cost functions are differentiable. The technology switching that previously led to discontinuity in the response of prices to shocks also leads to discontinuity in the response of output to shocks. This discontinuity manifests because switches in technology lead to jumps in the economy's Domar weights. The theorem highlights that scaling d by any $\eta > 0$ yields the same Domar weights but rotations of d may lead different Domar weights to govern shock propagation. Intuitively, when the economy is at a technology switching point, different directions of shocks will propagate differently.

To understand the implications of this result, observe that Proposition 7 implies that knowledge of $\mathcal{E}(x)$ is essential to know the effects of local productivity shocks. However, except when $\mathcal{E}(x)$ is a singleton, this implies that knowledge of Domar weights before a shock is insufficient to know the effects of a shock. Instead, the analyst must know how technology may change in order to predict how shocks propagate. For large shocks, it is well understood that Hulten (1978) may provide a poor approximation (Baqaee and Farhi, 2019). What is new is that Proposition 7 shows that the same conclusion can be true even for small shocks. Moreover, Proposition 6 clarifies exactly when the conclusion of Hulten (1978) fails: when technology switching is fast, technologies that prevailed before a shock can be a poor guide to technologies that prevail following a shock. Conversely, when the economy is fragile (and thus by Proposition 3 features a Gateaux differentiable T), the conclusion of Hulten (1978) is valid.

Illustrating the Generalized Hulten's Theorem in a Two-Sector Economy. To understand this result further, we explicitly apply for the formula in the example economy that we used to illustrate Theorem 1. In particular, we consider the local effects of a shock to upstream log productivity x_U starting from a steady state in which both the upstream and downstream sector are equally productive $(Z_U, Z_D) = (1,1)$. At the initial productivity vector, by symmetry the prices of both sectors are equal. Hence, the Cobb-Douglas expenditure share operator simply reduces to $L_{ij}(\tilde{W}, \Pi(0)) = \tilde{w}_{ij}$. Therefore, the Domar weight vector given technology choices $\tilde{W}$ is given by $\lambda = \xi' \Omega_{L(\tilde{W}, \Pi(0))} = \xi' \Omega_{\tilde{W}}$. Applying the fact that $\xi_D = 1$ (as the household only consumes downstream goods) and our earlier derivation that $\tilde{w}_{UU} = 1$ (as the upstream sector has no technology choice in this example), we have that the Domar weight vector corresponding to downstream technology choice $\tilde{w}_{DD}$ is given by:

$$\lambda(\tilde{w}_{DD}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}' \begin{pmatrix} 1 & 0 \\ \frac{1-\tilde{w}_{DD}}{2-\tilde{w}_{DD}} & \frac{1}{2-\tilde{w}_{DD}} \end{pmatrix} = \begin{pmatrix} \frac{1-\tilde{w}_{DD}}{2-\tilde{w}_{DD}} & \frac{1}{2-\tilde{w}_{DD}} \end{pmatrix} \tag{61}$$

As efficient technology choice for the downstream sector at symmetry simply maximizes downstream output, we have that $\tilde{w}_{DD}$ must lie in the set $S_D = \{\tilde{w}_{DD} \in [0,1] : G_D(\tilde{w}_{DD}, 1 - \tilde{w}_{DD}) = 1\}$. We denote the largest and smallest elements of this set by $\overline{w}_{DD}$ and $\underline{w}_{DD}$. The formula from Proposition 7 yields that the local effects of an upstream productivity shock x_U on log real GDP are given by:

$$\Upsilon_0(0; (x_U, 0)) = \max_{\tilde{w}_{DD} \in S_D} \left\{ \frac{1 - \tilde{w}_{DD}}{2 - \tilde{w}_{DD}} \left(-\frac{1}{2} x_U \right) \right\} = \begin{cases} \frac{1-\underline{w}_{DD}}{2-\underline{w}_{DD}} \left(-\frac{1}{2} x_U \right), x_U < 0, \\ \frac{1-\overline{w}_{DD}}{2-\overline{w}_{DD}} \left(-\frac{1}{2} x_U \right), x_U > 0. \end{cases} \tag{62}$$

where the second equality follows as the objective is an decreasing function of $\tilde{w}_{DD}$ when $x_U < 0$ and an increasing function of $\tilde{w}_{DD}$ when $x_U > 0$.

Intuitively, when there is an adverse shock to the upstream sector (*i.e.*, $x_U > 0$), the downstream sector switches to the technology (among all of those that are efficient) that is least reliant on upstream inputs. Conversely, when there is a positive shock to the upstream sector, it switches technology to be maximally exposed to the shock. To understand when there is a discontinuity in the effects of shocks, we therefore have to understand when S_D is singleton, *i.e.*, when there is a single efficient technology choice. If there is a single Leontief technology downstream with downstream input weight α, the unique efficient technology is $\tilde{w}_{DD} = \alpha$. Hence, $S_D = \{\alpha\}$ and there is no discontinuity. If there is a perfect substitutes technology downstream, then all technologies are efficient $S_D = [0,1]$, there are highly asymmetric effects: positive shocks propagate while negative shocks generate no adverse consequences. More generally, if there are finitely many Leontief technologies downstream (as in the introductory example from Section 2), negative shocks propagate according to the smallest Leontief weight on upstream and positive shocks propagate according to the largest Leontief weight on upstream. These examples illustrate that resilience and the ability to switch technologies generates these shock-dependent local responses. The fully general formula generalizes these examples to capture these technology switching phenomena with richer technological possibilities.

7 Application III: Microeconomic Volatility and GDP

To quantify the magnitude of an economy's resilience, we now apply our results to study how changes in microeconomic volatility affect the average GDP of the economy.

7.1 Local Resilience and Fragility

For this section, we assume that sectoral log productivities are stochastic and study how a change in the standard deviation of the log productivity of an arbitrary sector $i \in [n]$ affects average log GDP. To this end, suppose that sector i's log productivity can be written as $X_i = \mu_i + s_i \nu_i$, where $\mu_i \in \mathbb{R}$ represents mean log-productivity, $s_i \in \mathbb{R}_{++}$ denotes the standard deviation of log productivity and ν_i is a shock (in standardized units) to log productivity. We assume that ν_i is (jointly) independent across sectors and has probability density function φ_i, which is even, continuous, and has unit variance. With an abuse of notation, we henceforth suppress dependence on means, and let $\tilde{\varphi}_s(x)$ and, for all $i \in [n]$, $\tilde{\varphi}_{-i,s}(x_{-i})$ denote the induced densities of the random vectors $X = (X_j)_{j\in[n]}$ and $X = (X_j)_{j\neq i}$. A key example accommodated by this setting is that X is jointly normally distributed with mean μ and variance-covariance matrix $\operatorname{diag}(s_1^2, \ldots, s_n^2)$, where $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_{++}$.

Given this structure, we define expected log GDP given sectoral standard deviations $s \in \mathbb{R}^n_{++}$ as:

$$Y(s) = \mathbb{E}_s\left[\Upsilon(X)\right] = \int_{\mathbb{R}^n} \Upsilon(x)\tilde{\varphi}_s(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \Upsilon(x)\left(\prod_{i=1}^n \frac{1}{s_i}\varphi_i\left(\frac{x_i - \mu_i}{s_i}\right)\right)\mathrm{d}x \tag{63}$$

We say that the economy is locally (at s) fragile to increases in the volatility from sector i if $\frac{\partial Y(s)}{\partial s_i} \leq 0$ and the economy is locally resilient if $\frac{\partial Y(s)}{\partial s_i} \geq 0$. It also follows that if the economy is resilient (resp. fragile), then the economy is also everywhere locally resilient (resp. fragile) to every sector i.[12] We can consider $\frac{\partial Y(s)}{\partial s_i}$ as a quantitative measure of the local resilience of the economy to sector i. We now proceed to provide a general decomposition of this quantitative measure.

7.2 The Effects of Microeconomic Volatility on GDP

To state the result, we define $\mathcal{D}_T$ as the set of $x \in \mathbb{R}^n$ such that T is continuously differentiable at x. By Proposition 6, we have that $\mathcal{D}_T$ has measure one for all possible profiles of sectoral

[12] To see this, observe that, for every $i \in [n]$ and every convex function $\phi : \mathbb{R} \to \mathbb{R}$, the map

$$s_i \mapsto \int_{\mathbb{R}} \phi(\mu_i + s_i v_i)\varphi_i(v_i)\,\mathrm{d}v_i$$

is nondecreasing over $\mathbb{R}_+$. With this, the distribution of each X_i increases in the convex order whenever s_i increases.

standard deviations $s > 0$ and that for all $x \in \mathcal{D}_T$ there is a unique equilibrium. We also define by $\mathcal{H}_T$ as the set of points of $\mathbb{R}^n$ where T is twice continuously differentiable. In this section, we maintain the following technical assumptions: (i) the set $\mathcal{D}_T$ is open and (ii) $\mathcal{D}_T = \mathcal{H}_T$. Together with the full-measure property of $\mathcal{D}_T$, these assumptions say that T is *fundamentally* C^2 (Definition 5). Under the lens of Proposition 6, the first part of the assumption says that equilibrium uniqueness is robust to small perturbations of shocks. The second part of the assumption states that whenever a unique equilibrium exists, the log-cost operator admits a second-order expansion. Importantly, this does not imply that the second-order expansion approximates well the equilibrium *everywhere*, just generically. While this might seem a detail, especially since we are considering absolutely continuous shock distributions, our decomposition below shows that the (non-generic) discontinuous changes in the derivative of T still matter even when we consider infinitesimal changes in volatility s. Intuitively, while jumps are rare, they are locally infinitely larger than the smooth variation in the function.

Under our maintained assumptions, for every $x \in \mathcal{D}_T$, the Domar weights $\lambda(x) \in \Delta$ and their derivatives $\left(\frac{\partial \lambda_i(x)}{\partial x_i}\right)_{i\in[n]} \in \mathbb{R}^n$ are well defined. We set $\frac{\partial \lambda_i(x)}{\partial x_i} = 0$ for all $x \notin \mathcal{D}_T$ and all $i \in [n]$. Moreover, for every sector $i \in [n]$, define the Domar weight jump as $\Delta\lambda_i(x) = \lambda_i(x) - \lim_k \lambda_i(x - \frac{1}{k}e^i)$ for all $x \in \mathbb{R}^n$. These jump functions are well defined everywhere, and they are equal to 0 on $\mathcal{D}_T$. Finally, we define the function $m_i : \mathbb{R} \to \mathbb{R}_+$:

$$m_i(x_i) = \int_{\left|\frac{x_i - \mu_i}{s_i}\right|}^{\infty} r\varphi_i(r)\,\mathrm{d}r \tag{64}$$

With these objects in hand, we have that the local resilience of the economy is given by the following formula.

Proposition 8. *For all $s > 0$ and $i \in [n]$, we have:*

$$\frac{\partial Y(s)}{\partial s_i} = -(1-\beta)\mathbb{E}_s\left[\int_{\mathbb{R}} \frac{\partial \lambda_i(x_i, X_{-i})}{\partial x_i} m_i(x_i)\mathrm{d}x_i + \sum_{x_i\in\mathbb{R}} \Delta\lambda_i(x_i, X_{-i}) m_i(x_i)\right] \tag{65}$$

Moreover, we have that $\Delta\lambda_i(x) \leq 0$ for all $x \in \mathbb{R}^n$, and so the second term is negative.

To prove this result, we leverage techniques from convolution. The result expresses the effect of changing the standard deviation of sector i on expected log GDP as the sum of smooth and jump contributions, weighted by m_i and averaged over X_{-i}. These are the two components of the second-order distributional derivative of GDP with respect to x_i. The first term is the standard and familiar second-order term from Baqaee and Farhi (2019): the second-order effects of changing productivity shocks are captured by the change in the Domar weights. However, in general, this does not capture the total effect. The second term is the average of a singular (non-smooth) measure that captures points at which the Domar weights jump because firms discretely switch technologies. This term always contributes positively. Intuitively, technology switching always makes the economy more locally resilient because firms change technologies efficiently. Thus, neglecting technology switching will always *overstate* the extent of the economy's local fragility.

We now unpack these terms and how they contribute to local resilience and fragility under different technological assumptions. First, suppose that each sector has a sufficiently "smooth" production technology in the precise sense that its log-cost is Gateaux differentiable. In this case, we have there is always a uniquely optimal technological choice of firms and the singular measure in Proposition 8 is identically zero. That is, we have:

$$\frac{\partial Y(s)}{\partial s_i} = -(1-\beta)\mathbb{E}_s\left[\int_{\mathbb{R}} \frac{\partial \lambda_i(x_i, X_{-i})}{\partial x_i} m_i(x_i)\,\mathrm{d}x_i\right]. \tag{66}$$

If $X_i \sim N(\mu_i, s_i^2)$, this reduces to

$$\frac{\partial Y(s)}{\partial s_i} = -(1-\beta)s_i\mathbb{E}_s\left[\frac{\partial \lambda_i(X)}{\partial x_i}\right]. \tag{67}$$

For derivatives with respect to the variance $v_i = s_i^2$, the corresponding expression is divided by $2s_i$, as explained in Remark 1.

A direct corollary of this fact is that if all sectors are fragile, only the first term appears and, in this case, the total effect of microeconomic volatility is negative.[13] In this world of smooth technology, the Baqaee and Farhi (2019) logic of studying the derivatives of Domar weights is sufficient to characterize local resilience and fragility. However, even in this case, we highlight that the formula provided by Proposition 8 is novel and extends the logic from Baqaee and Farhi (2019): it provides the exact—rather than approximate—effects of changing volatility on expected GDP.

Second, to explore the opposite resilient case, suppose that each sector has an endogenous Cobb-Douglas technology with only finitely many technologies (*e.g.,* as in Acemoglu and Azar, 2020):

$$G_i(Q_i) = \max_{\alpha_i \in C_i} A(\alpha_i)\prod_{j=1}^{n}\left(\frac{Q_{ij}}{\alpha_{ij}}\right)^{\alpha_{ij}} \tag{68}$$

where C_i is a finite subset of Δ. On every region where the optimal technologies are unchanged, the economy has a fixed Cobb-Douglas production network and hence constant Domar weights. The weights can jump at the boundaries between these regions, but their smooth derivative

[13] This is true more generally since in fragile economies Υ is a concave function (Corollary 2).

term in Proposition 8 vanishes. Specializing that proposition therefore gives:

$$\frac{\partial Y(s)}{\partial s_i} = (1-\beta)\mathbb{E}_s\left[\sum_{t\in\mathbb{R}}\big[-\Delta\lambda_i(t,X_{-i})\big]m_i(t)\right] \geq 0. \tag{69}$$

Here the expectation is over X_{-i}, and, for each realization of X_{-i}, the sum collects the jumps as the coordinate $x_i = t$ varies. This is the jump contribution in Proposition 12 in the appendix, with the smooth term equal to zero. Since $\Delta\lambda_i \leq 0$ and $m_i \geq 0$, each jump contributes nonnegatively to the marginal effect of increasing s_i, making the economy everywhere locally resilient. The entire effect of increased volatility on expected log GDP therefore operates through technology switching, even though the switching boundaries themselves have probability zero under the maintained continuous productivity distributions.

Even beyond fragile and resilient economies, Proposition 8 formalizes and generalizes the qualitative phenomenon illustrated in Section 2, that is, the effect of microeconomic volatility in expected log-GDP is generally non-monotone. In particular, in the case of *finitely many* endogenous Leontief technologies considered in that section, the smooth term in Equation 65 is always negative and corresponds to the (smooth) volatility effect in the local Leontief economy, a fragile economy. Therefore, in this case, the total effect is the sum of two effects with opposite signs, respectively capturing the fragility (owing to the fact that technology is locally Leontief) and the resilience (arising from the ability to switch technologies) of the economy.

8 Application IV: Robust Bounds on GDP Volatility

In this section, we derive lower and upper bounds on the variance of real GDP under standard assumptions about the distribution of productivity shocks.

8.1 The Stochastic Properties of Productivity

Assume now that X, the profile of the sectors' inverse log-productivities, is a random vector with independent components. Importantly, we are not making any assumptions about the boundedness of the distributions' supports or the existence of densities. Let Υ denote the random variable describing the equilibrium real GDP of the economy at every possible realization of the random vector X. In fact, we let X be the only source of randomness for Υ. Our goal is to find bounds on $\mathrm{Var}[\Upsilon]$ given the original distribution of X and the equilibrium operator yielding Υ.

8.2 The Bounds on GDP Volatility

By Proposition 6, we know that there exists a full-measure set $\mathcal{D} \subseteq \mathbb{R}^n$ such that Υ is (Gateaux) differentiable at every $x \in \mathcal{D}$. For every $x \in \mathcal{D}$, let $\lambda(x) \in \Delta$ denote the Domar weight vector corresponding to the *unique* competitive equilibrium at x.[14] For every sector $i \in [n]$, define its smallest and largest Domar weights respectively as

$$\underline{\lambda}_i = \inf_{x \in \mathcal{D}} \lambda_i(x) \qquad \text{and} \qquad \overline{\lambda}_i = \sup_{x \in \mathcal{D}} \lambda_i(x). \tag{70}$$

Observe that, differently from any fixed Domar weight vector λ_E, we have that $\sum_{i=1}^n \underline{\lambda}_i \leq 1$ and $\sum_{i=1}^n \overline{\lambda}_i \geq 1$, usually with strict inequalities.[15]

With these definitions in hand, we now provide robust bounds and variance of aggregate GDP that can result from microeconomic productivity disturbances:

Proposition 9. *Under our maintained assumptions, we have*

$$(1-\beta)^2 \max_{i \in [n]} \underline{\lambda}_i^2 \mathrm{Var}[X_i] \leq \mathrm{Var}[\Upsilon] \leq (1-\beta)^2 \sum_{i=1}^n \overline{\lambda}_i^2 \mathrm{Var}[X_i]. \tag{71}$$

Before discussing the intuition behind and implications of these bounds, we remark that Equation 71 provides robust bounds on the volatility of the aggregate output that apply regardless of the distribution of productivity and firms' production technologies.

To unpack this result, we observe that these bounds in a fixed Cobb-Douglas production economy (with Domar weight vector $\lambda \in \Delta$) reduce to:

$$(1-\beta)^2 \max_{i \in [n]} \lambda_i^2 \mathrm{Var}[X_i] \leq \mathrm{Var}[\Upsilon] \leq (1-\beta)^2 \sum_{i=1}^n \lambda_i^2 \mathrm{Var}[X_i]. \tag{72}$$

Intuitively, there is a most influential sector i whose squared Domar weight λ_i^2 multiplied by its productivity volatility $\mathrm{Var}[X_i]$ is maximal. The volatility of aggregate output is always bounded from below by the volatility that arises from this most influential sector. For the upper bound, we observe that the volatility of GDP is simply less than the total contribution of each sector. While the lower bound is typically loose, we can see from this example that our upper bound is tight, as the volatility of GDP is exactly this upper bound in Cobb-Douglas economies.

In the general non-linear case, our result shows that this intuition from Cobb-Douglas economies carries over to arbitrary economies, with one caveat: instead of the fixed Domar

[14] This is well-defined again owing to Proposition 6.

[15] In fact, the main case where the previous conditions hold with equality is that of a production economy where all sectors have Cobb-Douglas production functions as in Acemoglu et al. (2012). In that case, there exists a single Domar weight vector that is independent of the realization of the sectors' productivities.

weights governing propagation, the lower bound is governed by the most influential sector's smallest possible Domar weight (across different productivity realizations) and the upper bound is governed by the largest possible Domar weight.

Thus, our result provides general conditions for when small microeconomic productivity disturbances can generate sizable macroeconomic fluctuations. If there is a sector which *always* has a non-negligible Domar weight, then microeconomic shocks are guaranteed to generate aggregate fluctuations. Relative to the insight of Acemoglu et al. (2012), our bounds imply that even with arbitrary technologies, so long as there exists a sector that cannot be replaced (even in extreme circumstances), microeconomic shocks have the potential to generate macroeconomic events.

9 Conclusion

In this paper, we have provided a global analysis of the canonical production network model with arbitrary constant returns to scale technology. Our main result provides a primitive formula for equilibrium prices that applies in any such economy. As a formula was previously known only in the Cobb-Douglas case (Acemoglu et al., 2012), this formula represents a substantial generalization and applies to popular special cases, such as nested CES economies (Baqaee and Farhi, 2019; Dew-Becker, 2023), endogenous Cobb-Douglas economies (Acemoglu and Azar, 2020; Kopytov et al., 2024), and many others (including those studied by Chodorow-Reich et al., 2026; Acemoglu and Tahbaz-Salehi, 2025).

Beyond the technical contribution of this paper, to demonstrate the general utility of this analysis, we used it in four applications. First, we studied the question of when the economy is resilient and when it is fragile. We found that the "min" and the "max" in the general representation correspond in a natural way to the competing forces of resilience and fragility. We also characterized the microeconomic properties of resilience and fragility at the sector level, revealing that standard specifications and approximation-based methods allow only for global resilience or fragility. Thus, such models cannot capture the qualitative possibility that changing input mix can lead to volatility having "S-shaped" effects on the economy, whereby small and large levels of volatility are bad, but intermediate levels are good.

Second, we provided a generalization of Hulten's (1978) theorem to cover the case of general (*i.e.,* non-differentiable) production technology. As we showed, such non-differentiabilities are a natural consequence of allowing for endogenous production networks. Moreover, the generalized theorem places important economic nuance on the received wisdom following Hulten (1978) that, for small shocks, it is not important to know anything about technology or the input-output network. Indeed, the presence of technology switching—which cannot be inferred from mere observation of the current Domar weights—leads current Domar weights to provide an inaccurate description of shock propagation, even for small shocks.

Third, we characterized the effects of microeconomic volatility on GDP. We showed that these effects can always be decomposed into two terms, reflecting how changes in productivity smoothly change the economy's Domar weights—providing an exact version of the logic from Baqaee and Farhi (2019)—and an additional term reflecting how jumps in Domar weights contribute. This change from jumps in Domar weights results from technology switching and always contributes positively. Hence, by this notion of local resilience, economies are necessarily weakly more resilient than would be suggested by second-order methods (and strictly more so exactly when technology switching occurs in equilibrium).

Finally, we further applied the generalized Hulten's (1978) theorem to provide robust upper and lower bounds on the variance of aggregate output that results from microeconomic productivity disturbances that hold regardless of the underlying production technology in the economy. These general bounds reveal that, in order to guarantee that microeconomic productivity disturbances matter in aggregate, there must generally be a sector whose sales share of GDP is substantial even when they are very unproductive and their goods are very expensive; *i.e.,* there must be a sector whose output is very hard to substitute.

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Appendices

A Preliminaries: nonexpansive operators and fixed points

In this appendix, with an abuse of notation, we let $x \in \mathbb{R}^n$ denote an arbitrary n-th vector, which does not need to be related to the vector of log-productivity shocks considered in the main text. We endow $\mathbb{R}^n$ with the supnorm $||x||_\infty = \max_{i\in[n]} |x_i|$. We say that an operator $S : \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive if $||S(x) - S(x')||_\infty \leq ||x - x'||_\infty$. Also, denote by D the set of constant vectors of $\mathbb{R}^n$.

We now recall a few definitions of operator properties that will be important in our analysis. We say that a selfmap S on $\mathbb{R}^n$ is normalized if $S(he) = he$ for all $h \in \mathbb{R}$, where $e = (1, \ldots, 1)$ is the n-dimensional unit vector. We say that such a selfmap is translation invariant if $S(p+he) = S(p) + he$ for all $p \in \mathbb{R}^n$ and $h \in \mathbb{R}$.

Given a nonexpansive operator $S : \mathbb{R}^n \to \mathbb{R}^n$, denote as $E(S)$ its fixed points, that is those points x such that $S(x) = x$. Fix $\beta \in (0, 1)$. We introduce the new operator $\tilde{S}_\beta : \mathbb{R}^n \to \mathbb{R}^n$ that maps each $x \in \mathbb{R}^n$ to the solution of the following equation (in $\tilde{x}$):

$$\tilde{x} = (1-\beta)\, x + \beta S\left(\tilde{x}\right). \tag{73}$$

Lemma 3 in Cerreia-Vioglio et al. (2025b) establishes that $\tilde{S}_\beta$ is well defined. In what follows, to lighten notation, we often denote $\tilde{S}_\beta(x) = \tilde{x}_\beta$. We define an additional ancillary operator $S_{\beta,x} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$S_{\beta,x}\left(x'\right) = (1-\beta)\, x + \beta S\left(x'\right) \quad \forall x' \in \mathbb{R}^n, \tag{74}$$

and, for every $m \in \mathbb{N}$, let $S^m_{\beta,x}(x') = S_{\beta,x} \circ \ldots \circ S_{\beta,x}(x')$ denote its m-th fold iteration evaluated at x'. It is standard to show that $\lim_m S^m_{\beta,x}(x') = \tilde{S}_\beta(x)$ for all $x', x \in \mathbb{R}^n$, see, e.g., Lemma 3 in Cerreia-Vioglio et al. (2025b).

We next state without proof a simple lemma which characterizes $\tilde{S}_\beta$ when S is affine, non-decreasing, and translation invariant.

Lemma 1. *Let $S(x) = Wx + u$ for some $W \in \mathcal{W}$ and $u \in \mathbb{R}^n$. We have*

$$\tilde{S}_\beta(x) = \Omega_W\left(x + \frac{\beta}{(1-\beta)}u\right) \qquad \forall x \in \mathbb{R}^n. \tag{75}$$

In this case, $\tilde{S}_\beta$ is affine, non-decreasing, and translation invariant.

Lemma 2. *Let S and R be nonexpansive. If either S or R is non-decreasing and $S \geq R$, then $\tilde{S}_\beta \geq \tilde{R}'_\beta$.*

Proof. Assume S is non-decreasing and fix $x \in \mathbb{R}^n$. We prove by induction that $S^t_{\beta,x}(x) \geq R^t_{\beta,x}(x)$ for all $t \in \mathbb{N}$. For $t = 1$, note that $S^1_{\beta,x}(x) = (1-\beta)x + \beta S(x) \geq (1-\beta)x + \beta R(x) = R^1_{\beta,x}(x)$. Next, we assume the statement is true for t and we prove it holds for $t+1$. Since S is non-decreasing and $S \geq R$, we have that

$$S^{t+1}_{\beta,x}(x) = S_{\beta,x}\left(S^t_{\beta,x}(x)\right) = (1-\beta)x + \beta S(S^t_{\beta,x}(x)) \geq (1-\beta)x + \beta S(R^t_{\beta,x}(x)) \tag{76}$$

$$\geq (1-\beta)x + \beta R(R^t_{\beta,x}(x)) = R_{\beta,x}\left(R^t_{\beta,x}(x)\right) = R^{t+1}_{\beta,x}(x), \tag{77}$$

proving the statement. By passing to the limit, $\tilde{S}_\beta(x) = \lim_t S^t_{\beta,x}(x) \geq \lim_t R^t_{\beta,x}(x) = \tilde{R}_\beta(x)$. If, instead, R were the one to be non-decreasing, it would then be enough to change in the inductive step $(1-\beta)x + \beta S(R^t_{\beta,x}(x))$ with $(1-\beta)x + \beta R(S^t_{\beta,x}(x))$, to reach the same conclusion. $\square$

B Proofs of the Results in Section 4

Proof of Proposition 1. (1) $\implies$ (2) Fix $P \in \mathbb{R}^n_{++}$. Fix also $i \in [n]$ and $\tilde{w}_i \in \Delta$ such that $G_i(\tilde{w}_i) \neq 0$. Define $Q_i = \frac{\tilde{w}_i}{G_i(\tilde{w}_i)} \in \mathbb{R}^n_+$. Since G_i is positive homogeneous, we have that $G_i(Q_i) = 1$. By definition of Q_i, it follows that

$$\frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)} = \sum_{j=1}^n Q_{ij} P_j \geq K_i(P). \tag{78}$$

Because $\tilde{w}_i$ was arbitrarily chosen, it follows that

$$K_i(P) \leq \min_{\tilde{w}_i \in \Delta : G_i(\tilde{w}_i) \neq 0} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)} = \min_{\tilde{w}_i \in \Delta} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)}. \tag{79}$$

Next, fix Q_i such that $G_i(Q_i) \geq 1$ and define $w_i = \frac{1}{\sum_{j=1}^n Q_{ij}} Q_i$. It follows that

$$\sum_{j=1}^n Q_{ij} P_j \geq \frac{\sum_{j=1}^n Q_{ij} P_j}{G_i(Q_i)} = \frac{\sum_{j=1}^n w_{ij} P_j}{G_i(w_i)} \geq \min_{\tilde{w}_i \in \Delta} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)} \tag{80}$$

where the equality follows from positive homogeneity of G_i. Because Q_i was arbitrarily chosen, it follows that $K_i(P) \geq \min_{\tilde{w}_i \in \Delta} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)}$, proving $K_i(P) = \min_{\tilde{w}_i \in \Delta} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)}$.

By the definition of T, we have

$$T_i(p) = \min_{\tilde{w}_i \in \Delta} \ln\left(\sum_{j=1}^n \tilde{w}_{ij} \exp(p_j)\right) + \ln\left(\frac{1}{G_i(\tilde{w}_i)}\right), \tag{81}$$

By Maccheroni et al. (2006), this establishes that T is normalized, monotone, and translation invariant, completing the proof of this part.

(2) $\implies$ (1) follows directly by Theorem 3.

The exact expression for T follows by combining the above with Proposition 1.4.2 in Dupuis and Ellis (2011).[16] $\square$

Define the function $H : [0, \infty] \times \Delta \times \Delta \to [0, \infty]$ as

$$H(\lambda, w_i, \tilde{w}_i) = \begin{cases} \lambda \sum_{j=1}^{k} w_{ij} \log\left(\frac{w_{ij}}{\tilde{w}_{ij}}\right) & \text{if} \quad w_i \ll \tilde{w}_i \quad \text{and} \quad \lambda < \infty, \\ 0 & \text{if} \quad w_i = \tilde{w}_i \quad \text{and} \quad \lambda = \infty, \\ \infty & \text{otherwise.} \end{cases} \tag{82}$$

Lemma 3. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be such that, for every $i \in [n]$, there exists $\lambda_i \in [0, \infty]$ and a lower semicontinuous function $c_i : \Delta \to [0, \infty]$ with*

$$T_i(x') = \min_{\tilde{w}_i \in \Delta} \max_{w_i \in \Delta} \{\langle w_i, x' \rangle - H(\lambda_i, w_i, \tilde{w}_i) + c_i(\tilde{w}_i)\} \qquad \forall x' \in \mathbb{R}^n. \tag{83}$$

We have that

$$\Pi(x) = \min_{\tilde{W} \in \mathcal{W}} \max_{W \in \mathcal{W}} \left\{\Omega_W\left(x - \tau^c(W||\tilde{W})\right)\right\} \qquad \forall x \in \mathbb{R}^n \tag{84}$$

where $\tau_i^c(W||\tilde{W}) = \frac{\beta}{1-\beta}(H(\lambda_i, w_i, \tilde{w}_i) - c_i(\tilde{w}_i))$ for all $i \in [n]$.

Proof. For every $i \in [n]$, fix $\lambda_i \in [0, \infty]$ and $c_i : \Delta \to [0, \infty]$ as in the statement. For all $\tilde{W}, W \in \mathcal{W}$, define $\alpha\left(W, \tilde{W}\right) \in \mathbb{R}^n$ as

$$\alpha\left(W, \tilde{W}\right)_i = -H(\lambda_i, w_i, \tilde{w}_i) + c_i(\tilde{w}_i) \qquad \forall i \in [n] \tag{85}$$

and the affine operator $T_{\tilde{W},W} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$T_{\tilde{W},W}(x') = Wx' + \alpha\left(W, \tilde{W}\right). \tag{86}$$

Also, for every $\tilde{W} \in \mathcal{W}$, define the operator $T^*_{\tilde{W}} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$T^*_{\tilde{W}}(x') = \max_{W \in \mathcal{W}} T_{\tilde{W},W}(x'). \tag{87}$$

[16] The steps of the proof of that result need to be adapted in the obvious way given that we have $\ln\left(\sum_{j=1}^{n} \tilde{w}_{ij} \exp(p_j)\right)$ in place of $-\ln\left(\sum_{j=1}^{n} \tilde{w}_{ij} \exp(-p_j)\right)$.

Observe that the operator in Equation 87 is well defined since

$$\left(\max_{W\in\mathcal{W}} T_{\tilde{W},W}\left(x'\right)\right)_i = \max_{w_i\in\Delta}\left\{\left\langle w_i, x'\right\rangle - H\left(\lambda_i, w_i, \tilde{w}_i\right) + c_i\left(\tilde{w}_i\right)\right\} \qquad \forall i\in[n], \forall x'\in\mathbb{R}^n. \tag{88}$$

Clearly, each $T^*_{\tilde{W}}$ is non-decreasing and translation invariant, hence nonexpansive. With this, we have

$$T\left(x'\right) = \min_{\tilde{W}\in\mathcal{W}} T^*_{\tilde{W}}\left(x'\right) \qquad \forall x'\in\mathbb{R}^n. \tag{89}$$

By Lemma 1, for all $\tilde{W}, W\in\mathcal{W}$, we have

$$\tilde{T}_{\tilde{W},W,\beta}\left(x\right) = \Omega_W\left(x + \frac{\beta}{1-\beta}\alpha\left(W,\tilde{W}\right)\right) \qquad \forall x\in\mathbb{R}^n. \tag{90}$$

Next, fix $\tilde{W}\in\mathcal{W}$ and observe that for all $W\in\mathcal{W}$ and $x'\in\mathbb{R}^n$, we have $T_{\tilde{W},W}\left(x'\right)\leq T^*_{\tilde{W}}\left(x'\right)$. By Lemma 2, we also have that

$$\Omega_W\left(x + \frac{\beta}{1-\beta}\alpha\left(W,\tilde{W}\right)\right) = \tilde{T}_{\tilde{W},W,\beta}\left(x\right) \leq \tilde{T}^*_{\tilde{W},\beta}\left(x\right) \tag{91}$$

for all $W\in\mathcal{W}$ and $x\in\mathbb{R}^n$. By Equation (87), for all $x\in\mathbb{R}^n$, there exists $\hat{W}\in\mathcal{W}$ such that

$$\tilde{T}^*_{\tilde{W},\beta}\left(x\right) = \left(1-\beta\right)x + \beta T^*_{\tilde{W}}\left(\tilde{T}^*_{\tilde{W},\beta}\left(x\right)\right) = \left(1-\beta\right)x + \beta T_{\tilde{W},\hat{W}}\left(\tilde{T}^*_{\tilde{W},\beta}\left(x\right)\right) \tag{92}$$

yielding that $\tilde{T}^*_{\tilde{W},\beta}\left(x\right) = \tilde{T}_{\tilde{W},\hat{W},\beta}\left(x\right)$, hence that

$$\tilde{T}^*_{\tilde{W},\beta}\left(x\right) = \max_{W\in\mathcal{W}}\left\{\Omega_W\left(x + \frac{\beta}{1-\beta}\alpha\left(W,\tilde{W}\right)\right)\right\}. \tag{93}$$

Because $\tilde{W}\in\mathcal{W}$ was arbitrarily chosen, it follows that Equation 93 holds for all $\tilde{W}\in\mathcal{W}$.

Next, fix $x\in\mathbb{R}^n$. Because $T\leq T^*_{\tilde{W}}$ for all $\tilde{W}\in\mathcal{W}$, Lemma 2 implies that

$$\Pi\left(x\right) \leq \tilde{T}^*_{\tilde{W},\beta}\left(x\right) \tag{94}$$

for all $\tilde{W}\in\mathcal{W}$. By (89), there exists $\tilde{W}^*\in\mathcal{W}$ such that

$$\Pi\left(x\right) = \left(1-\beta\right)x + \beta T\left(\Pi\left(x\right)\right) = \left(1-\beta\right)x + \beta T^*_{\tilde{W}^*}\left(\Pi\left(x\right)\right) \tag{95}$$

yielding that $\Pi\left(x\right) = \tilde{T}^*_{\tilde{W}^*,\beta}\left(x\right)$. This in turn yields that

$$\min_{\tilde{W}\in\mathcal{W}} \tilde{T}^*_{\tilde{W},\beta}\left(x\right) \leq \tilde{T}^*_{\tilde{W}^*,\beta}\left(x\right) = \Pi\left(x\right) \leq \min_{\tilde{W}\in\mathcal{W}} \tilde{T}^*_{\tilde{W},\beta}\left(x\right), \tag{96}$$

hence that

$$\Pi(x) = \min_{\tilde{W}\in\mathcal{W}} \tilde{T}^*_{\tilde{W},\beta}(x) = \min_{\tilde{W}\in\mathcal{W}} \max_{W\in\mathcal{W}} \left\{\Omega_W\left(x + \frac{\beta}{1-\beta}\alpha\left(W,\tilde{W}\right)\right)\right\} \tag{97}$$

$$= \min_{\tilde{W}\in\mathcal{W}} \max_{W\in\mathcal{W}} \left\{\Omega_W\left(x - \tau^c\left(W||\tilde{W}\right)\right)\right\}, \tag{98}$$

as desired. □

Proof of Theorem 1. By Proposition 1, observe that for every $i \in [n]$, T_i can be written as in Equation 83 with $\lambda_i = 1$ and $c_i(\tilde{w}_i) = \ln\left(\frac{1}{G_i(\tilde{w})}\right)$ for all $\tilde{w} \in \Delta$, which is clearly lower semicontinuous. Therefore, the result immediately follows by Lemma 3 and Equation 20. □

C Proofs of the Results in Section 5

Proof of Proposition 2. Fix $i \in [n]$ and a log-cost operator T_i.

(i) implies (ii). Because sector i is resilient and T_i is a log-cost operator, by Proposition 1 we have that T_i is normalized, non-decreasing, translation invariant, and concave. By Lemma 26 in Maccheroni et al. (2006), it follows that

$$T_i(p) = \min_{w_i\in\Delta} \left\{\langle w_i, p\rangle + d_i(w_i)\right\} \qquad \forall p \in \mathbb{R}^n, \tag{99}$$

where $d_i : \Delta \to [0,\infty]$ is lower semicontinuous, convex, and such that $\min_{w_i\in\Delta} d_i(w_i) = 0$. In particular, if T_i is positive homogeneous $d_i(w_i) \in \{0,\infty\}$ for all $w_i \in \Delta$. With this, we have

$$K_i(P) = \exp(T_i(\ln(P))) = \min_{w_i\in\Delta} \left\{\frac{\prod_{j=1}^n P_j^{w_{ij}}}{\exp(-d_i(w_i))}\right\} \qquad \forall P \in \mathbb{R}^n_{++}. \tag{100}$$

Next, define

$$G_i^*(Q_i) = \max_{w_i\in\Delta} \left\{\exp(-d_i(w_i)) \prod_{j=1}^n \left(\frac{Q_{ij}}{w_{ij}}\right)^{w_{ij}}\right\} \qquad \forall Q_i \in \mathbb{R}^n_+. \tag{101}$$

Observe that G_i is non-decreasing and such that

$$\max_{\tilde{w}_i\in\Delta} G_i^*(\tilde{w}_i) = \max_{w_i\in\Delta} \left\{\exp(-d_i(w_i)) \max_{\tilde{w}_i\in\Delta} \prod_{j=1}^n \left(\frac{\tilde{w}_{ij}}{w_{ij}}\right)^{w_{ij}}\right\} = 1. \tag{102}$$

Moreover, since the objective function in the maximization of Equation 101 is the product of two upper semicontinuous functions, it is upper semicontinuous in (Q_i, w_i). In turn, Lemma

17.30 in Aliprantis and Border (2006) implies that G_i is upper semicontinuous, and Proposition 5.94 in Moore (1999) implies that it is, in fact, continuous.

By letting $C_i = \{w_i \in \Delta : d_i(w_i) < \infty\}$ and $A_i(w_i) = \exp(-d_i(w_i))$ for all $w_i \in C_i$, we obtain

$$G_i^*(Q_i) = \max_{w_i \in C_i} \left\{ A_i(w_i) \prod_{j=1}^{n} \left(\frac{Q_{ij}}{w_{ij}} \right)^{w_{ij}} \right\} \qquad \forall Q_i \in \mathbb{R}_+^n. \tag{103}$$

Observe that, given the properties of d_i, we have that C_i is nonempty and convex A_i is upper semicontinuous, log-concave, and such that $\max_{w_i \in C_i} A_i(w_i) = 1$. Moreover, if T_i is positive homogeneous, and thus $d(w_i) \in \{0, \infty\}$ for all $w_i \in \Delta$, A_i is equal to 1 on its domain. Next, it is easy to see that, for all $P \in \mathbb{R}_{++}^n$,

$$\min_{Q_i \in \mathbb{R}_+^n : G_i^*(Q_i) \geq 1} \sum_{j=1}^{n} Q_{ij} P_j = \min_{w_i \in C_i} \left\{ \frac{\prod_{j=1}^{n} P_j^{w_{ij}}}{A_i(w_i)} \right\} = K_i(P), \tag{104}$$

yielding that T_i is generated by G_i^*.

(ii) implies (i) The cost operator generated by G_i is

$$K_i(P) = \min_{w_i \in C_i} \frac{\prod_{j=1}^{n} P_j^{w_{ij}}}{A_i(w_i)} \qquad \forall P \in \mathbb{R}_{++}^n. \tag{105}$$

In turn, this yields that

$$T_i(p) = \min_{w_i \in C_i} \left\{ \langle w_i, p \rangle + \ln\left(\frac{1}{A_i(w_i)} \right) \right\} \qquad \forall p \in \mathbb{R}^n, \tag{106}$$

which is a concave function, as desired. Moreover, if $A_i(w_i)$ is identically equal to 1 on its domain, $\ln\left(\frac{1}{A_i(w_i)}\right) \in \{0, \infty\}$ so T_i is positive homogeneous. □

Proof of Corollary 1. For every sector $i \in [n]$, Proposition 2 and Equation 106 give that

$$T_i(p) = \min_{w_i \in C_i} \left\{ \langle w_i, p \rangle + \ln\left(\frac{1}{A_i(w_i)} \right) \right\} \qquad \forall p \in \mathbb{R}^n. \tag{107}$$

By Equation 107, observe that for every $i \in [n]$, T_i can be written as in Equation 83 with $\lambda_i = \infty$ and

$$c_i(\tilde{w}_i) = \begin{cases} \ln\left(\frac{1}{A_i(\tilde{w}_i)}\right) & \text{if} \quad w \in C_i, \\ \infty & \text{otherwise.} \end{cases} \tag{108}$$

which is clearly lower semicontinuous. Therefore, the result immediately follows by Lemma 3. □

Proof of Proposition 3. It follows directly from Proposition 13. □

Proof of Proposition 4. First, assume that sector i is resilient. Fix $j \in [n]$ and observe that by Definition 3.1, Proposition 5.7, and Theorem 7.4 in Dal Maso (1993) we have:

$$\lim_{\lambda \to \infty} \frac{T_i(\lambda e_j)}{\lambda} = \lim_{\lambda \to \infty} \min_{w_i \in C_i} \left\{ w_{ij} + \frac{1}{\lambda} \ln \left(\frac{1}{A_i(w_i)} \right) \right\} = \min_{w_i \in C_i^R} w_{ij} \tag{109}$$

With this, sector i can abate shocks from sector j if and only if there exists $w_i \in C_i^R$ with $w_{ij} = 0$. Point 2 on the fragile-sector case follows from an entirely symmetric argument. □

Proof of Proposition 5. 1. It follows by Theorem 14 in Marinacci and Montrucchio (2008).

2. Immediate. □

D Proofs of the Results in Section 6

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be normalized, non-decreasing, and translation invariant. Define $\Pi^1 : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\Pi^1(x) = (1-\beta)x + \beta T(x) \qquad \forall x \in \mathbb{R}^n. \tag{110}$$

For all integers $t > 1$ define

$$\Pi^t(x) = (1-\beta)x + \beta T\left(\Pi^{t-1}(x)\right) \qquad \forall x \in \mathbb{R}^n. \tag{111}$$

We are going to say that $g : \mathbb{R}^n \to \mathbb{R}$ is *regular at* x if and only if

$$\lim_{h \downarrow 0} \frac{g(x+hd) - g(x)}{h} = \liminf_{x' \to x, h \downarrow 0} \frac{g(x'+hd) - g(x')}{h}.$$

We say that g is *regular* if and only if it is regular at each point of its domain. Define $M : \Delta \times \mathbb{R}^n \to \mathbb{R}$ by $M(\alpha, x) = \ln\left(\sum_{i=1}^n e^{x_i}\alpha_i\right)$ for all $\alpha \in \Delta$ and for all $x \in \mathbb{R}^n$. We define $\mathcal{M} : \mathcal{W} \times \mathbb{R}^n \to \mathbb{R}^n$ by $\mathcal{M}_i(W, x) = M(w_i, x)$ for all $W \in \mathcal{W}$, for all $x \in \mathbb{R}^n$, and for all $i \in [n]$. For each $g : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$ define by $\sigma_g(x)$ the subset of αs in Δ such that $g(x) = M(\alpha, x) + c(\alpha)$. Note that if T_i is generated by a production function G_i, then it easily follows by Proposition 1 that $\sigma_{T_i}(x)$ is nonempty and compact for all $x \in \mathbb{R}^n$.

Remark 1. *We list here some notation and make four observations. Given an operator $S : \mathbb{R}^n \to \mathbb{R}^n$, we denote the Jacobian of S at x by $J_S(x)$. If S is normalized, non-decreasing, and translation invariant and admits Jacobian at x, then $J_S(x)$ is a stochastic matrix. We endow the space of $n \times n$ matrices with the norm $\| \, \|_* : M_n(\mathbb{R}) \to [0, \infty)$ defined by*

$$\|W\|_* = \max_{i \in [n]} \sum_{j=1}^n |w_{ij}| \qquad \forall W \in M_n(\mathbb{R}). \tag{112}$$

This is the operator norm when W is viewed as a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathbb{R}^n$ is endowed with the supnorm $\| \|_\infty$. Observe that if W is a stochastic matrix, then $\|W\|_* = 1$. First, observe that when W and $\hat{W}$ are two stochastic matrices,[17] *we have that*

$$\left\|W^t - \hat{W}^t\right\|_* \leq t\left\|W - \hat{W}\right\|_* \qquad \forall t \in \mathbb{N}_0. \tag{113}$$

Second, consider the quantity $\kappa = (1-\beta)\sum_{t=0}^{\infty} \beta^t t$. Note that κ is well defined and finite. It is well defined because the partial sums are positive and increasing. Moreover, set $\gamma = \frac{1}{2}\frac{1}{\beta} + \frac{1}{2}$. Since $\beta \in (0,1)$, note that $\gamma \in (1, 1/\beta)$ and $\beta\gamma \in (0,1)$. It follows that eventually $\gamma^t \geq t$ and, in particular, eventually $(\beta\gamma)^t \geq \beta^t t$. This implies that κ is finite. Third, given $\beta \in (0,1)$, define $B : \mathcal{W} \to \mathcal{W}$ by $B(W) = \Omega_W$ for all $W \in \mathcal{W}$. Note that

$$\left\|B(W) - B\left(\hat{W}\right)\right\|_* = \left\|\Omega_W - \hat{W}_\beta\right\|_* = \left\|(1-\beta)\sum_{t=0}^{\infty}\beta^t\left(W^t - \hat{W}^t\right)\right\|_* \tag{114}$$

$$\leq (1-\beta)\sum_{t=0}^{\infty}\beta^t\left\|W^t - \hat{W}^t\right\|_* \leq (1-\beta)\sum_{t=0}^{\infty}\beta^t t\left\|W - \hat{W}\right\|_* \tag{115}$$

$$\leq \kappa\left\|W - \hat{W}\right\|_* \qquad \forall W, \hat{W} \in \mathcal{W}. \tag{116}$$

This proves that B is a Lipschitz continuous operator. Fourth, B is injective. By Horn and Johnson (2012), Theorem 8.3.4 and Problem 5.6.P26, observe that for each $W \in \mathcal{W}$ the spectral radius of βW is $\beta < 1$ and, in particular, $(I - \beta W)^{-1} = \sum_{t=0}^{\infty}\beta^t W^t$. We can conclude that $B(W) = B\left(\hat{W}\right)$ implies that $(I-\beta W)^{-1} = \left(I - \beta\hat{W}\right)^{-1}$, yielding that $I - \beta\hat{W} = I - \beta W$, that is, $W = \hat{W}$, proving injectivity.

Lemma 4. *Let $i \in [n]$. If T_i is generated by a production function G_i then T_i is regular.*

Proof. Fix $x \in \mathbb{R}^n$. Since K_i is concave and locally Lipschitz, we have that $-K_i$ is locally Lipschitz and regular a la Clarke at $\exp(x)$ (see Clarke, 1990, Definition 2.3.4 and Proposition 2.3.6). By Clarke (1990), p. 32 and Theorem 2.3.10, and since exp is continuously differentiable, it follows that $-K_i \circ \exp$ is regular a la Clarke at x. Define $\gamma : \mathbb{R}_{--} \to \mathbb{R}_{++}$ by $\gamma(s) = -\ln(-s)$ for all $s \in \mathbb{R}_{--}$. By Clarke (1990), p. 32 and Theorem 2.3.10, and since γ is continuously differentiable, we have that γ is regular a la Clarke. By Clarke (1990), point (i) of Theorem 2.3.9, and since γ is non-decreasing and $-K_i \circ \exp$ is regular a la Clarke at x, this implies that $\gamma \circ -K_i \circ \exp$ is regular a la Clarke at x. It is immediate to see that $g = -\gamma \circ -K_i \circ \exp$, proving that T_i is regular in our sense. □

[17] See footnote 32 of Cerreia-Vioglio et al. (2025b).

Lemma 5. *Let $i \in [n]$. If T_i is generated by a production function G_i then for each $x, d \in \mathbb{R}^n$*

$$\partial_C T_i(x) = \text{co}\left\{\nabla M(w_i, x) : w_i \in \sigma_{T_i}(x)\right\} \text{ and } \lim_{h \downarrow 0} \frac{T_i(x + hd) - T_i(x)}{h} = \min_{w_i \in \sigma_{T_i}(x)} \langle \nabla M(w_i, x), d \rangle. \tag{117}$$

Proof. The result follows from Clarke (2013), Theorem 10.22, whose assumptions are satisfied by Lemma 4. □

Corollary 3. *Fix $i \in [n]$. If T_i is generated by a production function G_i, then for each $y, z \in \mathbb{R}^n$ there exist $\lambda_{y,z} \in [0, 1]$ and $w_i \in \sigma_{T_i}(\lambda_{y,z} z + (1 - \lambda_{y,z}) y)$ such that*

$$T_i(z) - T_i(y) \geq \langle \nabla M(w_i, \lambda_{y,z} z + (1 - \lambda_{y,z}) y), z - y \rangle. \tag{118}$$

Proof. Fix $y, z \in \mathbb{R}^n$. By Lebourg's Mean Value Theorem (see, e.g., Clarke, 1990, Theorem 2.3.7) and since T_i is Lipschitz continuous, there exists $\lambda_{y,z} \in [0, 1]$ such that

$$T_i(z) - T_i(y) \geq \min_{w_i \in \partial_C T_i(\lambda_{y,z} z + (1 - \lambda_{y,z}) y)} \langle w_i, z - y \rangle. \tag{119}$$

By Lemma 5, we have that

$$T_i(z) - T_i(y) \geq \min_{w_i \in \partial_C T_i(\lambda_{y,z} z + (1 - \lambda_{y,z}) y)} \langle w_i, z - y \rangle = \min_{w_i \in \sigma_{T_i}(\lambda_{y,z} z + (1 - \lambda_{y,z}) y)} \langle \nabla M(w_i, \lambda_{y,z} z + (1 - \lambda_{y,z}) y), z - y \rangle. \tag{120}$$

By choosing $w_i \in \sigma_{T_i}(\lambda_{y,z} z + (1 - \lambda_{y,z}) y)$ which achieves the min above, the statement follows. □

Lemma 6. *Let $W \in \mathcal{W}$, $\beta \in (0, 1)$, and $d \in \mathbb{R}^n$. If $z \in \mathbb{R}^n$ is such that $z \leq (1 - \beta) d + \beta W z$ (resp. $\geq$), then $z \leq \Omega_W d$ (resp. $\geq$).*

Proof. We prove by induction that

$$z \leq (1 - \beta) \sum_{t=0}^{\tau} \beta^t W^t d + \beta^{\tau+1} W^{\tau+1} z \text{ (resp. } \geq \text{)} \quad \forall \tau \in \mathbb{N}_0. \tag{121}$$

Initial Step. $\tau = 0$. By assumption, we have that

$$z \leq (1 - \beta) d + \beta W z = (1 - \beta) \sum_{t=0}^{\tau} \beta^t W^t d + \beta^{\tau+1} W^{\tau+1} z \text{ (resp. } \geq \text{)}, \tag{122}$$

proving the initial step.

Inductive Step. We assume the inequality is true for τ and prove it holds for $\tau + 1$. By assumption, we have that

$$z \leq (1-\beta)\sum_{t=0}^{\tau}\beta^t W^t d + \beta^{\tau+1}W^{\tau+1}z \leq (1-\beta)\sum_{t=0}^{\tau}\beta^t W^t d + \beta^{\tau+1}W^{\tau+1}\left((1-\beta)\, d + \beta W z\right) \tag{123}$$

$$= (1-\beta)\sum_{t=0}^{\tau+1}\beta^t W^t d + \beta^{\tau+2}W^{\tau+2}z, \tag{124}$$

proving the inductive step.

The inequality (121) follows by induction. By passing to the limit in τ and since W is a stochastic matrix, we have that $z \leq \Omega_W d$ (resp. $\geq$). □

Given $x, d \in \mathbb{R}^n$ and $h > 0$, define $q_{h,x,d} = \frac{\Pi(x+hd)-\Pi(x)}{h}$. Given $x, d \in \mathbb{R}^n$, since Π is Lipschitz continuous of order 1, we have that $\{q_{h,x,d}\}_{h>0} \subseteq \left[-\left\|d\right\|_\infty, \left\|d\right\|_\infty\right]^n$.

Lemma 7. *Let T be generated by a production function G and $\beta \in (0,1)$. If $x, d \in \mathbb{R}^n$ and $\tilde{W} \in \Sigma\left(\Pi\left(x\right)\right)$ and $\{h_k\}_{k\in\mathbb{N}} \subseteq (0,\infty)$ is such that $\lim_k h_k = 0$ and $\lim_k q_{h_k,x,d} = q$, then*

$$q \leq \Omega_{J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(x))}d. \tag{125}$$

Proof. Fix $x, d \in \mathbb{R}^n$. Since we fixed x and d, we denote $q_{h,x,d}$ simply by q_h for all $h > 0$. Consider $\tilde{W} \in \Sigma\left(\Pi\left(x\right)\right)$. Note that, by Proposition 1, for each $x \in \mathbb{R}^n$

$$T\left(y\right) - T\left(x\right) \leq \mathcal{M}\left(\tilde{W}, y\right) - \mathcal{M}\left(\tilde{W}, x\right) \qquad \forall \tilde{W} \in \Sigma\left(x\right), \forall y \in \mathbb{R}^n. \tag{126}$$

Therefore, we have that

$$\Pi\left(x+hd\right) - \Pi\left(x\right) = (1-\beta)\left(x+hd\right) + \beta T\left(\Pi\left(x+hd\right)\right) - (1-\beta)\, x - \beta T\left(\Pi\left(x\right)\right) \tag{127}$$

$$= (1-\beta)\, hd + \beta\left[T\left(\Pi\left(x+hd\right)\right) - T\left(\Pi\left(x\right)\right)\right] \tag{128}$$

$$\leq (1-\beta)\, hd + \beta\left(\mathcal{M}\left(\tilde{W}, \Pi\left(x+hd\right)\right) - \mathcal{M}\left(\tilde{W}, \Pi\left(x\right)\right)\right) \qquad \forall h > 0. \tag{129}$$

By the Mean Value Theorem, we have that for each $h > 0$ and for each $i \in [n]$ there exists $\tau_{i,h} \in [0,1]$ such that

$$\mathcal{M}_i\left(\tilde{W}, \Pi\left(x+hd\right)\right) - \mathcal{M}_i\left(\tilde{W}, \Pi\left(x\right)\right) \tag{130}$$

$$= M\left(\tilde{w}_i, \Pi\left(x+hd\right)\right) - M\left(\tilde{w}_i, \Pi\left(x\right)\right) \tag{131}$$

$$= \left\langle \nabla M\left(\tilde{w}_i, \tau_{i,h}\Pi\left(x+hd\right) + \left(1-\tau_{i,h}\right)\Pi\left(x\right)\right), \Pi\left(x+hd\right) - \Pi\left(x\right)\right\rangle. \tag{132}$$

For each $h > 0$ define W_h the matrix whose i-th entry is $\nabla M\left(\tilde{w}_i, \tau_{i,h}\Pi\left(x+hd\right) + \left(1-\tau_{i,h}\right)\Pi\left(x\right)\right)$. Note that

$$\mathcal{M}\left(\tilde{W}, \Pi\left(x+hd\right)\right) - \mathcal{M}\left(\tilde{W}, \Pi\left(x\right)\right) = W_h\left(\Pi\left(x+hd\right) - \Pi\left(x\right)\right) \quad \forall h > 0. \tag{133}$$

Define also W_0 as the matrix whose i-th entry is $\nabla M\left(\tilde{w}_i, \Pi\left(x\right)\right)$, that is, $W_0 = J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi\left(x\right)\right)$. Since Π is Lipschitz continuous of order 1 and $(\omega, z) \mapsto \nabla M\left(\omega, z\right)$ is continuous on $\Delta \times \mathbb{R}^n$, we have that $\lim_{h\downarrow 0} W_h = W_0$. Since $h > 0$, it follows that

$$\frac{\Pi\left(x+hd\right) - \Pi\left(x\right)}{h} \leq \left(1-\beta\right)d + \beta\left(\frac{\mathcal{M}\left(\tilde{W}, \Pi\left(x+hd\right)\right) - \mathcal{M}\left(\tilde{W}, \Pi\left(x\right)\right)}{h}\right) \tag{134}$$

$$= \left(1-\beta\right)d + \beta W_h \frac{\Pi\left(x+hd\right) - \Pi\left(x\right)}{h} \qquad \forall h > 0, \tag{135}$$

that is,

$$q_h \leq \left(1-\beta\right)d + \beta W_h q_h \qquad \forall h > 0. \tag{136}$$

By Lemma 6 and since h was arbitrarily chosen and W_h is a stochastic matrix, we have that $q_h \leq \Omega_{W_h} d$ for all $h > 0$. By Remark 1 and since $\lim_k W_{h_k} = W_0$, $\lim_k q_{h_k} = q$, and $q_{h_k} \leq \Omega_{W_{h_k}} d$ for all $k \in \mathbb{N}$, it follows that $q \leq \Omega_{W_0} d$, proving (125) follows. $\square$

Lemma 8. *Let T be generated by a production function G and $\beta \in (0,1)$. If $x, d \in \mathbb{R}^n$ and $\{h_k\}_{k\in\mathbb{N}} \subseteq (0,\infty)$ is such that $\lim_k h_k = 0$ and $\lim_k q_{h_k,x,d} = q$, then there exists $\hat{W} \in \Sigma\left(\Pi\left(x\right)\right)$*

$$q \geq \Omega_{J_{\mathcal{M}\left(\hat{W},\cdot\right)}\left(\Pi\left(x\right)\right)} d. \tag{137}$$

In particular, $q = \min_{\bar{W}\in\Sigma(\Pi(x))} \Omega_{J_{\mathcal{M}\left(\bar{W},\cdot\right)}\left(\Pi\left(x\right)\right)} d$.

Proof. Fix $x, d \in \mathbb{R}^n$. Since we fixed x and d, we denote $q_{h,x,d}$ by simply q_h for all $h > 0$. By Corollary 3, for each $h > 0$ and for each $i \in [n]$ there exist $\lambda_{i,h} \overset{\text{def}}{=} \lambda_{i,\Pi(x),\Pi(x+hd)} \in [0,1]$ and $\hat{w}_{i,h} \in \sigma_{T_i}\left(\lambda_{i,h}\Pi\left(x+hd\right) + \left(1-\lambda_{i,h}\right)\Pi\left(x\right)\right)$ such that

$$T_i\left(\Pi\left(x+hd\right)\right) - T_i\left(\Pi\left(x\right)\right) \tag{138}$$

$$\geq \left\langle \nabla M\left(\hat{w}_{i,h}, \lambda_{i,h}\Pi\left(x+hd\right) + \left(1-\lambda_{i,h}\right)\Pi\left(x\right)\right), \Pi\left(x+hd\right) - \Pi\left(x\right)\right\rangle. \tag{139}$$

Define $\hat{W}_h$ the matrix whose i-th row is $\hat{w}_{i,h}$ and define W_h the matrix whose i-th row is

$\nabla M\left(\hat{w}_{i,h}, \lambda_{i,h}\Pi\left(x+hd\right)+\left(1-\lambda_{i,h}\right)\Pi\left(x\right)\right)$. This implies that

$$\begin{aligned}\Pi\left(x+hd\right)-\Pi\left(x\right) &= \left(1-\beta\right)\left(x+hd\right)+\beta T\left(\Pi\left(x+hd\right)\right)-\left(1-\beta\right)x-\beta T\left(\Pi\left(x\right)\right) &(140)\\ &= \left(1-\beta\right)hd+\beta\left[T\left(\Pi\left(x+hd\right)\right)-T\left(\Pi\left(x\right)\right)\right] &(141)\\ &\geq \left(1-\beta\right)hd+\beta W_h\left(\Pi\left(x+hd\right)-\Pi\left(x\right)\right). &(142)\end{aligned}$$

By dividing by h, we obtain that

$$q_h \geq \left(1-\beta\right)d+\beta W_h q_h \qquad \forall h>0. \tag{143}$$

By Lemma 6 and since h was arbitrarily chosen and W_h is a stochastic matrix, we have that $q_h \geq \Omega_{W_h} d$ for all $h>0$. Consider now $\{h_k\}_{k\in\mathbb{N}} \subseteq (0,\infty)$ such that $\lim_k h_k = 0$ and $\lim_k q_{h_k} = q$. Since $\left\{\hat{W}_{h_k}\right\}_{k\in\mathbb{N}} \subseteq \mathcal{W}$, we have that there exists a subsequence $\{h_{k_l}\}_{l\in\mathbb{N}} \subseteq \{h_k\}_{k\in\mathbb{N}}$ such that $\lim_l \hat{W}_{h_{k_l}} = \hat{W}$ and, in particular, $\lim_l \hat{w}_{i,h_{k_l}} = \hat{w}_i$ for all $i\in[n]$. Since Π is Lipschitz continuous, we have that $\hat{w}_i \in \sigma_{T_i}\left(\Pi\left(x\right)\right)$ for all $i\in[n]$ and, in particular, $\hat{W} \in \Sigma\left(\Pi\left(x\right)\right)$.[18] Since Π is Lipschitz continuous and $(\omega,z)\mapsto \nabla M\left(\omega,z\right)$ is continuous on $\Delta\times\mathbb{R}^n$, we have that

$$\lim_l \nabla M\left(\hat{w}_{i,h_{k_l}}, \lambda_{i,h_{k_l}}\Pi\left(x+h_{k_l}d\right)+\left(1-\lambda_{i,h_{k_l}}\right)\Pi\left(x\right)\right) = \nabla M\left(\hat{w}_i,\Pi\left(x\right)\right) \quad \forall i\in[n].$$

This implies that $\lim_l W_{h_{k_l}} = J_{\mathcal{M}\left(\hat{W},\cdot\right)}\left(\Pi\left(x\right)\right)$ with $\hat{W}\in\Sigma\left(\Pi\left(x\right)\right)$. By Remark 1 and since $\lim_l W_{h_{k_l}} = J_{\mathcal{M}\left(\hat{W},\cdot\right)}\left(\Pi\left(x\right)\right)$, $\lim_l q_{h_{k_l}} = q$, and $q_{h_{k_l}} \geq W_{h_{k_l},\beta} d$ for all $l\in\mathbb{N}$, it follows that $q \geq J_{\mathcal{M}\left(\hat{W},\cdot\right)}\left(\Pi\left(x\right)\right)_\beta d$ with $\hat{W}\in\Sigma\left(\Pi\left(x\right)\right)$, proving the first part of the statement. By Lemma 7, we have that $J_{\mathcal{M}\left(\hat{W},\cdot\right)}\left(\Pi\left(x\right)\right)_\beta d \geq q$ and, in particular,

$$q = J_{\mathcal{M}\left(\hat{W},\cdot\right)}\left(\Pi\left(x\right)\right)_\beta d \leq J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi\left(x\right)\right)_\beta d \quad \forall \tilde{W}\in\Sigma\left(\Pi\left(x\right)\right),$$

proving that $q = \min_{\tilde{W}\in\Sigma(x)} J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi\left(x\right)\right)_\beta d$. □

Proof of Theorem 2. Fix $x,d\in\mathbb{R}^n$. Consider a sequence $\{h_k\}_{k\in\mathbb{N}} \subseteq (0,\infty)$ such that $\lim_k h_k = 0$. Since we fixed x and d, we denote $q_{h,x,d}$ simply by q_h for all $h>0$. As noted, $\{q_{h_k}\}_{k\in\mathbb{N}}$ is a bounded sequence. Consider a subsequence $\left\{q_{h_{k_l}}\right\}_{l\in\mathbb{N}} \subseteq \{q_{h_k}\}_{k\in\mathbb{N}}$. Since $\{q_{h_k}\}_{k\in\mathbb{N}}$ is a bounded sequence, $\left\{q_{h_{k_l}}\right\}_{l\in\mathbb{N}}$ has a further subsequence $\left\{q_{h_{k_{l(m)}}}\right\}_{m\in\mathbb{N}}$ that converges.

By Lemma 8, it converges to $\min_{\tilde{W}\in\Sigma(\Pi(x))} J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi\left(x\right)\right)_\beta d$. By (Carothers, 2000, p. 47,

[18] Consider $\{x_l\}_{l\in\mathbb{N}} \subseteq \mathbb{R}^n$ and $\{\alpha_l\}_{l\in\mathbb{N}} \subseteq \Delta$ such that $\alpha_l \in \sigma_{T_i}\left(x_l\right)$ for all $l\in\mathbb{N}$, $\lim_l \alpha_l = \alpha$, and $\lim_l x_l = x$. Since T_i and M are continuous and G_i is continuous, we have that

$$T_i\left(x\right) = \liminf_l T_i\left(x_l\right) = \liminf_l \left[M\left(\alpha_l,x_l\right)+\ln G_i\left(\alpha\right)\right] \geq M\left(\alpha,x\right)+\ln G_i\left(\alpha\right) \geq T_i\left(x\right),$$

proving that $T_i\left(x\right) = M\left(\alpha,x\right)+\ln G_i\left(\alpha\right)$ and, in particular, $\alpha\in\sigma_{T_i}\left(x\right)$.

Exercise 39) we can conclude that

$$\lim_{k} \frac{\Pi\left(x+h_k d\right)-\Pi\left(x\right)}{h_k} = \min_{\tilde{W}\in\Sigma(\Pi(x))} \Omega_{L(\tilde{W},\Pi(x))} d.$$

Since $\{h_k\}_{k\in\mathbb{N}}$ and d were arbitrarily chosen, the statement follows. $\square$

Corollary 4. *Fix $i \in N$ and let T_i be generated by a production function G_i. The following statements are equivalent:*

(i) T_i is Gateaux differentiable at $x \in \mathbb{R}^n$;

(ii) $\sigma_{T_i}(x)$ is a singleton.

Proof. Fix $x \in \mathbb{R}^n$. Define $T_{i,\circ}(x;\cdot) : \mathbb{R}^n \to \mathbb{R}$ by $T_{i,\circ}(x;d) = \lim_{h\downarrow 0} \frac{T_i(x+hd)-T_i(x)}{h}$ for all $d \in \mathbb{R}^n$. By Lemma 5, it follows that $T_{i,\circ}(x;\cdot) = \min_{w\in\sigma_{T_i}(x)} \langle \nabla M(w_i,x), d\rangle$.

(i) implies (ii). Since T_i is Gateaux differentiable at $x \in \mathbb{R}^n$, $T_{i,\circ}(x;\cdot)$ is linear and, in particular, $\{\nabla M(w_i,x) : w_i \in \sigma_{T_i}(x)\}$ is a singleton. Since if $w_i \neq w_i'$ then $\nabla M(w_i,x) \neq \nabla M(w_i',x)$, we can conclude that $\sigma_{T_i}(x)$ is a singleton.

(ii) implies (i). If $\sigma_{T_i}(x)$ is a singleton, that is, $\sigma_{T_i}(x) = \{w_i\}$, then $T_{i,\circ}(x;d) = \langle \nabla M(w_i,x), \cdot\rangle$, proving the statement. $\square$

Lemma 9. *If $i \in [n]$ and $P \in \mathbb{R}^n_{++}$, then Equation 15 holds. Moreover, we have that:*

(i) If $(\hat{Q}_{i0}, \hat{Q}_i) \in \mathbb{R}^{n+1}_+$ solves (15), then $\hat{Q}_{i0} = \frac{1-\beta}{Z_i} K_i(P)^\beta$ and there exists $\gamma > 0$ such that $\left(\gamma\hat{Q}_{i1}, ..., \gamma\hat{Q}_{in}\right)$ solves (16).

(ii) If $Q_i \in \mathbb{R}^n_+$ solves (16), then there exist $\gamma > 0$ and $\hat{Q}_{i0} > 0$ such that $\left(\hat{Q}_{i0}, \gamma Q_{i1}, ..., \gamma Q_{in}\right) \in \mathbb{R}^{n+1}_+$ solves (15).

(iii) If $(\hat{Q}_{i0}, \hat{Q}_i) \in \mathbb{R}^{n+1}_+$ solves (15) and $P_i = \min_{(Q_{i0},Q_i)\in\mathbb{R}^{n+1}_+ : F_i(Q_{i0},Q_i)\geq 1} \sum_{j=1}^n Q_{ij}P_j + Q_{i0}$, then

$$\beta = \frac{\sum_{j=1}^n \hat{Q}_{ij} P_j}{P_i}. \tag{144}$$

Proof. Fix $P \in \mathbb{R}^n_{++}$ and $i \in [n]$. As a preliminary step, we show that the minimum in Equation 16 is attained. Observe first that the minimized function is continuous, and the set over which the minimization is performed is nonempty and includes some $\hat{Q}_i \in \Delta$. The set over which the minimization is performed is closed by the continuity of G_i. Moreover, for any $P \in \mathbb{R}^n_{++}$ the same value is obtained if we restrict the set over which the minimization is performed to $\{Q_i : G_i(Q_i) \geq 1 \text{ and } P_j Q_{ij} \leq \sum_{k=1}^n P_k \hat{Q}_{ik}, \forall j \in [n]\}$ which is also compact, and the minimum is thus attained there.

A similar argument shows that the minimum in Equation 15 is attained. We can thus now prove the proper lemma statement, assuming that the minima are attained.

Since G_i is positive homogeneous, note that

$$\min_{(Q_{i0},Q_i)\in\mathbb{R}_+^{n+1}:F_i(Q_{i0},Q_i)\geq 1}\sum_{j=1}^{n}Q_{ij}P_j+Q_{i0}=\min_{Q_{i0}\in\mathbb{R}_+,Q_i\in\mathbb{R}_+^n:Z_i\left(\frac{Q_{i0}}{1-\beta}\right)^{1-\beta}\left(\frac{G_i(Q_i)}{\beta}\right)^{\beta}\geq 1}\sum_{j=1}^{n}Q_{ij}P_j+Q_{i0} \tag{145}$$

$$=\min_{Q_{i0}\in\mathbb{R}_{++}}\min_{Q_i\in\mathbb{R}_+^n:Q_{i0}^{1-\beta}G_i(Q_i)^{\beta}\geq\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}}\sum_{j=1}^{n}Q_{ij}P_j+Q_{i0} \tag{146}$$

$$=\min_{Q_{i0}\in\mathbb{R}_{++}}\left\{Q_{i0}+\min_{Q_i\in\mathbb{R}_+^n:G_i(Q_i)\geq\left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{Q_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}}\sum_{j=1}^{n}Q_{ij}P_j\right\} \tag{147}$$

$$=\min_{Q_{i0}\in\mathbb{R}_{++}}\left\{Q_{i0}+\left(\frac{\beta^{\beta}\left(1-\beta\right)^{1-\beta}}{Z_i}\left(\frac{1}{Q_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}\min_{Q_i\in\mathbb{R}_+^n:G_i(Q_i)\geq 1}\sum_{j=1}^{n}Q_{ij}P_j\right\} \tag{148}$$

$$=\min_{Q_{i0}\in\mathbb{R}_{++}}\left\{Q_{i0}+\left(\frac{\beta^{\beta}\left(1-\beta\right)^{1-\beta}}{Z_i}\left(\frac{1}{Q_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}K_i\left(P\right)\right\} \tag{149}$$

$$=\min_{x\in\mathbb{R}_{++}}\left\{x+\left(A\left(\frac{1}{x}\right)^{1-\beta}\right)^{\frac{1}{\beta}}B.\right\} \tag{150}$$

where $x=Q_{i0}$, $A=\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}>0$, and $B=K_i\left(P\right)$. Define $f:\mathbb{R}_{++}\to\mathbb{R}_{++}$ by $f\left(x\right)=x+A^{\frac{1}{\beta}}x^{-\frac{1-\beta}{\beta}}B$ for all $x>0$. Since $P\in\mathbb{R}_{++}^n$ and G_i is positive homogeneous, we have that $B>0$. It follows that

$$f'\left(x\right)=1-\frac{1-\beta}{\beta}A^{\frac{1}{\beta}}Bx^{-\frac{1-\beta}{\beta}-1}=1-\frac{1-\beta}{\beta}A^{\frac{1}{\beta}}B\left(\frac{1}{x}\right)^{\frac{1}{\beta}}\quad\forall x>0. \tag{151}$$

It follows that f' is increasing and takes value 0 at $x^{\star}=\left(\frac{1-\beta}{\beta}\right)^{\beta}AB^{\beta}>0$ and, in particular, f is convex and minimized at $\left(\frac{1-\beta}{\beta}\right)^{\beta}AB^{\beta}$. Since $A=\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}$ and $B=K_i\left(P\right)$, we have that

$x^{\star} = \left(\frac{1-\beta}{\beta}\right)^{\beta} AB^{\beta} = \frac{1-\beta}{Z_i} K_i(P)^{\beta}$. Plugging $x^{\star}$ in f, we have that

$$f(x^{\star}) = \frac{1-\beta}{Z_i} K_i(P)^{\beta} + A^{\frac{1}{\beta}} \left(\frac{1-\beta}{Z_i} K_i(P)^{\beta}\right)^{-\frac{1-\beta}{\beta}} K_i(P) \tag{152}$$

$$= \frac{1-\beta}{Z_i} K_i(P)^{\beta} + \frac{\beta(1-\beta)^{\frac{1-\beta}{\beta}}}{Z_i^{\frac{1}{\beta}}} \frac{(1-\beta)^{-\frac{1-\beta}{\beta}}}{Z_i^{-\frac{1-\beta}{\beta}}} K_i(P)^{-(1-\beta)} K_i(P) \tag{153}$$

$$= \frac{1-\beta}{Z_i} K_i(P)^{\beta} + \frac{\beta}{Z_i} K_i(P)^{\beta} = \frac{1}{Z_i} K_i(P)^{\beta}, \tag{154}$$

proving the first part of the statement.

(i). Consider $(\hat{Q}_{i0}, \hat{Q}_i) \in \mathbb{R}_+^{n+1}$ that solves (15). Since $F_i\left(\hat{Q}_{i0}, \hat{Q}_i\right) \geq 1$, we have that $\hat{Q}_{i0} > 0$. It follows that $G_i\left(\hat{Q}_{i1}, ..., \hat{Q}_{in}\right) \geq \left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{\hat{Q}_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}$. Define $\gamma = \left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{\hat{Q}_{i0}}\right)^{1-\beta}\right)^{-\frac{1}{\beta}} > 0$. Since G_i is positive homogeneous, note that $G_i\left(\gamma\hat{Q}_{i1}, ..., \gamma\hat{Q}_{in}\right) \geq 1$. By contradiction, assume that $\left(\gamma\hat{Q}_{i1}, ..., \gamma\hat{Q}_{in}\right)$ does not solve (16). It follows that there exists $\bar{Q}_i \in \mathbb{R}_+^n$ such that $G_i\left(\bar{Q}_{i1}, ..., \bar{Q}_{in}\right) \geq 1$ and $\sum_{j=1}^{n} \bar{Q}_{ij} P_j < \sum_{j=1}^{n} \gamma\hat{Q}_{ij} P_j$. Consider $\left(\hat{Q}_{i0}, \gamma^{-1}\bar{Q}_{i1}, ..., \gamma^{-1}\bar{Q}_{in}\right) \in \mathbb{R}_+^{n+1}$. By the initial part of the proof and since G_i is positive homogeneous, we have that

$$\min_{(Q_{i0}, Q_i) \in \mathbb{R}_+^{n+1} : F_i(Q_{i0}, Q_i) \geq 1} \sum_{j=1}^{n} Q_{ij} P_j + Q_{i0} = \hat{Q}_{i0} + \sum_{j=1}^{n} \hat{Q}_{ij} P_j > \hat{Q}_{i0} + \sum_{j=1}^{n} \gamma^{-1} \bar{Q}_{ij} P_j \tag{155}$$

$$\geq \hat{Q}_{i0} + \min_{Q_i \in \mathbb{R}_+^n : G_i(Q_i) \geq \left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{\hat{Q}_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}} \sum_{j=1}^{n} Q_{ij} P_j \tag{156}$$

$$\geq \min_{Q_{i0} \in \mathbb{R}_{++}} \left\{ Q_{i0} + \min_{Q_i \in \mathbb{R}_+^n : G_i(Q_i) \geq \left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{Q_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}} \sum_{j=1}^{n} Q_{ij} P_j \right\} \tag{157}$$

$$= \min_{(Q_{i0}, Q_i) \in \mathbb{R}_+^{n+1} : F_i(Q_{i0}, Q_i) \geq 1} \sum_{j=1}^{n} Q_{ij} P_j + Q_{i0}, \tag{158}$$

a contradiction, proving the second part of the point. As for the first part, by the proof of the main statement and since $(\hat{Q}_{i0}, \hat{Q}_i) \in \mathbb{R}_+^{n+1}$ solves (15) and $\left(\gamma\hat{Q}_{i1}, ..., \gamma\hat{Q}_{in}\right)$ solves (16), we can conclude that

$$\min_{Q_{i0}\in\mathbb{R}_{++}}\left\{Q_{i0}+\left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{Q_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}K_i(P)\right\} \tag{159}$$

$$=\hat{Q}_{i0}+\sum_{j=1}^{n}\hat{Q}_{ij}P_j=\hat{Q}_{i0}+\gamma^{-1}\sum_{j=1}^{n}\gamma\hat{Q}_{ij}P_j \tag{160}$$

$$=\hat{Q}_{i0}+\left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{\hat{Q}_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}K_i(P) \tag{161}$$

proving that $\hat{Q}_{i0}$ minimizes the left-hand side of the expression. By the proof of the main statement, it follows that $\hat{Q}_{i0}=\frac{1-\beta}{Z_i}K_i(P)^{\beta}$.

(ii). Define $\hat{Q}_{i0}>0$ to be the unique point that minimizes $Q_{i0}+\left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{Q_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}K_i(P)$. By the proof of the main statement, it follows that $\hat{Q}_{i0}=\frac{1-\beta}{Z_i}K_i(P)^{\beta}$. Define $\gamma=\left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{\hat{Q}_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}$. Define $(\hat{Q}_{i0},\hat{Q}_i)=\left(\hat{Q}_{i0},\gamma Q_{i1},...,\gamma Q_{in}\right)\in\mathbb{R}_{+}^{n+1}$. By the proof of the main statement and since $Q_i\in\mathbb{R}_{+}^{n}$ solves (16), we have that $K_i(P)=\sum_{j=1}^{n}Q_{ij}P_j$ and

$$\min_{(Q_{i0},Q_i)\in\mathbb{R}_{+}^{n+1}:F_i(Q_{i0},Q_i)\geq 1}\sum_{j=1}^{n}Q_{ij}P_j+Q_{i0}=\min_{Q_{i0}\in\mathbb{R}_{++}}\left\{Q_{i0}+\left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{Q_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}K_i(P)\right\} \tag{162}$$

$$=\hat{Q}_{i0}+\left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{\hat{Q}_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}K_i(P) \tag{163}$$

$$=\hat{Q}_{i0}+\gamma\sum_{j=1}^{n}Q_{ij}P_j=\hat{Q}_{i0}+\sum_{j=1}^{n}\gamma Q_{ij}P_j \tag{164}$$

$$=\hat{Q}_{i0}+\sum_{j=1}^{n}\hat{Q}_{ij}P_j. \tag{165}$$

Since G_i is positive homogeneous and $G_i(Q_i)\geq 1$, we have that $G_i(\gamma Q_i)\geq\left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{\hat{Q}_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}$, yielding that $F_i\left(\hat{Q}_{i0},\hat{Q}_i\right)\geq 1$ and proving the point.

(iii). By the proof of the main statement and point (i) and since $(\hat{Q}_{i0},\hat{Q}_i)\in\mathbb{R}_{+}^{n+1}$ solves

(15) and $P_i = \min_{(Q_{i0},Q_i)\in\mathbb{R}^{n+1}_+ : F_i(Q_{i0},Q_i)\geq 1} \sum_{j=1}^n Q_{ij}P_j + Q_{i0}$, we have that

$$\frac{\sum_{j=1}^n \hat{Q}_{ij}P_j}{P_i} = \frac{\sum_{j=1}^n \hat{Q}_{ij}P_j + \hat{Q}_{i0} - \hat{Q}_{i0}}{P_i} = \frac{\sum_{j=1}^n \hat{Q}_{ij}P_j + \hat{Q}_{i0}}{P_i} - \frac{\hat{Q}_{i0}}{P_i} \tag{166}$$

$$= \frac{\min_{(Q_{i0},Q_i)\in\mathbb{R}^{n+1}_+ : F_i(Q_{i0},Q_i)\geq 1} \sum_{j=1}^n Q_{ij}P_j + Q_{i0}}{P_i} - \frac{\hat{Q}_{i0}}{P_i} \tag{167}$$

$$= 1 - \frac{\hat{Q}_{i0}}{P_i} = 1 - \frac{\frac{1-\beta}{Z_i} K_i\left(P\right)^\beta}{\frac{1}{Z_i} K_i\left(P\right)^\beta} = \beta, \tag{168}$$

proving the point. $\square$

Lemma 10. *Let $i \in [n]$. If K_i is generated by a production function G_i, we have that:*

(i) If $Q_i \in \mathbb{R}^n_+$ solves (16), then $\frac{1}{\sum_{j=1}^n Q_{ij}} Q_i \in \Delta$ solves the right-hand side problem of (31).

(ii) If $\tilde{w}_i \in \Delta$ solves the right-hand side problem of (31), then there exists $\gamma > 0$ such that $\gamma \tilde{w}_i \in \mathbb{R}^n_+$ solves (16).

Proof. (i). Assume $Q_i \in \mathbb{R}^n_+$ solves (16). Define $w_i = \frac{1}{\sum_{j=1}^n Q_{ij}} Q_i \in \Delta$. Since G_i is positive homogeneous and $G_i\left(Q_i\right) \geq 1$, observe that

$$K_i\left(P\right) = \sum_{j=1}^n Q_{ij}P_j \geq \frac{\sum_{j=1}^n Q_{ij}P_j}{G_i(Q_i)} = \frac{\sum_{j=1}^n w_{ij}P_j}{G_i(w_i)} \geq K_i\left(P\right), \tag{169}$$

proving that w_i solves the right-hand side problem of (31).

(ii). Assume that w_i solves the right-hand side problem of (31). It follows that $G_i\left(w_i\right) > 0$. Consider $\gamma = G_i\left(w_i\right)^{-1}$ and define $Q_i = \gamma w_i$. Since G_i is positive homogeneous, we have that $G_i\left(Q_i\right) = 1$ and

$$K_i\left(P\right) = \min_{\tilde{w}_i\in\Delta} \frac{\sum_{j=1}^n \tilde{w}_{ij}P_j}{G_i\left(\tilde{w}_i\right)} = \frac{\sum_{j=1}^n w_{ij}P_j}{G_i\left(w_i\right)} = \frac{\sum_{j=1}^n \gamma w_{ij}P_j}{G_i\left(\gamma w_i\right)} = \frac{\sum_{j=1}^n Q_{ij}P_j}{G_i\left(Q_i\right)} = \sum_{j=1}^n Q_{ij}P_j \geq K_i\left(P\right), \tag{170}$$

proving that $Q_i = \gamma w_i$ solves (16). $\square$

Corollary 5. *Let $i \in [n]$ and $P \in \mathbb{R}^n_{++}$. The following statements are true:*

(i) If $\hat{Q}_i \in \mathbb{R}^{n+1}_+$ solves (15), then $\frac{1}{\sum_{j=1}^n \hat{Q}_{ij}} \left(\hat{Q}_{i1}, ..., \hat{Q}_{in}\right)$ solves the right-hand side problem of (31).

(ii) If $w_i \in \Delta$ solves the right-hand side problem of (31), then there exist $\gamma > 0$ and $\hat{Q}_{i0} > 0$ such that $\hat{Q}_i = \left(\hat{Q}_{i0}, \gamma w_{i1}, ..., \gamma w_{in}\right) \in \mathbb{R}^{n+1}_+$ solves (15).

Proof. (i). Fix $i \in [n]$ and $P \in \mathbb{R}^n_{++}$. By Lemma 9, we have that there exists $\gamma > 0$ such that $\left(\gamma\hat{Q}_{i1}, ..., \gamma\hat{Q}_{in}\right)$ solves (16). By Lemma 10, it follows that $\frac{1}{\sum_{j=1}^n \gamma\hat{Q}_{ij}}\left(\gamma\hat{Q}_{i1}, ..., \gamma\hat{Q}_{in}\right) = \frac{1}{\sum_{j=1}^n \hat{Q}_{ij}}\left(\hat{Q}_{i1}, ..., \hat{Q}_{in}\right)$ solves the right-hand side problem of (31).

(ii). By Lemma 10, if $w_i \in \Delta$ solves the right-hand side problem of (31), then there exists $\gamma_1 > 0$ such that $\gamma_1\tilde{w}_i$ solves (16). By Lemma 9, there exist $\gamma_2 > 0$ and $\hat{Q}_{i0} > 0$ such that $\hat{Q}_i = \left(\hat{Q}_{i0}, \gamma_2\gamma_1 w_{i1}, ..., \gamma_2\gamma_1 w_{in}\right) \in \mathbb{R}^{n+1}_+$ solves (15). $\square$

We denote the set of all competitive equilibria by $\mathfrak{E}$. Given $\mathcal{E} \in \mathfrak{E}$, that is a tuple $\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right)$, we define $E_{\mathcal{E}}$ as the $n \times n$ matrices such that its ij-th entries is $\frac{\hat{Q}_{ij}\bar{P}_j}{F_i\left(\hat{Q}_i\right)\bar{P}_i}$We also define $\tilde{W}_{\mathcal{E}}$ in as the matrix whose ij-th entry is $\frac{\hat{Q}_{ij}}{\sum_{k=1}^n \hat{Q}_{ik}}$.

Proposition 10. *We have that:*

(i) $\mathfrak{E}$ *is nonempty.*

(ii) $\left\{\tilde{W}_{\mathcal{E}}\right\}_{\mathcal{E}\in\mathfrak{E}} = \Sigma\left(\Pi\left(\bar{x}\right)\right)$ *and* $\left\{L\left(\tilde{W}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)}, \ln\left(\bar{P}\right)\right)\right\}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)\in\mathfrak{E}} = \left\{J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi\left(\bar{x}\right)\right) : \tilde{W} \in \Sigma\left(\Pi\left(\bar{x}\right)\right)\right\}$.

(iii) If $\mathcal{E} \in \mathfrak{E}$, *then* $\lambda^T_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)} = \xi^T\left(I - \beta L\left(\tilde{W}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)}, \ln\left(\bar{P}\right)\right)\right)^{-1}$.

(iv) If $\left(\tilde{P}, \tilde{Q}, \tilde{Y}, \tilde{C}\right), \left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right) \in \mathfrak{E}$, *then* $\tilde{P} = \bar{P}$.

Proof. (i). Consider the (unique) equilibrium price $\bar{P}$ and a matrix $\hat{Q}$ such that $(\hat{Q}_{i0}, \hat{Q}_i)$ solves Equation 15 for all $i \in [n]$ with $P = \bar{P}$. Compute $\bar{C} \in \operatorname{argmax}_{C\in\mathbb{R}^n_+:\sum_{i=1}^n \bar{P}_i C_i \leq 1} U\left(C\right)$. Next, consider the equation $\lambda^{\mathrm{T}} = \bar{D}^{\mathrm{T}} + \lambda^{\mathrm{T}} E_{\bar{P},\hat{Q}}$ with unknown $\lambda \in \mathbb{R}^n$ where $\bar{D}$ is the vector whose j-th entry is $\bar{C}_j\bar{P}_j$. By point (iii) of Lemma 9 and since $\bar{P}$ is an equilibrium price, we have that $\beta = \sum_{j=1}^n \hat{Q}_{ij}\bar{P}_j/\bar{P}_i$ for all $i \in [n]$. Since $F_i\left(\hat{Q}_i\right) = 1$ for all $i \in [n]$, this implies that

$$\frac{\hat{Q}_{ij}\bar{P}_j}{\bar{P}_i} = \frac{\sum_{k=1}^n \hat{Q}_{ik}\bar{P}_k}{\bar{P}_i}\frac{\hat{Q}_{ij}\bar{P}_j}{\sum_{k=1}^n \hat{Q}_{ik}\bar{P}_k} = \beta\frac{\hat{Q}_{ij}\bar{P}_j}{\sum_{k=1}^n \hat{Q}_{ik}\bar{P}_k} \quad \forall i, j \in [n]\,, \tag{171}$$

that is $E_{\bar{P},\hat{Q}} = \beta L\left(\tilde{W}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)}, \ln\left(\bar{P}\right)\right)$, yielding that $\lambda^{\mathrm{T}} = \bar{D}^{\mathrm{T}} + \lambda^{\mathrm{T}}\beta L\left(\tilde{W}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)}, \ln\left(\bar{P}\right)\right)$. Since $L\left(\tilde{W}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)}, \ln\left(\bar{P}\right)\right)$ is a stochastic matrix, $\bar{D} \geq 0$, and $\beta \in (0, 1)$, it follows that this equation has a (unique) solution $\lambda^{\mathrm{T}} = \bar{D}^{\mathrm{T}}\left(I - \beta L\left(\tilde{W}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)}, \ln\left(\bar{P}\right)\right)\right)^{-1} \geq 0$. Since $\bar{P} \in \mathbb{R}^n_{++}$ and $\lambda \geq 0$, define $\bar{Y} \in \mathbb{R}^n_+$ by $\bar{Y}_i = \lambda_i\backslash\bar{P}_i$ for all $i \in [n]$. Since F_i is positive homogeneous and $F_i\left(\hat{Q}_i\right) = 1$ for all $i \in [n]$ and $\lambda^{\mathrm{T}} = \bar{D}^{\mathrm{T}} + \lambda^{\mathrm{T}} E_{\bar{P},\hat{Q}}$, it follows that for each $j \in [n]$

$$F_j\left(\bar{Y}_j\hat{Q}_{j0}, \bar{Y}_j\hat{Q}_j\right)\bar{P}_j = F_j\left(\hat{Q}_j\right)\left(\bar{P}_j\right) = \lambda_j = \bar{D}_j + \sum_{i=1}^n \lambda_i\frac{\hat{Q}_{ij}\bar{P}_j}{\bar{P}_i} = \bar{C}_j\bar{P}_j + \sum_{i=1}^n \bar{Y}_i\hat{Q}_{ij}\bar{P}_j \quad \forall j \in [n]\,. \tag{172}$$

By dividing by $\bar{P}_j$ and since $\bar{P}_j > 0$ for all $j \in [n]$, it follows that the market clearing conditions of the n sectors are satisfied. Since $\bar{C}_j \bar{P}_j + \sum_{i=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j = F_j\left(\bar{Y}_j \hat{Q}_{j0}, \bar{Y}_j \hat{Q}_j\right) \bar{P}_j$ for all $j \in [n]$, we obtain that

$$\sum_{i=1}^n \sum_{j=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j + \sum_{j=1}^n \bar{C}_j \bar{P}_j = \sum_{j=1}^n \sum_{i=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j + \sum_{j=1}^n \bar{C}_j \bar{P}_j = \sum_{j=1}^n F_j\left(\bar{Y}_j \hat{Q}_{j0}, \bar{Y}_j \hat{Q}_j\right) \bar{P}_j. \tag{173}$$

Since the preferences of the representative agent are monotone and $\bar{P}_i = \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j + \hat{Q}_{i0}$ for all $i \in [n]$, we have that

$$1 = \sum_{j=1}^n \bar{C}_j \bar{P}_j = \sum_{i=1}^n \left(F_i\left(\bar{Y}_i \hat{Q}_{i0}, \bar{Y}_i \hat{Q}_i\right) \bar{P}_i - \sum_{j=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j \right) = \sum_{i=1}^n \left(\bar{Y}_i \left[F_i\left(\hat{Q}_{i0}, \hat{Q}_i\right) \bar{P}_i - \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j \right] \right) \tag{174}$$

$$= \sum_{i=1}^n \left(\bar{Y}_i \left[\bar{P}_i - \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j \right] \right) = \sum_{i=1}^n \bar{Y}_i \hat{Q}_{i0} \quad \forall i \in [n]\,. \tag{175}$$

This proves that $\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right)$ satisfies condition 3 of a competitive equilibrium (cf. Definition 1). By construction of $\bar{P}$ and $\bar{C}$, conditions 1 and 4 of a competitive equilibrium are also satisfied. Finally, since $(\hat{Q}_{i0}, \hat{Q}_i)$ solves (15) for all $i \in [n]$ with $P = \bar{P}$, we have that condition 2 of a competitive equilibrium is satisfied, proving that $\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right) \in \mathfrak{E}$ and the latter set is nonempty.

(ii). Fix an equilibrium $\mathcal{E} = \left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right)$. It follows that $\bar{P} = \exp \Pi(\bar{x})$. By point (i) of Corollary 5 and since $(\hat{Q}_{i0}, \hat{Q}_i)$ solves (15) for all $i \in [n]$, then $\frac{1}{\sum_{k=1}^n \hat{Q}_{ik}} \left(\hat{Q}_{i1}, ..., \hat{Q}_{in}\right)$ solves the right-hand side problem of (31) with $\bar{p} = \Pi(\bar{x})$ for all $i \in [n]$, proving that $\tilde{W}_{\mathcal{E}} \in \Sigma(\Pi(\bar{x}))$. Define $W = J_{\mathcal{M}(\tilde{W}_{\mathcal{E}}, \cdot)}(\Pi(\bar{x}))$. It follows that

$$w_{ij} = \frac{\tilde{w}_{ij} e^{\Pi_j(\bar{x})}}{\sum_{k=1}^n \tilde{w}_{ik} e^{\Pi_k(\bar{x})}} = \frac{\tilde{w}_{ij} \bar{P}_j}{\sum_{k=1}^n \tilde{w}_{ik} \bar{P}_k} = \frac{\frac{\hat{Q}_{ij}}{\sum_{k=1}^n \hat{Q}_{ik}} \bar{P}_j}{\sum_{k=1}^n \frac{\hat{Q}_{ik}}{\sum_{k=1}^n \hat{Q}_{ik}} \bar{P}_k} = \frac{\hat{Q}_{ij} P_j}{\sum_{k=1}^n \hat{Q}_{ik} P_k}, \tag{176}$$

proving that $W = L\left(\tilde{W}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}, \ln(\bar{P})\right)$. We can conclude that $\left\{L\left(\tilde{W}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C})}, \ln(\bar{P})\right)\right\}_{(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}) \in \mathfrak{E}} \subseteq \left\{J_{\mathcal{M}(\tilde{W}, \cdot)}(\Pi(\bar{x})) : \tilde{W} \in \Sigma(\Pi(\bar{x}))\right\}$.

Consider $\tilde{W} \in \Sigma(\Pi(\bar{x}))$. By point (ii) of Corollary 5, it follows that there exist (unique) $\gamma > 0$ and (unique) $\hat{Q}_{i0} > 0$ such that $\left(\hat{Q}_{i0}, \gamma \tilde{w}_{i1}, ..., \gamma \tilde{w}_{in}\right) \in \mathbb{R}_+^{n+1}$ solves (15) for all $i \in [n]$ with $P = \bar{P} = \exp \Pi(\bar{x})$. By part 1 of this proof, there exist (unique) $\bar{Y} \in \mathbb{R}_+^n$ and $\bar{C} \in \mathbb{R}_+^n$

such that $\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right) \in \mathfrak{E}$. Finally, since $\tilde{W}$ is a stochastic matrix, observe that

$$\frac{\hat{Q}_{ij}}{\sum_{k=1}^{n} \hat{Q}_{ik}} = \frac{\gamma \tilde{w}_{ij}}{\sum_{k=1}^{n} \gamma \tilde{w}_{ik}} = \tilde{w}_{ij} \quad \forall i, j \in [n], \tag{177}$$

proving that $\tilde{W} \in \left\{\tilde{W}_{\mathcal{E}}\right\}_{\mathcal{E}\in\mathfrak{E}}$ and, in particular, $\left\{\tilde{W}_{\mathcal{E}}\right\}_{\mathcal{E}\in\mathfrak{E}} \supseteq \Sigma\left(\Pi\left(\bar{x}\right)\right)$. Finally, consider $W \in \left\{J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi\left(\bar{x}\right)\right) : \tilde{W} \in \Sigma\left(\Pi\left(\bar{x}\right)\right)\right\}$, that is, $W = J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi\left(\bar{x}\right)\right)$ for some $\tilde{W} \in \Sigma\left(\Pi\left(\bar{x}\right)\right)$. By (176) and since $\left\{\tilde{W}_{\mathcal{E}}\right\}_{\mathcal{E}\in\mathfrak{E}} = \Sigma\left(\Pi\left(\bar{x}\right)\right)$, we have that $\tilde{W} = \tilde{W}_{\mathcal{E}}$ for some $\mathcal{E} = \left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right) \in \mathfrak{E}$ and $W = \tilde{E}_{L\left(\tilde{W}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)}, \ln\left(\bar{P}\right)\right)}$, proving that

$$\left\{L\left(\tilde{W}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)}, \ln\left(\bar{P}\right)\right)\right\}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)\in\mathfrak{E}} \supseteq \left\{J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi\left(\bar{x}\right)\right) : \tilde{W} \in \Sigma\left(\Pi\left(\bar{x}\right)\right)\right\}. \tag{178}$$

(iii). Fix an equilibrium $\mathcal{E} = \left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right)$. It follows that

$$\bar{Y}_j \bar{P}_j = \bar{C}_j \bar{P}_j + \sum_{i=1}^{n} \bar{Y}_i \hat{Q}_{ij} \bar{P}_j = \bar{D}_j + \sum_{i=1}^{n} \left(\bar{Y}_i \bar{P}_i\right) \frac{\hat{Q}_{ij} \bar{P}_j}{\bar{P}_i} \quad \forall j \in [n], \tag{179}$$

that is, $\lambda_{\mathcal{E}}^{\mathrm{T}} = \bar{D}^{\mathrm{T}} + \lambda_{\mathcal{E}}^{\mathrm{T}} E_{\mathcal{E}}$. By the same arguments leading to (171), we can conclude that $\lambda_{\mathcal{E}}^{\mathrm{T}} = \bar{D}^{\mathrm{T}} + \lambda_{\mathcal{E}}^{\mathrm{T}} \beta L\left(\tilde{W}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)}, \ln\left(\bar{P}\right)\right)$, proving that $\lambda_{\mathcal{E}}^{\mathrm{T}} = \bar{D}^{\mathrm{T}}\left(I - \beta L\left(\tilde{W}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)}, \ln\left(\bar{P}\right)\right)\right)^{-1}$. Since $\bar{D}^{\mathrm{T}} = \xi^{\mathrm{T}}$, the statement follows.

(iv). It is obvious. $\square$

Corollary 6. *Let T be generated by a production function G. If $x \in \mathbb{R}^n$ and T is Gateaux differentiable at $\Pi\left(x\right)$, then $\Sigma\left(\Pi\left(x\right)\right) = \left\{\tilde{W}\right\}$ and*

$$\Pi_0(x; d) = \Omega_{J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi(x)\right)} d = \Omega_{J_T\left(\Pi(x)\right)} d \quad \forall d \in \mathbb{R}^n. \tag{180}$$

Proof. Since T is Gateaux differentiable at $\Pi\left(x\right)$, so is T_i for all $i \in [n]$. By Corollary 4, we have that $\sigma_{T_i}\left(\Pi\left(x\right)\right)$ is a singleton for all $i \in [n]$ and, in particular, $\Sigma\left(\Pi\left(x\right)\right)$ is a singleton. This implies that $J_T\left(\Pi\left(x\right)\right) = J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi\left(x\right)\right)$. By Theorem 2, the statement follows. $\square$

Remark 2. *Fix $x \in \mathbb{R}^n$. Consider two stochastic matrices W and $\hat{W}$ and assume that $\Omega_{J_{\mathcal{M}(W,\cdot)}(x)} = \Omega_{J_{\mathcal{M}(\hat{W},\cdot)}(x)}$. By the last part of Remark 1, it follows that $J_{\mathcal{M}(W,\cdot)}\left(x\right) = J_{\mathcal{M}\left(\hat{W},\cdot\right)}\left(x\right)$. By definition of $\mathcal{M}$, it follows that $\nabla M\left(w_i, x\right) = \nabla M\left(\hat{w}_i, x\right)$ for all $i \in [n]$, that is,*

$$\frac{e^{x_h} w_{ih}}{\sum_{j=1}^{n} e^{x_j} w_{ij}} = \frac{e^{x_h} \hat{w}_{ih}}{\sum_{j=1}^{n} e^{x_j} \hat{w}_{ij}} \qquad \forall i, h \in [n].$$

For each $i \in [n]$ set $\lambda_i = \sum_{j=1}^{n} e^{x_j} w_{ij} / \sum_{j=1}^{n} e^{x_j} \hat{w}_{ij}$. We obtain that $w_{ih} = \lambda_i \hat{w}_{ih}$ for all $i, h \in [n]$ and, in particular, $1 = \sum_{h=1}^{n} w_{ih} = \lambda_i \sum_{h=1}^{n} \hat{w}_{ih} = \lambda_i$ for all $i \in [n]$. We can conclude that $w_{ih} = \hat{w}_{ih}$ for all $i, h \in [n]$, that is, $W = \hat{W}$. ▲

Proof of Proposition 6. (i) is equivalent to (iii). By Corollary 4, the statement follows.

(iv) implies (iii). By point (ii) of Proposition 10, if $\mathfrak{E}$ is a singleton, then $\Sigma(\Pi(\bar{x})) = \left\{\tilde{W}_{\mathcal{E}}\right\}_{\mathcal{E}\in\mathfrak{E}}$ is a singleton.

(iii) implies (iv). By point (ii) of Proposition 10, if $\Sigma(\Pi(\bar{x}))$ is a singleton, so is $\left\{\tilde{W}_{\mathcal{E}}\right\}_{\mathcal{E}\in\mathfrak{E}}$. Assume there exist two equilibria $\left(\tilde{P}, \tilde{Q}, \tilde{Y}, \tilde{C}\right), \left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right) \in \mathfrak{E}$ that generate the unique $\tilde{W}_{\mathcal{E}}$. It follows that

$$\tilde{w}_{ij} = \frac{\hat{Q}_{ij}}{\sum_{k=1}^{n} \hat{Q}_{ik}} = \frac{\tilde{Q}_{ij}}{\sum_{k=1}^{n} \tilde{Q}_{ik}} \qquad \forall i, j \in [n]. \tag{181}$$

For each $i \in [n]$ define $\mu_i = \sum_{k=1}^{n} \tilde{Q}_{ik} / \sum_{k=1}^{n} \hat{Q}_{ik}$. By Equation 181, we have that $\left(\tilde{Q}_{i1}, ..., \tilde{Q}_{in}\right) = \mu_i \left(\hat{Q}_{i1}, ..., \hat{Q}_{in}\right)$ for all $i \in [n]$. By point (iv) of Proposition 10, we have that $\bar{P} = \tilde{P}$. By point (i) of Lemma 9 and and since $(\hat{Q}_{i0}, \hat{Q}_i), (\tilde{Q}_{i0}, \tilde{Q}_i) \in \mathbb{R}_{+}^{n+1}$ solve (15) for all $i \in [n]$ with $P = \bar{P} = \tilde{P}$, then $\tilde{Q}_{i0} = \hat{Q}_{i0} = \frac{1-\beta}{Z_i} K_i (P)^{\beta}$ for all $i \in [n]$ and

$$\hat{Q}_{i0} + \sum_{j=1}^{n} \hat{Q}_{ij} \bar{P}_j = \tilde{Q}_{i0} + \sum_{j=1}^{n} \tilde{Q}_{ij} \bar{P}_j \implies \sum_{j=1}^{n} \hat{Q}_{ij} \bar{P}_j = \sum_{j=1}^{n} \tilde{Q}_{ij} \bar{P}_j = \mu_i \sum_{j=1}^{n} \hat{Q}_{ij} \bar{P}_j \tag{182}$$

$$\implies \mu_i = 1 \qquad \forall i \in [n]. \tag{183}$$

This implies that $\hat{Q}_{ij} = \tilde{Q}_{ij}$ for all $i, j \in [n]$, proving that $\hat{Q} = \tilde{Q}$. By the proof of existence of a competitive equilibrium (cf. Proposition 10), we have that $\bar{Y} = \tilde{Y}$ is uniquely pinned down by $\bar{P} = \tilde{P}$ and $\hat{Q} = \tilde{Q}$. Given the preferences of the agent and $\bar{P} = \tilde{P}$, we have that $\bar{C} = \tilde{C}$, proving $\left(\tilde{P}, \tilde{Q}, \tilde{Y}, \tilde{C}\right) = \left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right)$ and uniqueness of the equilibrium.

(i) implies (ii). By Corollary 6, the implication follows.

(ii) implies (i). By Theorem 2 and since Π is Gateaux differentiable at $\bar{x}$, we have that there exists $W \in \mathcal{W}$ such that

$$\Omega_{J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(\bar{x}))} d \geq \Pi_0(\bar{x}, d) = W d \quad \forall d \in \mathbb{R}^n, \forall \tilde{W} \in \Sigma(\Pi(\bar{x})). \tag{184}$$

It follows that $\Omega_{J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(\bar{x}))} = W$ for all $\tilde{W} \in \Sigma(\Pi(\bar{x}))$. By Remark 2, we can conclude that $J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(\bar{x})) = J_{\mathcal{M}(\hat{W},\cdot)}(\Pi(\bar{x}))$, that is $\tilde{W} = \hat{W}$, for all $\tilde{W}, \hat{W} \in \Sigma(\Pi(\bar{x}))$, proving that $\Sigma(\Pi(\bar{x}))$ is a singleton and that T is Gateaux differentiable at $\Pi(\bar{x})$. □

Define $\mathcal{W}_N = \{W \in \mathcal{W} : G_i(w_i) = 1 \quad \forall i \in [n]\}$. In our world, it is immediate to see that $\Sigma(0) = \mathcal{W}_N$.

Corollary 7. *Let T be a production operator and $\beta \in (0,1)$. If $x \in \mathbb{R}^n$, then for each $d \in \mathbb{R}^n$ there exists $\tilde{W}_d \in \Sigma(\Pi(x))$ such that*

$$\lim_{h \downarrow 0} \frac{\Upsilon(x+hd) - \Upsilon(x)}{h} = -\xi^T J_{\mathcal{M}(\tilde{W}_d,\cdot)}(\Pi(x))_\beta d \geq -\xi^T J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(x))_\beta d \quad \forall \tilde{W} \in \Sigma(\Pi(x)).$$

In particular, we have that

$$\lim_{h \downarrow 0} \frac{\Upsilon(x+hd) - \Upsilon(x)}{h} = \max_{\tilde{W} \in \Sigma(\Pi(x))} -\xi^T J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(x))_\beta d.$$

Proof. Fix $x, d \in \mathbb{R}^n$. By Theorem 2, there exists $\tilde{W}_d \in \Sigma(\Pi(x))$ such that

$$\lim_{h \downarrow 0} \frac{\Pi(x+hd) - \Pi(x)}{h} = J_{\mathcal{M}(\tilde{W}_d,\cdot)}(\Pi(x))_\beta d \leq J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(x))_\beta d \quad \forall \tilde{W} \in \Sigma(\Pi(x)).$$

By definition of Υ and since $\xi \geq 0$, this implies that

$$\begin{aligned}\lim_{h \downarrow 0} \frac{\Upsilon(x+hd) - \Upsilon(x)}{h} &= \sum_{i=1}^n -\xi_i \lim_{h \downarrow 0} \frac{\Pi_i(x+hd) - \Pi_i(x)}{h} = -\xi^{\mathrm{T}} \lim_{h \downarrow 0} \frac{\Pi(x+hd) - \Pi(x)}{h} \\ &= -\xi^{\mathrm{T}} J_{\mathcal{M}(\tilde{W}_d,\cdot)}(\Pi(x))_\beta d \geq -\xi^{\mathrm{T}} J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(x))_\beta d \quad \forall \tilde{W} \in \Sigma(\Pi(x)),\end{aligned}$$

proving the statement.

Proof of Proposition 7. Fix $d \in \mathbb{R}^n$. By Corollary 7, we have that

$$\lim_{h \downarrow 0} \frac{\Upsilon(\bar{x}+hd) - \Upsilon(\bar{x})}{h} = \max_{\tilde{W} \in \Sigma(\Pi(\bar{x}))} -\xi^{\mathrm{T}} J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(\bar{x}))_\beta d.$$

By points 2 and 3 of Proposition 10, we have that

$$\lim_{h \downarrow 0} \frac{\Upsilon(\bar{x}+hd) - \Upsilon(\bar{x})}{h} = (1-\beta) \max_{\mathcal{E} \in \mathfrak{E}} -\lambda_{\mathcal{E}}^{\mathrm{T}} d,$$

proving the statement. □

E Proofs of the Results in Section 7

E.1 Semiconvexity of Υ

In this section, we are going to consider first a production operator T which is twice continuously differentiable and we are going to express the Hessian of Π as function of the Hessian of T. By Corollary 8, recall that Π inherits the differentiability properties. Before we start we introduce some notation. We consider the vector space $M_n(\mathbb{R})^n$ which is the space of vectors whose n components are $n \times n$ square matrices. Given a stochastic matrix W and $v \in M_n(\mathbb{R})^n$, the juxtaposition Wv is the vector whose i-th component is $\sum_{j=1}^n w_{ij} v_j$ and is itself an $n \times n$ square matrix. Finally, given a twice continuously differentiable operator $S : \mathbb{R}^n \to \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define by $H_S(x)$ the vector in $M_n(\mathbb{R})^n$ whose i-th component is the Hessian of S_i at x, denoted by $H_{S_i}(x)$.

Lemma 11. *Let T be a production operator and $\beta \in (0,1)$. If T is twice continuously differentiable, then*

$$H_\Pi(x) = \beta \sum_{t=0}^{\infty} \beta^t J_T(\Pi(x))^t v \quad \forall x \in \mathbb{R}^n \tag{185}$$

where $v \in M_n(\mathbb{R})^n$ is such that $v_i = J_\Pi(x)^T H_{T_i}(\Pi(x)) J_\Pi(x)$ for all $i \in [n]$.

Proof. Fix $x \in \mathbb{R}^n$. By definition of Π and since Π is twice continuously differentiable, we have that

$$J_\Pi(x) = (1-\beta) I + \beta J_T(\Pi(x)) J_\Pi(x).$$

Fix $i, j \in [n]$. It follows that

$$\frac{\partial \Pi_i}{\partial x_j}(x) = (1-\beta) e^i_j + \beta \sum_{l=1}^n \frac{\partial T_i}{\partial x_l}(\Pi(x)) \frac{\partial \Pi_l}{\partial x_j}(x). \tag{186}$$

Fix $h \in [n]$. We define $f_{il} : \mathbb{R}^n \to \mathbb{R}$ by $f_{il}(x) = \frac{\partial T_i}{\partial x_l}(\Pi(x)) \frac{\partial \Pi_l}{\partial x_j}(x)$ for all $x \in \mathbb{R}^n$ and for all $l \in [n]$. By the Chain Rule, we obtain that

$$\frac{\partial f_{il}}{\partial x_h}(x) = \sum_{m=1}^n \frac{\partial^2 T_i}{\partial x_l \partial x_m}(\Pi(x)) \frac{\partial \Pi_m}{\partial x_h}(x) \frac{\partial \Pi_l}{\partial x_j}(x) + \frac{\partial T_i}{\partial x_l}(\Pi(x)) \frac{\partial^2 \Pi_l}{\partial x_j \partial x_h}(x) \quad \forall l \in [n].$$

By (186), this implies that

$$\begin{aligned} \frac{\partial^2 \Pi_i}{\partial x_j \partial x_h}(x) &= \beta \sum_{l=1}^n \frac{\partial f_{il}}{\partial x_h}(x) \\ &= \beta \sum_{l=1}^n \sum_{m=1}^n \frac{\partial \Pi_l}{\partial x_j}(x) \frac{\partial^2 T_i}{\partial x_l \partial x_m}(\Pi(x)) \frac{\partial \Pi_m}{\partial x_h}(x) + \beta \sum_{l=1}^n \frac{\partial T_i}{\partial x_l}(\Pi(x)) \frac{\partial^2 \Pi_l}{\partial x_j \partial x_h}(x). \end{aligned}$$

Since i, j, and h were arbitrarily chosen, we have that

$$H_{\Pi_i}(x) = \beta J_\Pi(x)^{\mathrm{T}} H_{T_i}(\Pi(x)) J_\Pi(x) + \beta \sum_{l=1}^n \frac{\partial T_i}{\partial x_l}(\Pi(x)) H_{\Pi_l}(x) \quad \forall i \in [n].$$

If we define $\tilde{v} \in M_n(\mathbb{R})^n$ by $\tilde{v}_i = \frac{\beta}{1-\beta} J_\Pi(x)^{\mathrm{T}} H_{T_i}(\Pi(x)) J_\Pi(x)$ for all $i \in [n]$, then we have that

$$H_\Pi(x) = (1-\beta)\tilde{v} + \beta J_T(\Pi(x)) H_\Pi(x),$$

yielding that

$$H_\Pi(x) = \beta \sum_{t=0}^{\infty} \beta^t J_T(\Pi(x))^t v$$

and proving the statement. $\square$

Lemma 12. *Let T be a production operator, $\beta \in (0,1)$, and $\xi \in \Delta$. If T is twice continuously differentiable, then the map $\Upsilon : \mathbb{R}^n \to \mathbb{R}$, defined by $\Upsilon(x) = -\langle \xi, \Pi(x) \rangle$ for all $x \in \mathbb{R}^n$, is λ-semiconvex where $\lambda = \frac{\beta}{1-\beta}$, that is, $\Upsilon + \frac{\lambda}{2}\| \ \|_2^2$ is a convex function.*

Proof. Fix $x \in \mathbb{R}^n$. By Lemma 11, we have that

$$H_\Pi(x) = \frac{\beta}{1-\beta}(1-\beta)\sum_{t=0}^{\infty}\beta^t J_T(\Pi(x))^t v = \frac{\beta}{1-\beta} J_T(\Pi(x))_\beta v$$

where $v \in M_n(\mathbb{R})^n$ is such that $v_i = J_\Pi(x)^{\mathrm{T}} H_{T_i}(\Pi(x)) J_\Pi(x)$ for all $i \in [n]$. Since $J_T(\Pi(x))$ is a stochastic matrix, so is $J_T(\Pi(x))_\beta$ and, in particular, $\bar{\xi}$ defined as $\bar{\xi}^{\mathrm{T}} = \xi^{\mathrm{T}} J_T(\Pi(x))_\beta$ belongs to Δ. It follows that

$$\begin{aligned} H_\Upsilon(x) &= -\frac{\beta}{1-\beta}\xi^{\mathrm{T}} J_\Pi(x)_\beta v = -\frac{\beta}{1-\beta}\bar{\xi}^{\mathrm{T}} v = -\frac{\beta}{1-\beta}\sum_{i=1}^{n}\bar{\xi}_i J_\Pi(x)^{\mathrm{T}} H_{T_i}(\Pi(x)) J_\Pi(x) \\ &= -\frac{\beta}{1-\beta} J_\Pi(x)^{\mathrm{T}} \left(\sum_{i=1}^{n}\bar{\xi}_i H_{T_i}(\Pi(x))\right) J_\Pi(x). \end{aligned}$$

By Lemma 23, we have that $\max_{h\in B_\infty} \langle H_{T_i}(\Pi(x)) h, h\rangle \leq 1$ for all $i \in [n]$. Since $\bar{\xi} \in \Delta$, this implies that

$$\max_{h\in B_\infty}\left\langle \sum_{i=1}^{n}\bar{\xi}_i H_{T_i}(\Pi(x)) h, h\right\rangle \leq \sum_{i=1}^{n}\bar{\xi}_i \max_{h\in B_\infty}\langle H_{T_i}(\Pi(x)) h, h\rangle \leq 1.$$

Since $J_\Pi(x)$ is a stochastic matrix, we have that $\|J_\Pi(x) d\|_\infty \leq \|d\|_\infty$ for all $d \in \mathbb{R}^n$. To simplify notation, set $A = \sum_{i=1}^{n}\bar{\xi}_i H_{T_i}(\Pi(x))$, $B = J_\Pi(x)$, and $C = B^{\mathrm{T}}AB$. This implies that $\|Bd\|_\infty \leq \|d\|_\infty$ for all $d \in \mathbb{R}^n$ and

$$\max_{d\in B_\infty}\langle Cd, d\rangle = \max_{d\in B_\infty}\langle ABd, Bd\rangle = \max_{h\in\mathbb{R}^n : h=Bd \text{ and } d\in B_\infty}\langle Ah, h\rangle \leq \max_{h\in B_\infty}\langle Ah, h\rangle \leq 1.$$

Next, consider $d \in \mathbb{R}^n \backslash \{0\}$. If we set $\bar{d} = d \backslash \|d\|_2$, then $\|\bar{d}\|_\infty \leq \|\bar{d}\|_2 = 1$ and, in particular,

$\langle C\bar{d}, \bar{d}\rangle \leq 1$, that is, $\langle Cd, d\rangle \leq \|d\|_2^2$. Since d was arbitrarily chosen, we can conclude that $\langle Cd, d\rangle \leq \|d\|_2^2$ for all $d \in \mathbb{R}^n$. Since $H_\Upsilon(x) = -\frac{\beta}{1-\beta} J_\Pi(x)^{\mathrm{T}} \left(\sum_{i=1}^n \bar{\xi}_i H_{T_i}(\Pi(x))\right) J_\Pi(x) = -\frac{\beta}{1-\beta} B^{\mathrm{T}} A B = -\frac{\beta}{1-\beta} C$, we can conclude that

$$\langle -H_\Upsilon(x) d, d\rangle = \left\langle \frac{\beta}{1-\beta} Cd, d \right\rangle \leq \frac{\beta}{1-\beta} \|d\|_2^2 = \langle (\lambda I) d, d\rangle \qquad \forall d \in \mathbb{R}^n,$$

proving that $\lambda I + H_\Upsilon(x) \succeq 0$ where $\lambda = \frac{\beta}{1-\beta}$. Since x was arbitrarily chosen and Υ is twice continuously differentiable, the statement follows. □

Proposition 11. *If T is a production operator, $\beta \in (0,1)$, and $\xi \in \Delta$, then $\Upsilon : \mathbb{R}^n \to \mathbb{R}$, defined by $\Upsilon(x) = -\langle \xi, \Pi(x)\rangle$ for all $x \in \mathbb{R}^n$, is λ-semiconvex where $\lambda = \frac{\beta}{1-\beta}$, that is, $\Upsilon + \frac{\lambda}{2}\|\ \|_2^2$ is a convex function. In particular, we have that:*

1. *Π_i is λ-semiconcave for all $i \in [n]$.*
2. *For each $x, d \in \mathbb{R}^n$, the function $t \mapsto \Upsilon_\circ(x + td; d)$ is locally of bounded variation.*

Proof. By Proposition 15, there exists a sequence $\{T_m\}_{m\in\mathbb{N}} \subseteq C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ of production operators such that $\lim_m \Pi_m(x) = \Pi(x)$ for all $x \in \mathbb{R}^n$. Define $\Upsilon_m : \mathbb{R}^n \to \mathbb{R}$ by $\Upsilon_m(x) = -\langle \xi, \Pi_m(x)\rangle$ for all $x \in \mathbb{R}^n$ and for all $m \in \mathbb{N}$. By Lemma 12, we have that $\Upsilon_m + \frac{\lambda}{2}\|\ \|_2^2$ is convex for all $m \in \mathbb{N}$ and $\lim_m \Upsilon_m(x) = \Upsilon(x)$ for all $x \in \mathbb{R}^n$. Since convexity passes to the limit, we can conclude that $\Upsilon + \frac{\lambda}{2}\|\ \|_2^2$ is convex.

1. Fix $i \in [n]$ and set $\xi = e^i$. It follows that $\Upsilon = -\Pi_i$, proving the statement.

2. Fix $x, d \in \mathbb{R}^n$. Since $\Upsilon + \frac{\lambda}{2}\|\ \|_2^2$ is convex, we have that the function $t \mapsto \Upsilon_\circ(x + td; d) = [\Upsilon_\circ(x + td; d) + \lambda\langle x + td, d\rangle] - \lambda\langle x + td, d\rangle$, that is, it is the difference of two increasing functions, proving the statement.

Corollary 8. *If T is a production operator, then $\Pi^{-1} = \frac{1}{1-\beta}(I - \beta T)$, that is, $\Pi^{-1}(p) = \frac{1}{1-\beta}(p - \beta T(p))$ for all $p \in \mathbb{R}^n$. In particular, if T is twice continuously differentiable in a neighborhood U, so is Π in the neighborhood $V = \Pi^{-1}(U)$.*

Proof. Fix $p \in \mathbb{R}^n$. Since Π is bijective, there exists $x \in \mathbb{R}^n$ such that $\Pi(x) = p$, that is, $\Pi^{-1}(p) = x$. This implies that

$$\begin{aligned} p &= \Pi\left(\Pi^{-1}(p)\right) = \Pi(x) = (1-\beta)x + \beta T(\Pi(x)) \\ &= (1-\beta)\Pi^{-1}(p) + \beta T\left(\Pi\left(\Pi^{-1}(p)\right)\right) = (1-\beta)\Pi^{-1}(p) + \beta T(p), \end{aligned}$$

proving the statement. Since $\Pi^{-1} = \frac{1}{1-\beta}(I - \beta T)$ and T is twice continuously differentiable on U, so is Π^{-1}. Since T is a production operator, we have that $J_T(p)$ is a stochastic matrix and, in particular, $J_{\Pi^{-1}}(p) = \frac{1}{1-\beta}(I - \beta J_T(p))$ is invertible for all $p \in U$. By the Inverse Function Theorem, we can conclude that Π is twice continuously differentiable on $V = \Pi^{-1}(U)$. □

Definition 5 (Fundamentally C^2)**.** *For an operator $S : \mathbb{R}^n \to \mathbb{R}^n$, let $\mathcal{D}_S$ and $\mathcal{H}_S$ denote the sets of points where S is continuously differentiable and twice continuously differentiable, respectively. We say that S is* fundamentally C^2 *if $\mathcal{D}_S = \mathcal{H}_S$ and this common set is open and has full Lebesgue measure in $\mathbb{R}^n$.*

Lemma 13. *Let T be a production operator and $\beta \in (0, 1)$. If T is fundamentally C^2 and $d = e^j$ for some $j \in [n]$, then Π is fundamentally C^2, $\{t \in \mathbb{R} : x + td \in \mathcal{D}_\Pi\}$ is open for all $x \in \operatorname{span}\{d\}^\perp$, and $\{t \in \mathbb{R} : x + td \in \mathcal{D}_\Pi\}^c$ is countable for almost all $x \in \operatorname{span}\{d\}^\perp$.*[19]

Proof. By Corollary 8 and since T is twice continuously differentiable on the open set $\mathcal{D}_T = \mathcal{H}_T$, we have that Π is twice continuously differentiable on the open set $\Pi^{-1}(\mathcal{D}_T) = \Pi^{-1}(\mathcal{H}_T)$. Since $\mathcal{H}_\Pi \subseteq \mathcal{D}_\Pi$ and Π is a homeomorphism, this implies that $\mathcal{H}_\Pi \supseteq \Pi^{-1}(\mathcal{D}_T) = \mathcal{D}_\Pi \supseteq \mathcal{H}_\Pi$, proving that $\mathcal{H}_\Pi = \mathcal{D}_\Pi = \Pi^{-1}(\mathcal{D}_T)$ and Π is twice continuously differentiable on this latter set. By Proposition 11, we have that Π_i is λ-semiconcave for all $i \in [n]$ where $\lambda = \frac{\beta}{1-\beta}$. By (Cannarsa and Sinestrari, 2004, Propositions 1.1.3 and 4.1.3), this implies that $\mathcal{D}^c_{\Pi_i}$ is a countably H^{n-1}-rectifiable set for all $i \in [n]$. By (Cannarsa and Sinestrari, 2004, Definition A.3.4), we have that $\mathcal{D}^c_\Pi = \cup_{i=1}^n \mathcal{D}^c_{\Pi_i}$ is a countably H^{n-1}-rectifiable set. By (Evans, 2025, Definition 1.9 and Theorem 2.1), it follows that $\mathcal{D}^c_\Pi \subseteq \mathcal{N} \cup (\cup_{k\in\mathbb{N}}\mathcal{N}_k)$ where $\mathcal{N}$ is Borel measurable with $H^{n-1}(\mathcal{N}) = 0$ and $\mathcal{N}_k = f_k(\mathbb{R}^{n-1}) \subseteq \mathbb{R}^n$ and $f_k : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is Lipschitz continuous for all $k \in \mathbb{N}$. For each $k, m \in \mathbb{N}$ define $\mathcal{N}_{k,m} = f_k(\mathbb{R}^{n-1} \cap \{z \in \mathbb{R}^{n-1} : -m \le z_i \le m \quad \forall i \in [n-1]\})$ and $\bar{\mathcal{N}}_{k,m} = \mathcal{N} \cup \mathcal{N}_{k,m}$. By (Evans, 2025, Theorems 2.5 and 2.8) and since $H^{n-1}(\mathcal{N}) = 0$, we have that $H^{n-1}(\mathcal{N}_{k,m}) < \infty$ and $H^{n-1}(\bar{\mathcal{N}}_{k,m}) < \infty$ for all $k, m \in \mathbb{N}$. Fix $k, m \in \mathbb{N}$, $j \in [n]$, and $d = e^j$. Note that $\bar{\mathcal{N}}_{k,m}$ is a Borel set. Define $P : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the canonical projection deleting the j-th coordinate. Define $z \mapsto \operatorname{card}(\bar{\mathcal{N}}_{k,m} \cap P^{-1}(z))$ on $\mathbb{R}^{n-1}$. By (Evans, 2025, Lemma 5.8), we have that

$$\int_{\mathbb{R}^{n-1}} \operatorname{card}\left(\bar{\mathcal{N}}_{k,m} \cap P^{-1}(z)\right) dz \le H^{n-1}\left(\bar{\mathcal{N}}_{k,m}\right) < \infty.$$

Since k and m were arbitrarily chosen, it follows that $\left\{z \in \mathbb{R}^{n-1} : \operatorname{card}\left(\bar{\mathcal{N}}_{k,m} \cap P^{-1}(z)\right) = \infty\right\}$ has measure 0. Define $\mathcal{Z} = \cap_{k,m\in\mathbb{N}} \left\{z \in \mathbb{R}^{n-1} : \operatorname{card}\left(\bar{\mathcal{N}}_{k,m} \cap P^{-1}(z)\right) < \infty\right\}$, that is, $\mathcal{Z}^c = \cup_{k,m\in\mathbb{N}} \left\{z \in \mathbb{R}^{n-1} : \operatorname{card}\left(\bar{\mathcal{N}}_{k,m} \cap P^{-1}(z)\right) = \infty\right\}$. Note that $\mathcal{Z}^c$ has measure 0. Fix $z \in \mathcal{Z}$. Consider $x \in \mathbb{R}^n$ such that $P(x) = z$. We have that $\operatorname{card}\left(\bar{\mathcal{N}}_{k,m} \cap P^{-1}(z)\right) < \infty$, that is, $\left\{t \in \mathbb{R} : x + te^j \in \bar{\mathcal{N}}_{k,m}\right\}$ is finite for all $k, m \in \mathbb{N}$ and, in particular, $\cup_{k\in\mathbb{N}}\cup_{m\in\mathbb{N}}\left\{t \in \mathbb{R} : x + te^j \in \bar{\mathcal{N}}_{k,m}\right\}$ is countable. Since

$$\begin{aligned}\cup_{k\in\mathbb{N}} \cup_{m\in\mathbb{N}} \left\{t \in \mathbb{R} : x + te^j \in \bar{\mathcal{N}}_{k,m}\right\} &= \cup_{k\in\mathbb{N}} \left\{t \in \mathbb{R} : x + te^j \in \mathcal{N} \cup \mathcal{N}_k\right\} \\ &\supseteq \left\{t \in \mathbb{R} : x + te^j \in \mathcal{D}^c_\Pi\right\},\end{aligned}$$

[19] The set $\operatorname{span}\{d\}^\perp$ is isomorphic to $\mathbb{R}^{n-1}$. Thus, "almost all" is with respect to the Lebesgue measure over $\mathbb{R}^{n-1}$.

proving that the latter set is countable. □

Without loss of generality in what follows, we focus on the last coordinate of $\mathbb{R}^n$, that is, on the n-th sector. Consider a probability density function $\varphi : \mathbb{R} \to [0, \infty)$ which is continuous and even with standard deviation 1. For example, φ could be the normal distribution $\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ for all $t \in \mathbb{R}$. If we fix $(\mu, \varepsilon) \in \mathbb{R} \times (0, \infty)$, then $\varphi_{\mu,\varepsilon} : \mathbb{R} \to [0, \infty)$, defined by $\varphi_{\mu,\varepsilon}(t) = \varepsilon^{-1} \varphi\left(\frac{t-\mu}{\varepsilon}\right)$ for all $t \in \mathbb{R}$, is a probability density function with mean μ and standard deviation ε. Consider also a probability over $\mathbb{R}^{n-1}$ with density $\tilde{\varphi} : \mathbb{R}^{n-1} \to [0, \infty)$. Finally, define $\mathbb{E}\Upsilon : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ by for each $(\mu, \varepsilon) \in \mathbb{R} \times (0, \infty)$

$$\mathbb{E}\Upsilon(\mu, \varepsilon) = \int_{\mathbb{R}} ... \int_{\mathbb{R}} \Upsilon(x_1, ..., x_{n-1}, x_n)\, \tilde{\varphi}(x_1, ..., x_{n-1})\, \varphi_{\mu,\varepsilon}(x_n)\, dx_1 ... dx_{n-1} dx_n.$$

In other words, $\mathbb{E}\Upsilon(\mu, \varepsilon)$ is the expected log-GDP when the first $n-1$ normalized log-productivities, $x_i = -\ln Z_i \backslash (1-\beta)$, are jointly distributed according to $\tilde{\varphi}$ and x_n is independently distributed according to the rescaled φ, that is, with mean μ, standard deviation ε, and shape φ. The next result shows how infinitesimal changes in standard deviation ε affect this expectation. To this extent, we need to introduce some extra notation. We define $\Upsilon_\circ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\Upsilon_\circ(x; d) = \lim_{h \downarrow 0} \frac{\Upsilon(x + hd) - \Upsilon(x)}{h} \qquad \forall (x, d) \in \mathbb{R}^n \times \mathbb{R}^n.$$

By Proposition 7, we have that $\Upsilon_\circ$ is well defined. Note that if $d = e^n$ and $x \in \mathcal{D}_\Pi$, then

$$\Upsilon_\circ(x; e^n) = \frac{\partial \Upsilon}{\partial x_n}(x) = -(1-\beta)\, \lambda_n(x) \qquad \forall x \in \mathcal{D}_\Pi$$

where $\lambda_n(x)$ is the n-th Domar weight.[20] If T is a fundamentally C^2 production operator, we have that $\frac{\partial \lambda_n}{\partial x_n}(x)$ exists for all $x \in \mathcal{D}_\Pi$. In such a case, we define $\frac{\partial \lambda_n}{\partial x_n}$ over $\mathbb{R}^n$ by setting it equal to 0 outside $\mathcal{D}_\Pi$. Given λ_n and x, we define $\Delta\lambda_n(x) = \lambda_n(x) - \lim_k \lambda_n\left(x - \frac{1}{k} e^n\right)$ for all $x \in \mathbb{R}^n$.

Lemma 14. *Let $f : \mathbb{R} \to \mathbb{R}$ be an L-Lipschitz function such that $f'_+, f'_- : \mathbb{R} \to \mathbb{R}$ are well defined functions such that $\left\{t \in \mathbb{R} : f'_+(t) \neq f'_-(t)\right\}$ is a countable set. If $\gamma : \mathbb{R} \to \mathbb{R}$ is right-continuous, locally of bounded variation, and such that $\left\{t \in \mathbb{R} : \gamma(t) \neq f'_+(t)\right\}$ is a countable set, then*

$$\frac{\partial u}{\partial \varepsilon}(\mu, \varepsilon) = \int_{\mathbb{R}} s\gamma(\mu + \varepsilon s)\, \varphi(s)\, ds \in \mathbb{R} \qquad \forall (\mu, \varepsilon) \in \mathbb{R} \times (0, \infty) \tag{187}$$

and

$$\frac{\partial u}{\partial \varepsilon}(\mu, \varepsilon) = \lim_k \int_{(\mu - \varepsilon k, \mu + \varepsilon k]} \left(\int_{\left[\left|\frac{t-\mu}{\varepsilon}\right|, k\right]} s\varphi(s)\, ds \right) d\gamma(t) \quad \forall (\mu, \varepsilon) \in \mathbb{R} \times (0, \infty). \tag{188}$$

[20] For notational convenience, we drop the subscript $\mathcal{E}$.

Moreover, we have that:

1. *If* $t \mapsto \gamma(t) + \lambda t$ *is increasing (resp.* $t \mapsto \gamma(t) - \lambda t$ *is decreasing) with* $\lambda > 0$*, continuously differentiable on an open set* O *such that* $O^c = \{t_k\}_{k\in\mathbb{N}}$ *is countable, and* $M : [0,\infty) \to [0,\infty)$ *is defined by* $M(t) = \int_{[t,\infty)} s\varphi(s)\,ds$ *for all* $t \in [0,\infty)$*, then* $\Delta\gamma \geq 0$ *(resp.* $\Delta\gamma \leq 0$*) and*

$$\frac{\partial u}{\partial \varepsilon}(\mu,\varepsilon) = \int_{\mathbb{R}} M\left(\left|\frac{t-\mu}{\varepsilon}\right|\right)\gamma'(t)\,dt + \sum_{k=1}^{\infty} M\left(\left|\frac{t_k-\mu}{\varepsilon}\right|\right)\Delta\gamma(t_k) \quad \forall(\mu,\varepsilon) \in \mathbb{R}\times(0,\infty).$$

2. *If* γ *is as in Point 1 and* $\varphi(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2\backslash 2}$ *for all* $t \in \mathbb{R}$*, then*

$$\frac{\partial u}{\partial \varepsilon}(\mu,\varepsilon) = \varepsilon\int_{\mathbb{R}} \varphi_{\mu,\varepsilon}(t)\gamma'(t)\,dt + \varepsilon\sum_{k=1}^{\infty}\varphi_{\mu,\varepsilon}(t_k)\Delta\gamma(t_k) \quad \forall(\mu,\varepsilon)\in\mathbb{R}\times(0,\infty). \tag{189}$$

Proof. Fix $(\mu,\varepsilon) \in \mathbb{R}\times(0,\infty)$. Since f is L-Lipschitz, we have that

$$\left|\frac{f(\mu+(\varepsilon+\delta)s) - f(\mu+\varepsilon s)}{\delta}\right| \leq L|s| \qquad \forall s\in\mathbb{R}, \forall\delta\in\mathbb{R}\backslash\{0\}.$$

Since f is continuous and f'_- and f'_+ are well defined, if we consider a sequence $\{\delta_k\}_{k\in\mathbb{N}} \subseteq (0,\infty)$ (resp. $\subseteq (-\infty,0)$), we have that

$$\lim_k \frac{f(\mu+(\varepsilon+\delta_k)s) - f(\mu+\varepsilon s)}{\delta_k} = \begin{cases} sf'_+(\mu+\varepsilon s) & s>0 \\ 0 & s=0 \\ sf'_-(\mu+\varepsilon s) & s<0 \end{cases} \left(\text{resp. } \begin{cases} sf'_-(\mu+\varepsilon s) & s>0 \\ 0 & s=0 \\ sf'_+(\mu+\varepsilon s) & s<0 \end{cases}\right),$$

yielding that $s \mapsto \frac{f(\mu+(\varepsilon+\delta_k)s)-f(\mu+\varepsilon s)}{\delta_k}$ is measurable for all $k\in\mathbb{N}$ and so is the pointwise limit of this sequence. Moreover, since $\int_{\mathbb{R}}|t|\varphi(t)\,dt < \infty$, we have that $s \mapsto \left|\frac{f(\mu+(\varepsilon+\delta_k)s)-f(\mu+\varepsilon s)}{\delta_k}\right|$ is bounded by $s \mapsto L|s|$ for all $k\in\mathbb{N}$ and, in particular, it is integrable for all $k\in\mathbb{N}$ and so is the pointwise limit of this sequence. By Lebesgue's Dominated Convergence Theorem, this implies that if $\{\delta_k\}_{k\in\mathbb{N}} \subseteq (0,\infty)$ (resp. $\{\delta_k\}_{k\in\mathbb{N}} \subseteq (-\infty,0)$), then

$$\begin{aligned} \lim_k \frac{u(\mu,\varepsilon+\delta_k) - u(\mu,\varepsilon)}{\delta_k} &= \lim_k \int_{\mathbb{R}}\left(\frac{f(\mu+(\varepsilon+\delta_k)s) - f(\mu+\varepsilon s)}{\delta_k}\right)\varphi(s)\,ds \\ &= \int_{\mathbb{R}}\lim_k\left(\frac{f(\mu+(\varepsilon+\delta_k)s) - f(\mu+\varepsilon s)}{\delta_k}\right)\varphi(s)\,ds \\ &= \int_{(-\infty,0]} sf'_-(\mu+\varepsilon s)\varphi(s)\,ds + \int_{(0,\infty)} sf'_+(\mu+\varepsilon s)\varphi(s)\,ds \\ (\text{resp.} &= \int_{(-\infty,0]} sf'_+(\mu+\varepsilon s)\varphi(s)\,ds + \int_{(0,\infty)} sf'_-(\mu+\varepsilon s)\varphi(s)\,ds). \end{aligned}$$

Since $\left\{t \in \mathbb{R} : f'_{+}(t) \neq f'_{-}(t)\right\}$ and $\left\{t \in \mathbb{R} : \gamma(t) \neq f'_{+}(t)\right\}$ are countable sets, we can conclude that

$$\begin{aligned}
&\int_{(-\infty,0]} s f'_{-}(\mu+\varepsilon s)\,\varphi(s)\,ds + \int_{(0,\infty)} s f'_{+}(\mu+\varepsilon s)\,\varphi(s)\,ds \\
&= \int_{(-\infty,0]} s f'_{+}(\mu+\varepsilon s)\,\varphi(s)\,ds + \int_{(0,\infty)} s f'_{-}(\mu+\varepsilon s)\,\varphi(s)\,ds \\
&= \int_{\mathbb{R}} s f'_{+}(\mu+\varepsilon s)\,\varphi(s)\,ds = \int_{\mathbb{R}} s\gamma(\mu+\varepsilon s)\,\varphi(s)\,ds.
\end{aligned}$$

Since $\{\delta_k\}_{k\in\mathbb{N}}$ was arbitrarily chosen, this implies that $u(\mu,\cdot) : (0,\infty) \to \mathbb{R}$ has left- and right-derivative at ε, the two side-derivatives coincide, and their value is $\int_{\mathbb{R}} s\gamma(\mu+\varepsilon s)\,\varphi(s)\,ds$. Since μ and ε were arbitrarily chosen, (187) follows. Fix again $(\mu,\varepsilon) \in \mathbb{R}\times(0,\infty)$. Define $\Phi : \mathbb{R} \to [0,\infty)$ by $\Phi(t) = \int_{[0,t]} s\varphi(s)\,ds$ for all $t \in [0,\infty)$ and $\Phi(t) = 0$ for all $t \in (-\infty,0)$. Define $\Psi : \mathbb{R} \to (-\infty,0]$ by $\Psi(t) = \int_{[t,0]} s\varphi(s)\,ds$ for all $t \in (-\infty,0]$, $\Psi(0) = 0$, and $\Psi(t) = 0$ for all $t \in (0,\infty)$. By the Fundamental Theorem of Calculus and since the Riemann and the Lebesgue integral coincide over closed and bounded intervals for continuous integrands (see, e.g., (Folland, 1999, Theorem 2.28)), note that Φ is continuously differentiable, increasing, and such that $\Phi(0) = 0$ and $\Phi'(t) = t\varphi(t)$ (resp. $= 0$) for all $t \in [0,\infty)$ (resp. $t \in (-\infty,0)$). Note also that $\Psi(t) = -\Phi(-t)$ for all $t \in \mathbb{R}$ and, in particular, Ψ is continuously differentiable, increasing, and such that $\Psi(0) = 0$ and, $\Psi'(t) = -t\varphi(t)$ (resp. $= 0$) for all $t \in (-\infty,0]$ (resp. $t \in (0,\infty)$). Define $\Phi_{\mu,\varepsilon} : \mathbb{R} \to [0,\infty)$ by $\Phi_{\mu,\varepsilon}(t) = \Phi\left(\frac{t-\mu}{\varepsilon}\right)$ for all $t \in \mathbb{R}$ and $\Psi_{\mu,\varepsilon} : \mathbb{R} \to (-\infty,0]$ by $\Psi_{\mu,\varepsilon}(t) = \Psi\left(\frac{t-\mu}{\varepsilon}\right)$ for all $t \in \mathbb{R}$. Given γ, consider a decomposition (γ_1,γ_2). Fix $i \in \{1,2\}$. By the Change of Variable Formula and Integration by Parts and since γ_i, $\Phi_{\mu,\varepsilon}$, and $\Psi_{\mu,\varepsilon}$ are right-continuous and increasing, we have that

$$\begin{aligned}
\int_{(0,k]} s\gamma_i(\mu+\varepsilon s)\,\varphi(s)\,ds &= \int_{(0,k]} \gamma_i(\mu+\varepsilon s)\,\Phi'(s)\,ds = \int_{(\mu,\mu+\varepsilon k]} \gamma_i(t)\,\frac{1}{\varepsilon}\Phi'\left(\frac{t-\mu}{\varepsilon}\right)dt \\
&= \int_{(\mu,\mu+\varepsilon k]} \gamma_i(t)\,\Phi'_{\mu,\varepsilon}(t)\,dt = \int_{(\mu,\mu+\varepsilon k]} \gamma_i(t)\,d\Phi_{\mu,\varepsilon}(t) \\
&= \Phi_{\mu,\varepsilon}(\mu+\varepsilon k)\,\gamma_i(\mu+\varepsilon k) - \Phi_{\mu,\varepsilon}(\mu)\,\gamma_i(\mu) - \int_{(\mu,\mu+\varepsilon k]} \Phi_{\mu,\varepsilon}(t)\,d\gamma_i(t) \\
&= \Phi_{\mu,\varepsilon}(\mu+\varepsilon k)\,\gamma_i(\mu+\varepsilon k) - \int_{(\mu,\mu+\varepsilon k]} \Phi_{\mu,\varepsilon}(t)\,d\gamma_i(t) \\
&= \Phi(k)\,\gamma_i(\mu+\varepsilon k) - \int_{(\mu,\mu+\varepsilon k]} \Phi\left(\frac{t-\mu}{\varepsilon}\right)d\gamma_i(t) \\
&= \Phi(k)\,\gamma_i(\mu+\varepsilon k) - \int_{(\mu,\mu+\varepsilon k]} \Phi\left(\left|\frac{t-\mu}{\varepsilon}\right|\right)d\gamma_i(t) \qquad \forall k \in \mathbb{N}
\end{aligned}$$

and

$$\begin{aligned}
\int_{(-k,0]} s\gamma_i(\mu+\varepsilon s)\varphi(s)\,ds &= -\int_{(-k,0]} \gamma_i(\mu+\varepsilon s)\Psi'(s)\,ds = -\int_{(\mu-\varepsilon k,\mu]} \gamma_i(t)\frac{1}{\varepsilon}\Psi'\left(\frac{t-\mu}{\varepsilon}\right)dt \\
&= -\int_{(\mu-\varepsilon k,\mu]} \gamma_i(t)\Psi'_{\mu,\varepsilon}(t)\,dt = -\int_{(\mu-\varepsilon k,\mu]} \gamma_i(t)\,d\Psi_{\mu,\varepsilon}(t) \\
&= -\left[\Psi_{\mu,\varepsilon}(\mu)\gamma_i(\mu) - \Psi_{\mu,\varepsilon}(\mu-\varepsilon k)\gamma_i(\mu-\varepsilon k) - \int_{(\mu-\varepsilon k,\mu]} \Psi_{\mu,\varepsilon}(t)\,d\gamma_i(t)\right] \\
&= \Psi_{\mu,\varepsilon}(\mu-\varepsilon k)\gamma_i(\mu-\varepsilon k) + \int_{(\mu-\varepsilon k,\mu]} \Psi_{\mu,\varepsilon}(t)\,d\gamma_i(t) \\
&= \Psi(-k)\gamma_i(\mu-\varepsilon k) + \int_{(\mu-\varepsilon k,\mu]} \Psi\left(\frac{t-\mu}{\varepsilon}\right)d\gamma_i(t) \\
&= -\Phi(k)\gamma_i(\mu-\varepsilon k) - \int_{(\mu-\varepsilon k,\mu]} \Phi\left(\frac{\mu-t}{\varepsilon}\right)d\gamma_i(t) \\
&= -\Phi(k)\gamma_i(\mu-\varepsilon k) - \int_{(\mu-\varepsilon k,\mu]} \Phi\left(\left|\frac{t-\mu}{\varepsilon}\right|\right)d\gamma_i(t) \qquad \forall k \in \mathbb{N}.
\end{aligned}$$

By adding these two terms, it follows that

$$\begin{aligned}
&\int_{(-k,k]} s\gamma_i(\mu+\varepsilon s)\varphi(s)\,ds \\
&= -\Phi(k)\gamma_i(\mu-\varepsilon k) - \int_{(\mu-\varepsilon k,\mu]} \Phi\left(\left|\frac{t-\mu}{\varepsilon}\right|\right)d\gamma_i(t) + \Phi(k)\gamma_i(\mu+\varepsilon k) - \int_{(\mu,\mu+\varepsilon k]} \Phi\left(\left|\frac{t-\mu}{\varepsilon}\right|\right)d\gamma_i(t) \\
&= \Phi(k)\left[\gamma_i(\mu+\varepsilon k) - \gamma_i(\mu-\varepsilon k)\right] - \int_{(\mu-\varepsilon k,\mu+\varepsilon k]} \Phi\left(\left|\frac{t-\mu}{\varepsilon}\right|\right)d\gamma_i(t) \\
&= \int_{(\mu-\varepsilon k,\mu+\varepsilon k]} \left[\Phi(k) - \Phi\left(\left|\frac{t-\mu}{\varepsilon}\right|\right)\right]d\gamma_i(t) \\
&= \int_{(\mu-\varepsilon k,\mu+\varepsilon k]} \left(\int_{\left(\left|\frac{t-\mu}{\varepsilon}\right|,k\right]} s\varphi(s)\,ds\right)d\gamma_i(t) \\
&= \int_{(\mu-\varepsilon k,\mu+\varepsilon k]} \left(\int_{\left[\left|\frac{t-\mu}{\varepsilon}\right|,k\right]} s\varphi(s)\,ds\right)d\gamma_i(t) \qquad \forall k \in \mathbb{N}.
\end{aligned}$$

Since i was arbitrarily chosen and each integral is finite, we can conclude that

$$\int_{(-k,k]} s\gamma(\mu+\varepsilon s)\varphi(s)\,ds = \int_{(\mu-\varepsilon k,\mu+\varepsilon k]} \left(\int_{\left[\left|\frac{t-\mu}{\varepsilon}\right|,k\right]} s\varphi(s)\,ds\right)d\gamma(t)$$

By passing to the limit and the Lebesgue's Dominated Converge Theorem and since $\int_{\mathbb{R}} sf'_+(\mu+\varepsilon s)\varphi(s)\,ds \in$

$\mathbb{R}$, we can conclude that

$$\int_{\mathbb{R}} s\gamma\left(\mu+\varepsilon s\right)\varphi\left(s\right)ds=\lim_{k}\int_{(\mu-\varepsilon k,\mu+\varepsilon k]}\left(\int_{\left[\left|\frac{t-\mu}{\varepsilon}\right|,k\right]} s\varphi\left(s\right)ds\right)d\gamma\left(t\right).$$

Since μ and ε were arbitrarily chosen, (188) follows.

1. Given γ, consider the decomposition (γ_1,γ_2) such that $\gamma_1(t)=\gamma(t)+\lambda t$ and $\gamma_2(t)=\lambda t$ for all $t\in\mathbb{R}$. Since γ_1 is increasing and continuous on O, we have that $\Delta\gamma_1$ is nonnegative and such that $\Delta\gamma_1=\Delta\gamma$ as well as $\Delta\gamma_1(t)=0$ for all $t\in O$, proving that $\Delta\gamma$ is nonnegative and such that $\Delta\gamma(t)=0$ for all $t\in O$. By definition of μ_{γ_1}, we have that

$$\mu_{\gamma_1}\left(\{t\}\right)=\lim_{k}\mu_{\gamma_1}\left((t-1\backslash k,t]\right)=\lim_{k}\left[\gamma_1\left(t\right)-\gamma_1\left(t-1\backslash k\right)\right]=\Delta\gamma_1\left(t\right)=\Delta\gamma\left(t\right)\qquad\forall t\in\mathbb{R}.$$

Since γ_1 is continuously differentiable on the open set O and O^c is countable, we have that γ_1' is well defined and continuous on O and so is γ'. By convention, recall that we set $\gamma_1'(t)=0$ for all $t\in O^c$. By Lebesgue's Decomposition Theorem we have that $\mu_{\gamma_1}=\mu_{\gamma_1,\mathrm{ac}}+\mu_{\gamma_1,\mathrm{s}}$ where $\mu_{\gamma_1,\mathrm{ac}}$ (resp. $\mu_{\gamma_1,\mathrm{s}}$) is absolutely continuous (resp. singular) with respect to the Lebesgue measure. By the Radon-Nikodym Theorem there exists a measurable function $\tilde{\gamma}:\mathbb{R}\to[0,\infty)$ such that $\mu_{\gamma_1,\mathrm{ac}}(A)=\int_A\tilde{\gamma}(t)\,dt$ for all $A\in\mathcal{B}(\mathbb{R})$. It follows that $\tilde{\gamma}$ can be replaced by γ_1' and $\mu_{\gamma_1,\mathrm{ac}}(O^c)=0$ as well as $\mu_{\gamma_1,\mathrm{s}}(O)=0$. Since $\mu_{\gamma_1,\mathrm{s}}$ is countably additive and $O^c=\{t_k\}_{k\in\mathbb{N}}$ is countable, it follows that

$$\begin{aligned}\mu_{\gamma_1}\left(A\right)&=\mu_{\gamma_1,\mathrm{ac}}\left(A\right)+\mu_{\gamma_1,\mathrm{s}}\left(A\right)=\mu_{\gamma_1,\mathrm{ac}}\left(A\right)+\mu_{\gamma_1,\mathrm{s}}\left(A\cap O^c\right)\\&=\int_A\gamma_1'\left(t\right)dt+\sum_{k:t_k\in A}\mu_{\gamma_1,\mathrm{s}}\left(\{t_k\}\right)\quad\forall A\in\mathcal{B}\left(\mathbb{R}\right).\end{aligned}$$

Since $0\le\mu_{\gamma_1,\mathrm{s}}\le\mu_{\gamma_1}$, note that $\mu_{\gamma_1,\mathrm{s}}(A)=\lim_m\sum_{k=1}^m\Delta\gamma(t_k)\,\delta_{t_k}(A)$ for all $A\in\mathcal{B}(\mathbb{R})$, that is, $\mu_{\gamma_1,\mathrm{s}}=\sum_{k=1}^\infty\Delta\gamma(t_k)\,\delta_{t_k}$ where δ_{t_k} is the Dirac measure at t_k for all $k\in\mathbb{N}$. Fix $k\in\mathbb{N}$. Define $M_k:\mathbb{R}\to[0,\infty)$ by $M_k(t)=\int_{\left[\left|\frac{t-\mu}{\varepsilon}\right|,k\right]}s\varphi(s)\,ds=\int 1_{\left[\left|\frac{t-\mu}{\varepsilon}\right|,k\right]}(s)\,s\varphi(s)\,ds$ for all $t\in\mathbb{R}$. We can conclude that

$$\begin{aligned}&\int_{(\mu-\varepsilon k,\mu+\varepsilon k]}\left(\int_{\left[\left|\frac{t-\mu}{\varepsilon}\right|,k\right]}s\varphi\left(s\right)ds\right)d\gamma_1\left(t\right)\\&=\int_{\mathbb{R}}1_{(\mu-\varepsilon k,\mu+\varepsilon k]}M_k d\mu_{\gamma_1}=\int_{\mathbb{R}}1_{(\mu-\varepsilon k,\mu+\varepsilon k]}M_k d\mu_{\gamma_1,\mathrm{ac}}+\int_{\mathbb{R}}1_{(\mu-\varepsilon k,\mu+\varepsilon k]}M_k d\mu_{\gamma_1,\mathrm{s}}\\&=\int_{\mathbb{R}}1_{(\mu-\varepsilon k,\mu+\varepsilon k]}\left(t\right)M_k\left(t\right)\gamma_1'\left(t\right)dt+\int_{\mathbb{R}}1_{(\mu-\varepsilon k,\mu+\varepsilon k]}M_k d\mu_{\gamma_1,\mathrm{s}}\end{aligned}$$

By Tonelli's Theorem and since φ is even and

$$\begin{aligned}
&1_{[\mu-\varepsilon k,\mu+\varepsilon k]}(t)\,1_{\left[\left|\frac{t-\mu}{\varepsilon}\right|,k\right]}(s)\,s\varphi(s)\\
&=1_{[\mu-\varepsilon k,\mu+\varepsilon k]}(t)\,1_{[\mu-\varepsilon s,\mu+\varepsilon s]}(t)\,1_{[0,k]}(s)\,s\varphi(s)\geq 0\qquad \forall (s,t)\in\mathbb{R}\times\mathbb{R},
\end{aligned}$$

we have that

$$\begin{aligned}
&\int_{(\mu-\varepsilon k,\mu+\varepsilon k]}\left(\int_{\left[\left|\frac{t-\mu}{\varepsilon}\right|,k\right]} s\varphi(s)\,ds\right)d\gamma_2(t)\\
&=\lambda\int_{(\mu-\varepsilon k,\mu+\varepsilon k]}\left(\int_{\left[\left|\frac{t-\mu}{\varepsilon}\right|,k\right]} s\varphi(s)\,ds\right)dt\\
&=\lambda\int_{[\mu-\varepsilon k,\mu+\varepsilon k]}\left(\int_{\left[\left|\frac{t-\mu}{\varepsilon}\right|,k\right]} s\varphi(s)\,ds\right)dt\\
&=\lambda\int_{\mathbb{R}}1_{[\mu-\varepsilon k,\mu+\varepsilon k]}(t)\left(\int_{\mathbb{R}}1_{\left[\left|\frac{t-\mu}{\varepsilon}\right|,k\right]}(s)\,s\varphi(s)\,ds\right)dt\\
&=\lambda\int_{\mathbb{R}}\left(\int_{\mathbb{R}}1_{[\mu-\varepsilon k,\mu+\varepsilon k]}(t)\,1_{\left[\left|\frac{t-\mu}{\varepsilon}\right|,k\right]}(s)\,s\varphi(s)\,ds\right)dt\\
&=\lambda\int_{\mathbb{R}}\left(\int_{\mathbb{R}}1_{[\mu-\varepsilon k,\mu+\varepsilon k]}(t)\,1_{[\mu-\varepsilon s,\mu+\varepsilon s]}(t)\,1_{[0,k]}(s)\,s\varphi(s)\,ds\right)dt\\
&=\lambda\int_{\mathbb{R}}\left(\int_{\mathbb{R}}1_{[\mu-\varepsilon k,\mu+\varepsilon k]}(t)\,1_{[\mu-\varepsilon s,\mu+\varepsilon s]}(t)\,1_{[0,k]}(s)\,s\varphi(s)\,dt\right)ds\\
&=\lambda\int_{\mathbb{R}}\left[1_{[0,k]}(s)\,s\varphi(s)\left(\int_{\mathbb{R}}1_{[\mu-\varepsilon k,\mu+\varepsilon k]}(t)\,1_{[\mu-\varepsilon s,\mu+\varepsilon s]}(t)\,dt\right)\right]ds\\
&=\lambda\int_{\mathbb{R}}\left[1_{[0,k]}(s)\,s\varphi(s)\left(\int_{\mathbb{R}}1_{[\max\{\mu-\varepsilon k,\mu-\varepsilon s\},\min\{\mu+\varepsilon s,\mu+\varepsilon k\}]}dt\right)\right]ds\\
&=\lambda\int_{\mathbb{R}}1_{[0,k]}(s)\,s\varphi(s)\,(2\varepsilon s)\,ds=2\lambda\varepsilon\int_{[0,k]}s^2\varphi(s)\,ds=\lambda\varepsilon\int_{[-k,k]}s^2\varphi(s)\,ds
\end{aligned}$$

By the Monotone Convergence Theorem and since k was arbitrarily chosen, $M_{k+1}\geq M_k\geq 0$ for all $k\in\mathbb{N}$, and $\int_{\mathbb{R}}s^2\varphi(s)\,ds=1$, we obtain that $\lim_k M_k(t)=M\left(\left|\frac{t-\mu}{\varepsilon}\right|\right)$ for all $t\in\mathbb{R}$ and

$$\begin{aligned}
&\lim_k\int_{(\mu-\varepsilon k,\mu+\varepsilon k]}\left(\int_{\left[\left|\frac{t-\mu}{\varepsilon}\right|,k\right]} s\varphi(s)\,ds\right)d\gamma_1(t)\\
&=\int_{\mathbb{R}}M\left(\left|\frac{t-\mu}{\varepsilon}\right|\right)\gamma_1'(t)\,dt+\int_{\mathbb{R}}M\left(\left|\frac{t-\mu}{\varepsilon}\right|\right)d\mu_{\gamma_1,\mathrm{s}}\\
&=\int_{\mathbb{R}}M\left(\left|\frac{t-\mu}{\varepsilon}\right|\right)\gamma_1'(t)\,dt+\sum_{k=1}^{\infty}M\left(\left|\frac{t_k-\mu}{\varepsilon}\right|\right)\Delta\gamma(t_k)
\end{aligned}$$

and

$$\lim_k \int_{(\mu-\varepsilon k,\mu+\varepsilon k]} \left[\int_{\left[\left| \frac{t-\mu}{\varepsilon} \right|, k \right]} s\varphi(s)\, ds \right] d\gamma_2(t) = \int_{\mathbb{R}} M\left(\left| \frac{t-\mu}{\varepsilon} \right| \right) \lambda dt \tag{190}$$

and

$$\lim_k \int_{(\mu-\varepsilon k,\mu+\varepsilon k]} \left[\int_{\left[\left| \frac{t-\mu}{\varepsilon} \right|, k \right]} s\varphi(s)\, ds \right] d\gamma_2(t) = \lim_k \lambda\varepsilon \int_{[-k,k]} s^2 \varphi(s)\, ds = \lambda\varepsilon \in \mathbb{R}. \tag{191}$$

By (190) and (191) and since $\gamma_1'(t) = \gamma'(t) + \lambda$ for all $t \in \mathbb{R}$, it follows that

$$\lim_k \int_{(\mu-\varepsilon k,\mu+\varepsilon k]} \left(\int_{\left[\left| \frac{t-\mu}{\varepsilon} \right|, k \right]} s\varphi(s)\, ds \right) d\gamma(t) = \int_{\mathbb{R}} M\left(\left| \frac{t-\mu}{\varepsilon} \right| \right) \gamma'(t)\, dt + \sum_{k=1}^{\infty} M\left(\left| \frac{t_k-\mu}{\varepsilon} \right| \right) \Delta\gamma(t_k),$$

proving the point (a symmetric argument holds when $t \mapsto \gamma(t) - \lambda t$ is decreasing).

2. Observe that $\varphi'(s) = -s\varphi(s)$ for all $s \in \mathbb{R}$. By the Monotone Convergence Theorem and the Fundamental Theorem of Calculus and since the Riemann and the Lebesgue integral coincide over closed and bounded intervals for continuous integrands and $\varphi \geq 0$, it follows that

$$\begin{aligned} M(t) &= \lim_k \int_{[t,k]} s\varphi(s)\, ds = -\lim_k \int_{[t,k]} \varphi'(s)\, ds = -\lim_k \int_t^k \varphi'(s)\, ds \\ &= \lim_k [\varphi(t) - \varphi(k)] = \varphi(t) \quad \forall t \in [0,\infty). \end{aligned}$$

Since φ is even, it follows that $M\left(\left|\frac{t-\mu}{\varepsilon}\right|\right) = \varphi\left(\left|\frac{t-\mu}{\varepsilon}\right|\right) = \varphi\left(\frac{t-\mu}{\varepsilon}\right) = \varepsilon\varphi_{\mu,\varepsilon}(t)$ for all $t \in \mathbb{R}$, proving the statement. □

Proposition 12. *Let T be a production operator, $\beta \in (0,1)$, $\xi \in \Delta$, and $\varphi : \mathbb{R} \to [0,\infty)$ continuous, even, and such that $\int_{\mathbb{R}} \varphi(t)\, dt = 1$ and $\int_{\mathbb{R}} t^2\varphi(t)\, dt = 1$. If T is fundamentally C^2, then for each $(\mu,\varepsilon) \in \mathbb{R} \times (0,\infty)$*

$$\begin{aligned} &\frac{\partial \mathbb{E}\Upsilon}{\partial \varepsilon}(\mu,\varepsilon) = \\ &\quad -(1-\beta) \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} M\left(\left| \frac{x_n-\mu}{\varepsilon} \right| \right) \frac{\partial \lambda_n}{\partial x_n}(x_1,\ldots,x_n)\, \tilde{\varphi}(x_1,\ldots,x_{n-1})\, dx_1 \ldots dx_{n-1} dx_n \\ &\quad -(1-\beta) \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \sum_{x_n \in \mathbb{R}} M\left(\left| \frac{x_n-\mu}{\varepsilon} \right| \right) \Delta\lambda_n(x_1,\ldots,x_n)\, \tilde{\varphi}(x_1,\ldots,x_{n-1})\, dx_1 \ldots dx_{n-1} \end{aligned}$$

where $M : [0,\infty) \to [0,\infty)$ is defined by $M(t) = \int_{[t,\infty)} s\varphi(s)\, ds$ for all $t \in [0,\infty)$ and $\Delta\lambda_n(x) \leq 0$ for all $x \in \mathbb{R}^n$.

Proof. Given a vector $z \in \mathbb{R}^{n-1}$ we define $\tilde{z} \in \mathbb{R}^n$ to be such that $\tilde{z}_i = z_i$ for all $i \in [n-1]$ and

$\tilde{z}_n = 0$. Define $\hat{u} : \mathbb{R}^{n-1} \times \mathbb{R} \times (0, \infty) \to \mathbb{R}$ by

$$\hat{u}(z, \mu, \varepsilon) = \int_{\mathbb{R}} \Upsilon(\tilde{z} + te^n) \varphi_{\mu,\varepsilon}(t)\, dt \qquad \forall (\mu, \varepsilon) \in \mathbb{R} \times (0, \infty).$$

Define by $\mathcal{Z}$ the set of points z in $\mathbb{R}^{n-1}$ such that $O = \{t \in \mathbb{R} : \tilde{z} + te^n \in \mathcal{D}_\Pi\}$ is open and $\{t \in \mathbb{R} : \tilde{z} + te^n \in \mathcal{D}_\Pi\}^c$ is countable. By Lemma 13 and since T is fundamentally C^2, observe that $\mathcal{Z}^c$ has Lebesgue measure 0 in $\mathbb{R}^{n-1}$. Fix $z \in \mathcal{Z}$. Define $f_z : \mathbb{R} \to \mathbb{R}$ by $f_z(t) = \Upsilon(\tilde{z} + te^n)$ for all $t \in \mathbb{R}$. Since Υ is Lipschitz continuous of order 1, so is f. By Propositions 6 and 7 and since $\{t \in \mathbb{R} : \tilde{z} + te^n \in \mathcal{D}_\Pi\}^c$ is countable, we have that f_z has left- and right-derivative at each point of $\mathbb{R}$ with

$$f'_{z,+}(t) = \Upsilon_\circ(\tilde{z} + te^n, e^n) \text{ and } f'_{z,-}(t) = -\Upsilon_\circ(\tilde{z} + te^n, -e^n) \qquad \forall t \in \mathbb{R}$$

and, in particular, $\{t \in \mathbb{R} : f'_{z,-}(t) \neq f'_{z,+}(t)\}$ is countable. By Proposition 11, we have that $\Upsilon + \frac{\lambda}{2} \|\ \|_2^2$ is convex with $\lambda > 0$ and, in particular, $g_z : \mathbb{R} \to \mathbb{R}$, defined by $g_z(t) = \Upsilon(\tilde{z} + te^n) + \frac{\lambda}{2} \|\tilde{z} + te^n\|_2^2$ for all $t \in \mathbb{R}$, is convex too. This implies that the right-derivative $g'_{z,+}$ is well defined, increasing, right-continuous, and such that $g'_{z,+}(t) = f'_{z,+}(t) + \lambda t$ for all $t \in \mathbb{R}$. Set $\gamma_z = f'_{z,+}$. Since $f'_{z,+}(t) = g'_{z,+}(t) - \lambda t$ for all $t \in \mathbb{R}$, this implies that γ_z is right-continuous and locally of bounded variation. Since T is fundamentally C^2, we have that γ is continuously differentiable on O and O^c is countable. By Lemma 14 and since z was arbitrarily chosen, it follows that

$$\frac{\partial \hat{u}}{\partial \varepsilon}(z, \mu, \varepsilon) = \int_{\mathbb{R}} M\left(\left|\frac{t-\mu}{\varepsilon}\right|\right) \gamma'_z(t)\, dt + \sum_{k=1}^{\infty} M\left(\left|\frac{t_k-\mu}{\varepsilon}\right|\right) \Delta\gamma_z(t_k) \quad \forall (z, \mu, \varepsilon) \in \mathcal{Z} \times \mathbb{R} \times (0, \infty).$$

Next, observe that

$$\begin{aligned} \mathbb{E}\Upsilon(\mu, \varepsilon) &= \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Upsilon(x_1, ..., x_{n-1}, x_n) \varphi_{\mu,\varepsilon}(x_n)\, dx_n \right) \tilde{\varphi}(x_1, ..., x_{n-1})\, dx_1 ... dx_{n-1} \\ &= \int_{\mathbb{R}^{n-1}} \hat{u}(z, \mu, \varepsilon) \tilde{\varphi}(z)\, dz \quad \forall (\mu, \varepsilon) \in \mathbb{R} \times (0, \infty). \end{aligned}$$

By Lebesgue's Dominated Convergence Theorem and since $\mathcal{Z}^c$ has measure 0 and $\left|\frac{\hat{u}(z,\mu,\varepsilon+\delta) - \hat{u}(z,\mu,\varepsilon)}{\delta}\right| \leq 1$ for all $z \in \mathbb{R}^{n-1}$, for all $\delta \in (-\varepsilon, \infty) \setminus \{0\}$, and for all $\varepsilon \in (0, \infty)$, we can conclude that for each $(\mu, \varepsilon) \in \mathbb{R} \times (0, \infty)$

$$\begin{aligned}
&\frac{\partial \mathbb{E}\Upsilon}{\partial \varepsilon}(\mu, \varepsilon) \\
&= \int_{\mathbb{R}^{n-1}} \frac{\partial \hat{u}}{\partial \varepsilon}(z, \mu, \varepsilon) \tilde{\varphi}(z)\, dz \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} M\left(\left|\frac{t-\mu}{\varepsilon}\right|\right) \gamma_z'(t) \tilde{\varphi}(z)\, dz dt + \int_{\mathbb{R}^{n-1}} \sum_{k=1}^{\infty} M\left(\left|\frac{t_k-\mu}{\varepsilon}\right|\right) \Delta\gamma_z(t_k) \tilde{\varphi}(z)\, dz,
\end{aligned}$$

proving the statement. □

Remark 1. *As mentioned above, given f, $u(\mu, \varepsilon)$ is the expected value of f when the probability distribution function has shape φ, mean μ, and standard deviation ε. Thus, $\frac{\partial u}{\partial \varepsilon}$ is quantifying how infinitesimal changes in standard deviation impact this expectation. Clearly, given the formulas above, one can study also how infinitesimal changes in* variance *impact the same expectation. If we call v the variance, then we can define $\tilde{u} : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ by $\tilde{u}(\mu, v) = u(\mu, \sqrt{v})$ for all $(\mu, v) \in \mathbb{R} \times (0, \infty)$ which is the expected value of f when the probability distribution function has shape φ, mean μ, and variance v. By the Chain Rule and under the assumptions of Lemma 14, we have*

$$\frac{\partial \tilde{u}}{\partial v}(\mu, v) = \frac{1}{2\sqrt{v}} \frac{\partial u}{\partial \varepsilon}(\mu, \sqrt{v}) \qquad \forall (\mu, v) \in \mathbb{R} \times (0, \infty).$$

In particular, point 2 of the same lemma becomes

$$\frac{\partial \tilde{u}}{\partial v}(\mu, v) = \frac{1}{2}\left(\int_{\mathbb{R}} \varphi_{\mu,\sqrt{v}}(t) \gamma'(t)\, dt + \sum_{k=1}^{\infty} \varphi_{\mu,\sqrt{v}}(t_k) \Delta\gamma(t_k)\right) \qquad \forall (\mu, v) \in \mathbb{R} \times (0, \infty)$$

where $\varphi_{\mu,\sqrt{v}}$ is the normal density with mean μ and variance v. ▲ △

Proof of Proposition 8. It follows combining Proposition 12, Lemma 14, and Remark 1. □

F Proofs of the Results in Section 8

Define $\hat{\Upsilon} : \mathbb{R}^n \to \mathbb{R}$ as $\hat{\Upsilon}(x) = -\Upsilon(x)$. It is immediate to verify that $\hat{\Upsilon}$ is non-decreasing, normalized, and translation invariant, hence Lipschitz continuous of order 1. By Proposition 7, for every $i \in [n]$ and $x \in \mathcal{D}$, we have $\frac{\partial}{\partial x_i}\hat{\Upsilon}(x) = (1-\beta)\lambda_i(x)$. Therefore, we have $(1-\beta)\underline{\lambda}_i = \inf_{x \in \mathcal{D}} \frac{\partial}{\partial x_i}\hat{\Upsilon}(x)$ and $(1-\beta)\overline{\lambda}_i = \sup_{x \in \mathcal{D}} \frac{\partial}{\partial x_i}\hat{\Upsilon}(x)$ for all $i \in [n]$.

Lemma 15. *For every $i \in [n]$, there exists a non-decreasing function $R_i : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\hat{\Upsilon}(x) = (1-\beta)\underline{\lambda}_i x_i + (1-(1-\beta)\underline{\lambda}_i) R_i(x) \qquad \forall x \in \mathbb{R}^n. \tag{192}$$

Proof. The proof is identical to that of the beginning of the proof of Part 2 of Theorem 3 in Cerreia-Vioglio, Corrao, and Lanzani (2024). □

Proof of Proposition 9. Clearly, we have $\mathrm{Var}[\Upsilon] = \mathrm{Var}[\hat{\Upsilon}]$. We start with the lower bound. Consider any $j \in \arg\max_{i\in[n]} \mathrm{Var}[X_i]\underline{\lambda}_i^2$. We have

$$\mathrm{Var}[\hat{\Upsilon}(X)] = \mathrm{Var}[(1-\beta)\underline{\lambda}_j X_j + (1-(1-\beta)\underline{\lambda}_j)R_j(X)] \tag{193}$$

$$= (1-\beta)^2\underline{\lambda}_j^2\mathrm{Var}[X_j] + (1-(1-\beta)^2\underline{\lambda}_j)^2\mathrm{Var}[R_j(X)] \tag{194}$$

$$+ 2(1-\beta)\underline{\lambda}_j(1-(1-\beta)\underline{\lambda}_j)\mathrm{Cov}[X_j, R_j(X)] \tag{195}$$

$$\geq (1-\beta)^2\underline{\lambda}_j^2\mathrm{Var}[X_j] \tag{196}$$

where the first equality follows from Lemma 15, the second equality is standard, the first inequality follows from the fact that $\mathrm{Var}[R_j(X)] \geq 0$ and that, by Harris' inequality, $\mathrm{Cov}[X_j, R_j(X)] \geq 0$, and the second inequality follows by assumption.

The proof of the upper bound is in Proposition 1 of Cerreia-Vioglio, Corrao, Flynn, and Lanzani (2025a). □

G Relaxing Cobb-Douglas in Labor

Suppose that the equilibrium log-price equation for each firm $i \in [n]$ can be written as

$$p_i = -\ln(Z_i) + T_i(p_0, p) \tag{197}$$

where $p_0 \in \mathbb{R}$ is the log-price of the primary input and $T_i : \mathbb{R}^{n+1} \to \mathbb{R}$. This generalizes Equation 18 by allowing T_i to also depend on the price of labor. Next, suppose that there exist $\bar{\beta} \in (0,1)$ and a normalized, non-decreasing, and translation invariant $S : \mathbb{R}^{n+1} \to \mathbb{R}^n$ such that

$$T_i(p_0, p) = \left(1-\bar{\beta}\right)p_0 + \bar{\beta}S_i(p_0, p) \tag{198}$$

for all $(p_0, p) \in \mathbb{R}^{n+1}$ and $i \in [n]$. Assume also that S is a valid log-cost operator.[21] The previous fixed-point equation then becomes

$$p_i = \left(1-\bar{\beta}\right)(x_i + p_0) + \bar{\beta}S_i(p_0, p) \qquad \forall i \in [n] \tag{199}$$

[21]This is tantamount to assuming that the production function of sector i can be written as

$$F_i(Q_0, Q) = Z_i\left(\frac{Q_0}{1-\bar{\beta}}\right)^{1-\bar{\beta}}\left(\frac{G_i(Q_0, Q)}{\bar{\beta}}\right)^{\bar{\beta}}$$

where $G_i : \mathbb{R}^{n+1} \to \mathbb{R}$ is positive homogeneous, non-decreasing, continuous, and with $\max_{\left\{Q_0, Q : Q_0 + \sum_{i=1}^n Q_i = 1\right\}} G_i(Q_0, Q) = 1$.

where, as in the main text, $x_i = \frac{-\ln(Z_i)}{\left(1-\bar{\beta}\right)}$ for all $i \in [n]$. Let $\Pi\left(p_0, x\right) \in \mathbb{R}^n$ denote the unique solution of the previous fixed-point equation indexed to $(p_0, x) \in \mathbb{R}^{n+1}$. This is well-defined because $y \mapsto \left(1-\bar{\beta}\right)\left(x+p_0 e\right)+\bar{\beta} S\left(p_0, y\right)$ is a contraction, hence admits a unique fixed point. In our main analysis, we always focus on $\Pi\left(0, x\right)$ since it is always possible to normalize $p_0 = 0$.

Consider the cloned economy where, for each original sector $i \in [n]$, we generate two sectors given by the original one i and a new cloned sector $i+n$. With this, the new cloned economy has $2n$ sectors. Define $\tilde{S} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ as

$$\tilde{S}_j\left(\tilde{p}\right) = \begin{cases} S_j\left(\tilde{p}_{j+n}, \tilde{p}_1, ..., \tilde{p}_n\right) & \text{if} \quad j \in \{1, ..., n\}, \\ \tilde{p}_j & \text{if} \quad j \in \{n+1, ..., 2n\}. \end{cases} \tag{200}$$

Intuitively, to fit this more general dependence on labor in the baseline framework of the paper, for every sector i, we are adding an extra sector $i+n$ that transforms the household's labor supply in labor services sold to sector i (this is a fairly standard approach, see, e.g., La'O and Tahbaz-Salehi (2022)). It is immediate to see that $\tilde{S}$ is normalized, monotone, and translation invariant. Assume that $\tilde{S}$ represents the log-cost operator of our cloned economy.

The equilibrium price equation in the cloned economy parametrized to the log-cost of labor $p_0 \in \mathbb{R}$ and the cloned log-productivity $\tilde{x} \in \mathbb{R}^{2n}$ is

$$\tilde{p} = \left(1-\bar{\beta}\right)\left(\tilde{x}+p_0 e\right)+\bar{\beta}\tilde{S}\left(\tilde{p}\right). \tag{201}$$

Call $\tilde{\Pi}\left(p_0, \tilde{x}\right) \in \mathbb{R}^{2n}$ the unique solution of the previous fixed-point equation.

Lemma 16. *Define*

$$\tilde{x}_j = \begin{cases} x_j & \textit{if} \quad j \in \{1, .., n\}, \\ 0 & \textit{if} \quad j \in \{n+1, ...2n\}. \end{cases} \tag{202}$$

We have

$$\tilde{\Pi}_j\left(0, \tilde{x}\right) = \begin{cases} \Pi_j\left(0, x\right) & \textit{if} \quad j \in \{1, .., n\}, \\ 0 & \textit{if} \quad j \in \{n+1, ...2n\}. \end{cases} \tag{203}$$

Proof. That $\tilde{\Pi}_j\left(0, \tilde{x}\right) = 0$ for all $j \in \{n+1, ...2n\}$ follows from

$$\tilde{\Pi}_j\left(0, \tilde{x}\right) = \left(1-\bar{\beta}\right)\left(\tilde{x}_j+0\right)+\bar{\beta}\tilde{S}_j\left(\tilde{\Pi}\left(0, \tilde{x}\right)\right) = \bar{\beta}\tilde{\Pi}_j\left(0, \tilde{x}\right) \qquad \forall j \in \{n+1, ...2n\}. \tag{204}$$

With this, we have that for all $j \in [n]$

$$\tilde{\Pi}_j\left(0, \tilde{x}\right) = \left(1-\bar{\beta}\right)\left(\tilde{x}_j+0\right)+\bar{\beta}\tilde{S}_j\left(\tilde{\Pi}\left(0, \tilde{x}\right)\right) = \left(1-\bar{\beta}\right)x_j+\bar{\beta}S_j\left(0, \tilde{\Pi}\left(0, \tilde{x}\right)_1, ..., \tilde{\Pi}\left(0, \tilde{x}\right)_n\right). \tag{205}$$

Since this coincides with the system of equations in (199) that defines $\Pi_j\left(0, x\right)$ and it admits a unique solution, we obtain $\tilde{\Pi}_j\left(0, \tilde{x}\right) = \Pi_j\left(0, x\right)$ for all $j \in \{1, ...n\}$. □

Crucially, the economy described by the log-cost operator of Equation (200) satisfies our Assumption 1, and therefore, all our results can be applied to it verbatim. Given the relation established between the prices of the cloned and original economy, these results are easily transferred to the latter, too.

H Production Theory

Given $n \in \mathbb{N}$, we define $[n] = \{1, ..., n\}$ and we consider $\mathbb{R}^n$ endowed with the supnorm $\| \|_\infty$. We denote by Δ the simplex of nonnegative vectors of $\mathbb{R}^n$ whose components sum up to 1. We define by $\operatorname{int}\Delta$ the subset of vectors in Δ whose components are all strictly positive. Given two vectors w and $\tilde{w}$ in Δ, we write $w \ll \tilde{w}$ whenever, given $j \in [n]$, $\tilde{w}_j = 0$ implies $w_j = 0$. As usual, $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathbb{R}^n$. We denote by e the vector whose components are all 1. We define by $\mathcal{G}$ the collection of all production functions $G : \mathbb{R}^n_+ \to \mathbb{R}_+$ that are nondegenerate,[22] monotone, upper semicontinuous, and positive homogeneous.[23] We denote by $\mathcal{G}_{\text{nor}}$ the collection of all production functions $G : \mathbb{R}^n_+ \to \mathbb{R}_+$ that are monotone, upper semicontinuous, positive homogeneous, and such that $\max_{w \in \Delta} G(w) = 1$. Clearly, we have that $\mathcal{G}_{\text{nor}} \subseteq \mathcal{G}$. We define by $\mathcal{G}_{\text{con}}$ (resp. $\mathcal{G}_{\text{qcon}}$) the subset of functions in $\mathcal{G}$ that are concave (resp. quasiconcave). We set $\mathcal{G}_{\text{nor}-\text{con}} = \mathcal{G}_{\text{nor}} \cap \mathcal{G}_{\text{con}}$. Given $G \in \mathcal{G}$, we define the expenditure function of G at 1, denoted $K_G : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$, by

$$K_G(P) = \min_{Q \in \mathbb{R}^n_+ : G(Q) \geq 1} \sum_{j=1}^{n} Q_j P_j \qquad \forall P \in \mathbb{R}^n_{++}.$$

We will show that indeed K_G is well defined (cf. Lemma 17). We will also show that K_G is monotone, continuous, concave, and positive homogeneous. We denote the class of functions $K : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ with these four properties by $\mathcal{K}$. We also denote by $\mathcal{K}_{\text{nor}}$ the subset of functions of $\mathcal{K}$ that are such that $K(e) = 1$. A goal of this note is to show that $G \mapsto K_G$ is a bijection from $\mathcal{G}_{\text{nor}-\text{con}}$ onto $\mathcal{K}_{\text{nor}}$. Finally, given a function $K \in \mathcal{K}$, we define $\kappa : \mathbb{R}^n \to \mathbb{R}$ by $\kappa(p) = \ln(K(\exp(p)))$ for all $p \in \mathbb{R}^n$ where $\exp(p)$ is the vector whose j-th component is e^{p_j} for all $j \in [n]$. We conclude the paper by characterizing convexity and concavity of κ in terms of G. Throughout the note, we assume that $0 \cdot \pm\infty = 0$, $0/0 = 0$, $\infty^0 = 1$, $0^0 = 1$, $\ln\infty = \infty$, and $\ln 0 = -\infty$. Given a nonempty set X and a function $f : X \to \mathbb{R}$ and $t \in \mathbb{R}$, we denote by $\{f \geq t\} = \{x \in X : f(x) \geq t\}$ the upper contour set of f at t. Similarly, we define the lower contour set $\{f \leq t\}$.

[22] That is, $G(e) > 0$.
[23] That is, $G(\lambda Q) = \lambda G(Q)$ for all $\lambda > 0$ and for all $Q \in \mathbb{R}^n_+$.

Remark 2. *In what follows, whenever $G \in \mathcal{G}$, it will automatically be granted that G is continuous* by the classical result of Moore (1999). $\triangle$

Remark 3. *An important class of production functions are Cobb-Douglas production functions. Given a vector $w \in \Delta$ and a number $A > 0$, define $G_w : \mathbb{R}^n_+ \to \mathbb{R}_+$ by $G_w(Q) = A\Pi_{j:w_j>0}\left(\frac{Q_j}{w_j}\right)^{w_j}$ for all $Q \in \mathbb{R}^n_+$. It is well known that $G_w \in \mathcal{G}_{\text{con}}$ for all $w \in \Delta$. In what follows, it might be useful to think of A and G_w also as functions of w. We begin by considering a convex subset $\tilde{C}$ of Δ and $A : \operatorname{cl}\tilde{C} \to [0,1]$ such that $A(w) > 0$ if and only if $w \in \tilde{C}$. We also define $d : \operatorname{cl}\tilde{C} \to [0,\infty]$ by $d(w) = -\ln A(w)$ for all $w \in \operatorname{cl}\tilde{C}$. Note that $A : C \to [0,1]$ is upper semicontinuous, log-concave, and such that $\max_{w\in C} A(w) = 1$ if and only if d is normalized, lower semicontinuous, and convex. To proceed in considering the function $(w,Q) \mapsto G_w(Q)$, we introduce a final object: the function $f : [0,1] \times [0,\infty) \to \mathbb{R}_+$ defined by*

$$f(s,t) = \begin{cases} \left(\frac{t}{s}\right)^s & \text{if } s > 0 \\ 1 & \text{if } s = 0 \end{cases} \qquad \forall (s,t) \in [0,1] \times [0,\infty).$$

Given our conventions, note that

$$\begin{aligned} G_w(Q) &= A(w)\,\Pi_{j:w_j>0}\left(\frac{Q_j}{w_j}\right)^{w_j} = A(w)\,\Pi_{j=1}^n\left(\frac{Q_j}{w_j}\right)^{w_j} \\ &= A(w)\,\Pi_{j=1}^n f(w_j, Q_j) \quad \forall (w,Q) \in \operatorname{cl}\tilde{C} \times \mathbb{R}^n_+. \end{aligned}$$

Note that f is upper semicontinuous. Since the product of nonnegative upper semicontinuous functions is upper semicontinuous, If A is upper semicontinuous, then we have that $(w,Q) \mapsto G_w(Q)$ is upper semicontinuous. Finally, consider $t > 0$ and $P \in \mathbb{R}^n_+\setminus\{0\}$. It is well known that

$$\sup_{Q\in\mathbb{R}^n_+:\langle Q,P\rangle\le t} G_w(Q) = tA(w)\,\Pi_{j=1}^n P_j^{-w_j}.$$

If A is upper semicontinuous and log-concave, then we have that $w \mapsto te^{-d(w)+\sum_{j=1}^n \ln(P_j)w_j}$ is upper semicontinuous and quasiconcave on $\tilde{C}$. $\triangle$

H.1 Expenditure functions

We move to study the expenditure functions induced by our class of production functions. Given $G \in \mathcal{G}$, recall that $K_G : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ is defined by

$$K_G(P) = \min_{Q\in\mathbb{R}^n_+:G(Q)\ge 1} \sum_{j=1}^n Q_j P_j \qquad \forall P \in \mathbb{R}^n_{++}.$$

We also define $U_G : \mathbb{R}_+ \times \mathbb{R}^n_+ \to [0, \infty]$ and $U_{G,\text{int}} : \mathbb{R}_{++} \times \mathbb{R}^n_+ \to [0, \infty]$ by

$$U_G(t, P) = \sup \left\{ G(Q) : Q \in \mathbb{R}^n_+ \text{ and } \langle Q, P \rangle \leq t \right\} \quad \forall (t, P) \in \mathbb{R}_+ \times \mathbb{R}^n_+$$

and

$$U_{G,\text{int}}(t, P) = \sup \left\{ G(Q) : Q \in \mathbb{R}^n_{++} \text{ and } \langle Q, P \rangle \leq t \right\} \quad \forall (t, P) \in \mathbb{R}_{++} \times \mathbb{R}^n_+.$$

We next show that expenditure functions inherit most of the functional properties defining the functions in $\mathcal{G}$. In particular, the next lemma shows that if $G \in \mathcal{G}$ (resp. $\mathcal{G}_{\text{nor}}$), then $K_G \in \mathcal{K}$ (resp. $\mathcal{K}_{\text{nor}}$).

Lemma 17. *If $G \in \mathcal{G}$ (resp. $G \in \mathcal{G}_{\text{nor}}$), then K_G is well defined and belongs to $\mathcal{K}$ (resp. $\mathcal{K}_{\text{nor}}$).*

Proof. Since G is continuous and positive homogeneous, we have that $G(0) = 0$. Fix $P \in \mathbb{R}^n_{++}$. Since G is nondegenerate and positive homogeneous, denote by $\hat{Q}$ an element in $\mathbb{R}^n_+$ such that $G\left(\hat{Q}\right) = 1$. Since G is monotone, continuous, positive homogeneous, and such that $G(0) = 0$, we have that

$$\inf_{Q \in \mathbb{R}^n_+ : G(Q) \geq 1} \sum_{j=1}^n Q_j P_j = \inf_{Q \in \mathbb{R}^n_+ : G(Q) = 1} \sum_{j=1}^n Q_j P_j = \inf_{Q \in \mathbb{R}^n_+ : G(Q) = 1 \text{ and } \sum_{j=1}^n Q_j P_j \leq \sum_{j=1}^n \hat{Q}_j P_j} \sum_{j=1}^n Q_j P_j$$
$$= \min_{Q \in \mathbb{R}^n_+ : G(Q) = 1 \text{ and } \sum_{j=1}^n Q_j P_j \leq \sum_{j=1}^n \hat{Q}_j P_j} \sum_{j=1}^n Q_j P_j > 0,$$

proving that the inf is attained and K_G is well defined. Next, for each $Q \in \{G \geq 1\}$ define $\ell_Q : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ by $\ell_Q(P) = \langle Q, P \rangle$ for all $P \in \mathbb{R}^n_{++}$. Since $K_G = \inf_{Q \in \{G \geq 1\}} \ell_Q$ and each ℓ_Q is monotone, continuous, affine, and positive homogeneous, we have that K_G is monotone, concave, and positive homogeneous. By (Borwein and Lewis, 2006, Theorem 4.1.3) and since K_G is concave, it follows that K_G is continuous, proving the main statement. Finally, if $\max_{w \in \Delta} G(w) = 1$, then $\hat{Q} = \ddot{w}$ can be chosen to be in Δ. This implies that $1 = \sum_{j=1}^n \ddot{w}_j = \sum_{j=1}^n \ddot{w}_j e_j \geq K_G(e)$ where $\ddot{w} \in \Delta$ is such that $G(\ddot{w}) = 1$. Next, consider $Q \in \mathbb{R}^n_+$ such that $G(Q) \geq 1$ and $\sum_{j=1}^n Q_j = K_G(e) \leq 1$. Since $G(Q) \geq 1$, it follows that $Q \neq 0$ and $\sum_{j=1}^n Q_j = \|Q\|_1 \in (0, 1]$. Since $\max_{w \in \Delta} G(w) = 1$ and $G(Q) \geq 1$, this implies that $\hat{w} = Q / \|Q\|_1 \in \Delta$ is such that $1 \geq G(\hat{w}) = G(Q) / \|Q\|_1 \geq 1$, proving that $G(Q) / \|Q\|_1 = 1$, that is, $1 \leq G(Q) = \|Q\|_1 \leq 1$. It follows that $K_G(e) = \sum_{j=1}^n Q_j = \|Q\|_1 = 1$, proving the statement. $\square$

We next study some of the properties of the indirect utility U_G and relate it to K_G. Those are classic results. We begin with an ancillary fact.

Lemma 18. *If $G \in \mathcal{G}$, then $U_G(t, P) \in (0, \infty)$ for all $(t, P) \in \mathbb{R}_{++} \times \mathbb{R}^n_{++}$ and $U_G(t, P) = U_{G,\text{int}}(t, P)$ for all $(t, P) \in \mathbb{R}_{++} \times \mathbb{R}^n_+ \setminus \{0\}$.*

Proof. Consider $(t,P) \in \mathbb{R}_{++} \times \mathbb{R}^n_{++}$ set $\lambda = \left(\sum_{j=1}^n P_j\right)^{-1} t \in (0,\infty)$ and $Q = \lambda e \in \mathbb{R}^n_{++}$. Since G is nondegenerate and positive homogeneous, we have that $\langle Q,P\rangle = t$ and $0 < \lambda G(e) = G(\lambda e) = G(Q) \leq U_G(t,P)$. At the same time, since $(t,P) \in \mathbb{R}_{++} \times \mathbb{R}^n_{++}$, we have that $\left\{Q \in \mathbb{R}^n_+ : \langle Q,P\rangle \leq t\right\}$ is closed and bounded and, in particular, compact. Since G is upper semicontinuous and real-valued, it follows that the sup defining $U_G(t,P)$ is attained and real-valued too. Since (t,P) was arbitrarily chosen, the first part of the statement follows. Consider $(t,P) \in \mathbb{R}_{++} \times \mathbb{R}^n_+ \backslash \{0\}$. Since $P \in \mathbb{R}^n_+ \backslash \{0\}$, define $\sigma = \sum_{j=1}^n P_j > 0$. Clearly, we have that $\left\{Q \in \mathbb{R}^n_+ : \langle Q,P\rangle \leq t\right\}$ is a closed set and, in particular, $\mathrm{cl}\left\{Q \in \mathbb{R}^n_{++} : \langle Q,P\rangle \leq t\right\} \subseteq \left\{Q \in \mathbb{R}^n_+ : \langle Q,P\rangle \leq t\right\}$. Consider $Q \in \mathbb{R}^n_+$ such that $\langle Q,P\rangle \leq t$. We have two cases: either $\langle Q,P\rangle < t$ or $\langle Q,P\rangle = t$. In the first case, define $s = t - \langle Q,P\rangle > 0$ and for each $k \in \mathbb{N}$ define $Q^k = Q + \frac{s}{\sigma k} e \in \mathbb{R}^n_{++}$. Note that $\left\langle Q^k,P\right\rangle = \langle Q,P\rangle + \frac{s}{k} \leq \langle Q,P\rangle + s = t$ for all $k \in \mathbb{N}$ and $\lim_k Q^k = Q$, proving that $Q \in \mathrm{cl}\left\{Q \in \mathbb{R}^n_{++} : \langle Q,P\rangle \leq t\right\}$. In the second case, since $t > 0$ and $P \in \mathbb{R}^n_+ \backslash \{0\}$, the condition $\langle Q,P\rangle = t$ implies that $Q_{\bar{j}} P_{\bar{j}} > 0$ for some $\bar{j} \in [n]$ and, in particular, $Q_{\bar{j}} > 0$ and $P_{\bar{j}} > 0$ for some $\bar{j} \in [n]$. For each $k \in \mathbb{N}$ define $s_k = \frac{1}{k+1} Q_{\bar{j}} P_{\bar{j}} > 0$ and define $Q^k = Q - \frac{1}{k+1} Q_{\bar{j}} e^{\bar{j}} + \frac{s_k}{\sigma} e \in \mathbb{R}^n_{++}$. Note that $\left\langle Q^k,P\right\rangle = \langle Q,P\rangle - s_k + s_k = \langle Q,P\rangle = t$ for all $k \in \mathbb{N}$ and $\lim_k Q^k = Q$, proving that $Q \in \mathrm{cl}\left\{Q \in \mathbb{R}^n_{++} : \langle Q,P\rangle \leq t\right\}$. Since Q was arbitrarily chosen, we have that $\left\{Q \in \mathbb{R}^n_+ : \langle Q,P\rangle \leq t\right\} \subseteq \mathrm{cl}\left\{Q \in \mathbb{R}^n_{++} : \langle Q,P\rangle \leq t\right\}$ and, in particular, $\left\{Q \in \mathbb{R}^n_+ : \langle Q,P\rangle \leq t\right\} = \mathrm{cl}\left\{Q \in \mathbb{R}^n_{++} : \langle Q,P\rangle \leq t\right\}$. Since G is continuous, we have that

$$U_G(t,P) = \sup_{Q \in \mathbb{R}^n_+ : \langle Q,P\rangle \leq t} G(Q) = \sup_{Q \in \mathbb{R}^n_{++} : \langle Q,P\rangle \leq t} G(Q) = U_{G,\mathrm{int}}(t,P).$$

Since (t,P) was arbitrarily chosen, the second part of the statement follows. □

Lemma 19. *If $G \in \mathcal{G}$, then $U_G(K_G(P),P) = 1$ for all $P \in \mathbb{R}^n_{++}$.*

Proof. Fix $P \in \mathbb{R}^n_{++}$. Consider $\bar{Q} \in \mathbb{R}^n_+$ such that $G\left(\bar{Q}\right) \geq 1$ and $K_G(P) = \sum_{j=1}^n \bar{Q}_j P_j$. Since $\sum_{j=1}^n \bar{Q}_j P_j \leq K_G(P)$, we have that $U(K_G(P),P) \geq G\left(\bar{Q}\right) \geq 1$. By contradiction, assume that $U_G(K_G(P),P) > 1$. Since G is upper semicontinuous and $P \in \mathbb{R}^n_{++}$, consider $\hat{Q} \in \mathbb{R}^n_+$ such that $\sum_{j=1}^n \hat{Q}_j P_j \leq K_G(P)$ and $G\left(\hat{Q}\right) = U_G(K_G(P),P) > 1$. By definition of K_G and since $G\left(\hat{Q}\right) > 1$, this would imply that $\sum_{j=1}^n \hat{Q}_j P_j \geq K_G(P)$, yielding that $K_G(P) = \sum_{j=1}^n \hat{Q}_j P_j$. Define $\gamma = 1/G\left(\hat{Q}\right) \in (0,1)$ and $\tilde{Q} = \gamma \hat{Q} \in \mathbb{R}^n_+$. Since G is positive homogeneous and $\sum_{j=1}^n \hat{Q}_j P_j = K_G(P) > 0$, we would have that $G\left(\tilde{Q}\right) = 1$ and

$$\sum_{j=1}^n \tilde{Q}_j P_j = \gamma \sum_{j=1}^n \hat{Q}_j P_j < \sum_{j=1}^n \hat{Q}_j P_j = K_G(P) \leq \sum_{j=1}^n \tilde{Q}_j P_j,$$

a contradiction. □

H.2 Expenditure functions and lattice operations

We next explore the question of whether lattice operations on production functions translate in lattice operations over expenditure functions. This will play a role in characterizing when a (log-expenditure) function κ is either convex or concave. We first show that the expenditure function of a production function which is the infimum of production functions is the maximum of the corresponding expenditure functions, provided some extra conditions are met. To this extent, given a function from a nonempty set C of a normed vector space to the real line, we say that it is lower (resp. upper) coercive if and only if its lower (resp. upper) contour sets are compact.

Lemma 20. *Let $\tilde{C}$ be a nonempty precompact convex subset of a normed vector space and $\mathfrak{G} : \tilde{C} \times \mathbb{R}^n_+ \to \mathbb{R}_+$ be such that $w \mapsto \mathfrak{G}(w, Q)$ is lower coercive and quasiconvex for all $Q \in \mathbb{R}^n_{++}$ and $\mathfrak{G}(w, \cdot) \in \mathcal{G}_{\text{con}}$ for all $w \in \tilde{C}$. If $G : \mathbb{R}^n_+ \to \mathbb{R}_+$ is such that*

$$G(Q) = \inf_{w \in \tilde{C}} \mathfrak{G}(w, Q) \qquad \forall Q \in \mathbb{R}^n_+,$$

then the inf *is attained on $\mathbb{R}^n_{++}$ and G is well defined and belongs to $\mathcal{G}_{\text{con}}$. Moreover, we have that $K_G = \max_{w \in \tilde{C}} K_w$ where $K_w = K_{\mathfrak{G}(w,\cdot)}$.*

Proof. Since $Q \mapsto \mathfrak{G}(w, Q)$ is nonnegative, monotone, upper semicontinuous, concave, and positive homogeneous for all $w \in \tilde{C}$, it is plain that G is well defined, monotone, upper semicontinuous, concave, and positive homogeneous. By Remark 2, G is continuous. Since $w \mapsto \mathfrak{G}(w, Q)$ is lower coercive for all $Q \in \mathbb{R}^n_{++}$, we have that $G(Q) = \min_{w \in \tilde{C}} \mathfrak{G}(w, Q)$ for all $Q \in \mathbb{R}^n_{++}$. In particular, this implies that G is nondegenerate and, in particular, $G \in \mathcal{G}$. Define $\bar{\mathfrak{G}} : \operatorname{cl} \tilde{C} \times \mathbb{R}^n_+ \to [0, \infty]$ by

$$\bar{\mathfrak{G}}(w, Q) = \begin{cases} \mathfrak{G}(w, Q) & \text{if } w \in \tilde{C} \\ \infty & \text{if } w \notin \tilde{C} \end{cases} \qquad \forall (w, Q) \in \operatorname{cl} \tilde{C} \times \mathbb{R}^n_+.$$

By definition of $\bar{\mathfrak{G}}$ and since $w \mapsto \mathfrak{G}(w, Q)$ is lower coercive and quasiconvex for all $Q \in \mathbb{R}^n_{++}$ on $\tilde{C}$, we have that $w \mapsto \bar{\mathfrak{G}}(w, Q)$ is lower semicontinuous and quasiconvex on $\operatorname{cl} \tilde{C}$ for all $Q \in \mathbb{R}^n_{++}$. Similarly, $Q \mapsto \bar{\mathfrak{G}}(w, Q)$ is monotone, continuous, concave, and positive homogeneous on $\mathbb{R}^n_+$ for all $w \in \operatorname{cl} \tilde{C}$. Fix $P \in \mathbb{R}^n_{++}$. Note that if $Q \in \mathbb{R}^n_+$ and $G(Q) \geq 1$, then $\mathfrak{G}(w, Q) \geq 1$ for all $w \in \tilde{C}$, proving that

$$\sum_{j=1}^{n} Q_j P_j \geq K_w(P) \quad \forall w \in \tilde{C} \implies \sum_{j=1}^{n} Q_j P_j \geq \sup_{w \in \tilde{C}} K_w(P) \implies K_G(P) \geq \sup_{w \in \tilde{C}} K_w(P).$$

Define $U_w = U_{\mathfrak{G}(w,\cdot)}$ and $U_{w,\text{int}} = U_{\mathfrak{G}(w,\cdot),\text{int}}$ for all $w \in \tilde{C}$. Observe that $G(Q) = \min_{w \in \tilde{C}} \mathfrak{G}(w, Q) = \min_{w \in \tilde{C}} \bar{\mathfrak{G}}(w, Q) = \min_{w \in \operatorname{cl} \tilde{C}} \bar{\mathfrak{G}}(w, Q)$ for all $Q \in \mathbb{R}^n_{++}$. By Sion's Maxmin Theorem and

Lemma 18 and since $w \mapsto \bar{\mathfrak{G}}(w, Q)$ is lower semicontinuous and quasiconvex for all $Q \in \mathbb{R}^n_{++}$ and $Q \mapsto \bar{\mathfrak{G}}(w, Q)$ is upper semicontinuous and quasiconcave for all $w \in \operatorname{cl} \tilde{C}$, note that

$$\begin{aligned} U_G(t, P) = U_{G,\mathrm{int}}(t, P) &= \sup_{Q \in \mathbb{R}^n_{++} : \sum_{j=1}^n Q_j P_j \le t} \inf_{w \in \operatorname{cl} \tilde{C}} \bar{\mathfrak{G}}(w, Q) = \inf_{w \in \operatorname{cl} \tilde{C}} \sup_{Q \in \mathbb{R}^n_{++} : \sum_{j=1}^n Q_j P_j \le t} \bar{\mathfrak{G}}(w, Q) \\ &= \min_{w \in \operatorname{cl} \tilde{C}} \sup_{Q \in \mathbb{R}^n_{++} : \sum_{j=1}^n Q_j P_j \le t} \bar{\mathfrak{G}}(w, Q) = \min_{w \in \tilde{C}} \sup_{Q \in \mathbb{R}^n_{++} : \sum_{j=1}^n Q_j P_j \le t} \bar{\mathfrak{G}}(w, Q) \\ &= \min_{w \in \tilde{C}} U_{w,\mathrm{int}}(t, P) = \min_{w \in \tilde{C}} U_w(t, P) \quad \forall t > 0, \end{aligned}$$

proving that $U_G(t, P) = \min_{w \in \tilde{C}} U_w(t, P)$ for all $t > 0$. By Lemma 19, we have that $1 = U_G(K_G(P), P) = \min_{w \in \tilde{C}} U_w(K_G(P), P)$ and $U_w(K_w(P), P) = 1$ for all $w \in \tilde{C}$ and, in particular, there exists $\ddot{w} \in \tilde{C}$ such that $1 = U_G(K_G(P), P) = U_{\ddot{w}}(K_G(P), P)$. Since $P \in \mathbb{R}^n_{++}$ and $\mathfrak{G}(\ddot{w}, \cdot)$ is upper semicontinuous, this implies that $U_{\ddot{w}}(1, P) \in (0, \infty)$. Since $U_{\ddot{w}}(K_{\ddot{w}}(P), P) = 1$ and $t \mapsto U_{\ddot{w}}(t, \ddot{w})$ is real-valued and positive homogeneous, it follows that

$$K_{\ddot{w}}(P) U_{\ddot{w}}(1, P) = U_{\ddot{w}}(K_{\ddot{w}}(P), P) = 1 = U_{\ddot{w}}(K_G(P), P) = K_G(P) U_{\ddot{w}}(1, P),$$

proving that $K_G(P) = K_{\ddot{w}}(P) \le \sup_{w \in \tilde{C}} K_w(P)$. Since $K_G(P) \ge \sup_{w \in \tilde{C}} K_w(P)$ and P was arbitrarily chosen, we can conclude that

$$K_G(P) = \max_{w \in \tilde{C}} K_w(P) \quad \forall P \in \mathbb{R}^n_{++},$$

proving the statement. □

H.3 Representation result

We can now move to our first representation result. We provide a representation of the expenditure function in terms of its generating production function. The main representation result will follow and build on this and provide a canonical representation in terms of an alternative and *unique concave* production function.

Lemma 21. *If $G \in \mathcal{G}$, then*

$$K_G(P) = \min_{w \in \Delta} \frac{\langle w, P \rangle}{G(w)} \quad \forall P \in \mathbb{R}^n_{++}.$$

Moreover, $K_G(e) = 1$ if and only if $\max_{w \in \Delta} G(w) = 1$ (that is $G \in \mathcal{G}_{\mathrm{nor}}$).

Proof. Fix $P \in \mathbb{R}^n_{++}$ and $w \in \Delta$ such that $G(w) \neq 0$. Define $Q = \frac{1}{G(w)} w \in \mathbb{R}^n_{+}$. Since G is positive homogeneous, we have that $G(Q) = 1$. By definition of Q and K_G, it follows that

$$\frac{\langle w, P\rangle}{G(w)} = \sum_{j=1}^{n} Q_j P_j \geq K_G(P).$$

Since w was arbitrarily chosen, it follows that $\min_{w\in\Delta} \frac{\langle w,P\rangle}{G(w)} \geq \min_{w\in\Delta: G(w)\neq 0} \frac{\langle w,P\rangle}{G(w)} \geq K_G(P)$. Next, fix $Q \in \mathbb{R}^n_+$ such that $G(Q) \geq 1$. Since G is nondegenerate and positive homogeneous, this implies that $\sum_{j=1}^n Q_j > 0$.[24] Define $w = \frac{1}{\sum_{j=1}^n Q_j} Q \in \Delta$. Since $G(Q) \geq 1$, it follows that

$$\sum_{j=1}^{n} Q_j P_j \geq \frac{\sum_{j=1}^n Q_j P_j}{G(Q)} = \frac{\sum_{j=1}^n w_j P_j}{G(w)}$$

where the equality follows from positive homogeneity of G. Because Q was arbitrarily chosen, it follows that $K_G(P) \geq \min_{w\in\Delta} \frac{\langle w,P\rangle}{G(w)}$, proving the main statement. Finally, observe that $K_G(e) = 1$ if and only if $\min_{w\in\Delta} \frac{1}{G(w)} = 1$ if and only if $\max_{w\in\Delta} G(w) = 1$.

□

Theorem 3. *Let $K : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$. The following statements are equivalent:*

(i) $K \in \mathcal{K}_{\text{nor}}$*;*

(ii) *K is monotone, continuous, quasiconcave, positive homogeneous, and such that $K(e) = 1$;*

(iii) *There exists an upper semicontinuous and concave function $\Gamma : \Delta \to [0, \infty)$ such that $\max_{w\in\Delta} \Gamma(w) = 1$ and*

$$K(P) = \min_{w\in\Delta} \frac{\langle w, P\rangle}{\Gamma(w)} \qquad \forall P \in \mathbb{R}^n_{++}. \tag{206}$$

(iv) *There exists $G^\star \in \mathcal{G}_{\text{nor}-\text{con}}$ such that $K = K_{G^\star}$.*

(v) *There exists $G^\star \in \mathcal{G}_{\text{nor}-\text{con}}$ such that*

$$K(P) = \min_{w\in\Delta} \frac{\langle w, P\rangle}{G^\star(w)} \qquad \forall P \in \mathbb{R}^n_{++}.$$

Moreover, we have that:

1. *$\Gamma : \Delta \to [0, \infty)$ is such that*

$$\Gamma(w) = \frac{1}{\sup\{K(P) : P \in \mathbb{R}^n_{++} \text{ and } \langle w, P\rangle \leq 1\}} \qquad \forall w \in \Delta$$

and it is the unique upper semicontinuous and concave function $\tilde{\Gamma}$ satisfying (206) and such that $\max_{w\in\Delta} \tilde{\Gamma}(w) = 1$.

[24] Otherwise, since $Q \in \mathbb{R}^n_+$, we would have that $Q = 0$ and, in particular, $G(Q) = 0$, a contradiction.

2. *$G^\star$ is the unique element in $\mathcal{G}_{\text{nor}-\text{con}}$ such that $K = K_{G^\star}$.*
3. *$\Gamma(w) = G^\star(w)$ for all $w \in \Delta$. In particular, Γ is continuous and $G^\star(0) = 0$ and $G^\star(Q) = \|Q\|_1 \Gamma(Q/\|Q\|_1)$ for all $Q \in \mathbb{R}^n_+ \setminus \{0\}$.*
4. *If $G \in \mathcal{G}_{\text{nor}}$ and $K = K_G$, then $G^\star = \min\left\{\hat{G} : \hat{G} \in \mathcal{G}_{\text{nor}-\text{con}} \text{ and } \hat{G} \geq G\right\}$.*

Proof. (i) implies (ii). By definition of $\mathcal{K}_{\text{nor}}$, it is obvious.

(ii) implies (iii) and point 1. Call $\bar{K}$ the extension of K to $\mathbb{R}^n$ such that $\bar{K}(P) = 0$ for all $P \notin \mathbb{R}^n_{++}$. Since $K > 0$ and K is monotone and positive homogeneous, it is immediate to see that $\bar{K}$ is monotone and positive homogeneous. Since $\mathbb{R}^n_{++}$ is open and K is continuous on $\mathbb{R}^n_{++}$ and $K > 0$, we have that

$$\left\{\bar{K} > t\right\} = \left\{P \in \mathbb{R}^n_{++} : K(P) > t\right\} \qquad \forall t \geq 0.$$

It follows that the latter set is relatively open in $\mathbb{R}^n_{++}$, proving that the latter and the former are open. Moreover, since $\bar{K} \geq 0$, we have that $\left\{\bar{K} > t\right\} = \mathbb{R}^n$ for all $t < 0$, proving that $\bar{K}$ is lower semicontinuous. Since K is quasiconcave on $\mathbb{R}^n_{++}$ and $K > 0$, we have that

$$\left\{\bar{K} \geq t\right\} = \left\{P \in \mathbb{R}^n_{++} : K(P) \geq t\right\} \qquad \forall t > 0.$$

It follows that the first set is convex. Moreover, since $\bar{K} \geq 0$, we have that $\left\{\bar{K} \geq t\right\} = \mathbb{R}^n$ for all $t \leq 0$, proving that $\bar{K}$ is quasiconcave. By (Cerreia-Vioglio et al., 2011, Theorem 8.1 and Proposition 8.1 and their proofs) and since $\bar{K}$ is monotone, lower semicontinuous, quasiconcave, positive homogeneous, and such that $\bar{K}(e) = K(e) = 1$, we have that

$$\bar{K}(P) = \min_{w \in \tilde{\Delta}} \left\{ \frac{\langle w, P\rangle^+}{c_1(w)} - \frac{\langle w, P\rangle^-}{c_2(w)} \right\} \qquad \forall P \in \mathbb{R}^n \tag{207}$$

where $\tilde{\Delta}$ is a nonempty, closed, and convex subset of Δ, $c_1 : \tilde{\Delta} \to [0, \infty)$ is upper semicontinuous and concave, $c_2 : \tilde{\Delta} \to (0, \infty]$ is lower semicontinuous and convex. Moreover, $\tilde{\Delta} = \left\{w \in \Delta : \text{dom}\, \bar{H}(\cdot, w) \neq \emptyset\right\}$ where $\bar{H} : \mathbb{R} \times \Delta \to (-\infty, \infty]$ is defined by

$$\bar{H}(t, w) = \sup\left\{\bar{K}(P) : \langle w, P\rangle \leq t\right\} \qquad \forall (t, w) \in \mathbb{R} \times \Delta$$

and c_1 and c_2 are unique with $\bar{H}(1, w) = \frac{1}{c_1(w)}$ for all $w \in \tilde{\Delta}$. Since $\bar{K}(P) = 0$ for all $P \notin \mathbb{R}^n_{++}$, it is immediate to see that $\bar{H}(t, w) = 0$ for all $w \in \Delta$ and for all $t \leq 0$, that is,

$(-\infty, 0] \subseteq \operatorname{dom} \bar{H}(\cdot, w)$ for all $w \in \Delta$, proving that $\tilde{\Delta} = \Delta$. In particular, this implies that

$$c_1(w) = \frac{1}{\bar{H}(1, w)} = \frac{1}{\sup\left\{\bar{K}(P) : \langle w, P\rangle \leq 1\right\}}$$
$$= \frac{1}{\sup\left\{K(P) : P \in \mathbb{R}^n_{++} \text{ and } \langle w, P\rangle \leq 1\right\}} \qquad \forall w \in \Delta.$$

By (207) and since $\bar{K}(e) = 1$ and $\bar{K}(-e) = 0$, we have that $1 = \min_{w\in\Delta} \frac{1}{c_1(w)} = \frac{1}{\max_{w\in\Delta} c_1(w)}$ and $0 = \min_{w\in\Delta} -\frac{1}{c_2(w)} = -\frac{1}{\min_{w\in\Delta} c_2(w)}$, that is, $\max_{w\in\Delta} c_1(w) = 1$ and $\min_{w\in\Delta} c_2(w) = \infty$. By restricting (207) to $\mathbb{R}^n_{++}$ and setting $\Gamma = c_1$, Equation (206) follows. Next, consider $\tilde{\Gamma} : \Delta \to [0, \infty)$ such that $\tilde{\Gamma}$ is upper semicontinuous, concave, and such that $\max_{w\in\Delta} \tilde{\Gamma}(w) = 1$ and $\tilde{\Gamma}$ satisfies (206). By the uniqueness part of (Cerreia-Vioglio et al., 2011, Proposition 8.1) and since $c_2 = \infty$ and $\bar{K}(P) = 0$ for all $P \notin \mathbb{R}^n_{++}$, this implies that

$$\bar{K}(P) = \min_{w\in\Delta}\left\{\frac{\langle w, P\rangle^+}{\tilde{\Gamma}(w)} - \frac{\langle w, P\rangle^-}{c_2(w)}\right\} \quad \forall P \in \mathbb{R}^n,$$

proving that $\Gamma = c_1 = \tilde{\Gamma}$ and uniqueness.

(iii) implies (iv). Define $\hat{K} : \mathbb{R}^n_+ \to \mathbb{R}_+$ by $\hat{K}(P) = \min_{w\in\Delta:\Gamma(w)>0} \frac{\langle w,P\rangle}{\Gamma(w)}$ for all $P \in \mathbb{R}^n_+$. Since $\hat{K}$ is the inf of monotone, continuous, affine, and positive homogeneous functionals, we have that $\hat{K}$ is monotone, upper semicontinuous, concave, and positive homogeneous with $\hat{K}(e) = K(e) = 1$. By Remark 2, $\hat{K}$ is continuous. Define $\bar{G} : \mathbb{R}^n \to \mathbb{R}$ by

$$\bar{G}(Q) = \min_{w\in\Delta}\left\{\frac{\langle w, Q\rangle^+}{c_1(w)} - \frac{\langle w, Q\rangle^-}{c_2(w)}\right\} \qquad \forall Q \in \mathbb{R}^n$$

where $c_1 = \hat{K}_{|\Delta}$ and $c_2 = \infty$. By (Cerreia-Vioglio et al., 2011, Proposition 8.1) and since $\max_{w\in\Delta} \hat{K}_{|\Delta}(w) > 0$, we have that $\bar{G}$ is nondegenerate, monotone, lower semicontinuous, quasiconcave, and positive homogeneous. Define $G^\star = \bar{G}_{|\mathbb{R}^n_+}$. Since $c_2 = \infty$, note that $G^\star(Q) = \bar{G}(Q) = \min_{w\in\Delta} \frac{\langle w,Q\rangle}{c_1(w)}$ for all $Q \in \mathbb{R}^n_+$. Since $G^\star$ is the inf of upper semicontinuous and affine functions, it follows that $G^\star$ is concave and upper semicontinuous, proving that $G^\star \in \mathcal{G}_{\text{con}}$. By (Cerreia-Vioglio et al., 2011, Theorem 8.1 and Proposition 8.1) and their proofs and the uniqueness of the representation and since $\bar{G}(Q) = 0$ for all $Q \notin \mathbb{R}^n_+$, we also have that

$$\frac{t}{c_1(w)} = \sup\left\{\bar{G}(Q) : \langle w, Q\rangle \leq t\right\} = U_{G^\star}(t, w) \quad \forall t > 0, \forall w \in \Delta.$$

By Lemma 19 and since $G^\star \in \mathcal{G}_{\text{con}}$, we have that $U_{G^\star}(K_{G^\star}(P), P) = 1$ for all $P \in \mathbb{R}^n_{++}$ and, in particular,

$$\frac{K_{G^\star}(w)}{c_1(w)} = U_{G^\star}(K_{G^\star}(w), w) = 1 \quad \forall w \in \operatorname{int} \Delta.$$

By definition of c_1 and since $\hat{K}$ and $K_{G^\star}$ are positive homogeneous and $K_{G^\star} = c_1 = \hat{K}$ on $\operatorname{int}\Delta$, this implies that $K_{G^\star}(P) = \hat{K}(P) = K(P)$ for all $P \in \mathbb{R}^n_{++}$. By Lemma 21 and since $G^\star \in \mathcal{G}_{\text{con}}$ and $1 = K(e) = K_{G^\star}(e)$, we have that $\max_{w\in\Delta} G^\star(w) = 1$ and, in particular, $G^\star \in \mathcal{G}_{\text{nor}-\text{con}}$, proving the implication.

(iv) implies (v). By Lemma 21, the implication follows.

(v) implies (i). Since K is the inf of monotone, continuous, affine, and positive homogeneous functionals, we have that K is monotone, upper semicontinuous, concave, and positive homogeneous. By (Borwein and Lewis, 2006, Theorem 4.1.3) and since K is concave, it follows that K is continuous. Finally, by Lemma 21 and since $\max_{w\in\Delta} G^\star(w) = 1$, we have that $K(e) = 1$, proving the implication.

3. By point (iii) and (v) and point 1 and since $G^\star$ is continuous and concave and such that $\max_{w\in\Delta} G^\star(w) = 1$, we have that

$$K(P) = \min_{w\in\Delta} \frac{\langle w, P\rangle}{\Gamma(w)} = \min_{w\in\Delta} \frac{\langle w, P\rangle}{G^\star(w)} \quad \forall P \in \mathbb{R}^n_{++},$$

proving that $\Gamma = G^\star$. Since $G^\star$ is continuous and positive homogeneous, the rest of the point follows.

2. Consider any other function $G \in \mathcal{G}_{\text{nor}-\text{con}}$ satisfying either point (iv) or (v). By the same argument above and the equivalence with (iii), we have that $G(w) = \Gamma(w) = G^\star(w)$ for all $w \in \Delta$. By positive homogeneity, we can conclude that $G(Q) = G^\star(Q)$ for all $Q \in \mathbb{R}^n_+ \setminus \{0\}$. By continuity, we have that $G(0) = 0 = G^\star(0)$, proving uniqueness of $G^\star$.

4. Define $\bar{K} : \mathbb{R}^n \to \mathbb{R}$ by $\bar{K}(P) = 0$ for all $P \notin \mathbb{R}^n_{++}$ and $\bar{K}(P) = K_G(P)$ for all $P \in \mathbb{R}^n_{++}$. Define $\hat{H} : \mathbb{R} \times \Delta \to (-\infty, \infty]$ by

$$\hat{H}(t, w) = \begin{cases} \frac{t}{G(w)} & t > 0 \\ 0 & t \leq 0 \end{cases} \qquad \forall (t, w) \in \mathbb{R} \times \Delta.$$

By Lemma 21, it is immediate to see that

$$\bar{K}(P) = \min_{w\in\Delta} \hat{H}(\langle w, P\rangle, w) \qquad \forall P \in \mathbb{R}^n.$$

By points 1 and 3 and definition of $\bar{K}$, we have that

$$\begin{aligned} \frac{1}{G^\star(\tilde{w})} &= \sup\left\{K(P) : P \in \mathbb{R}^n_{++} \text{ and } \langle \tilde{w}, P\rangle \leq 1\right\} = \sup\left\{\bar{K}(P) : \langle \tilde{w}, P\rangle \leq 1\right\} \\ &= \sup_{P\in\mathbb{R}^n:\langle \tilde{w},P\rangle\leq 1} \inf_{w\in\Delta} \hat{H}(\langle w, P\rangle, w) \leq \inf_{w\in\Delta} \sup_{P\in\mathbb{R}^n:\langle \tilde{w},P\rangle\leq 1} \hat{H}(\langle w, P\rangle, w) = \frac{1}{G(\tilde{w})} \quad \forall \tilde{w} \in \Delta, \end{aligned}$$

proving that $G^\star \geq G$ on Δ. Since G and $G^\star$ are continuous and positive homogeneous, it follows that $G^\star \geq G$. Next, consider $\tilde{G} \in \mathcal{G}_{\text{nor}-\text{con}}$ and $\tilde{G} \geq G$. It follows that $K_{\tilde{G}} \leq K_G = K_{G^\star}$. By points 1 and 3, we have that

$$\begin{aligned}
\frac{1}{G^\star(w)} &= \sup\left\{K_{G^\star}(P) : P \in \mathbb{R}^n_{++} \text{ and } \langle w, P\rangle \leq 1\right\} \\
&\geq \sup\left\{K_{\tilde{G}}(P) : P \in \mathbb{R}^n_{++} \text{ and } \langle w, P\rangle \leq 1\right\} \\
&= \frac{1}{\tilde{G}(w)} \qquad \forall w \in \Delta,
\end{aligned}$$

proving that $\tilde{G} \geq G^\star$ on Δ. Since $\tilde{G}$ and $G^\star$ are continuous and positive homogeneous, it follows that $\tilde{G} \geq G^\star$, proving the point.

H.4 Fragile Sectors

By Lemma 21, if $G \in \mathcal{G}_{\text{nor}}$, then

$$K_G(P) = \min_{w\in\Delta} \frac{\langle w, P\rangle}{G(w)} \quad \forall P \in \mathbb{R}^n_{++}$$

and

$$\kappa_G(p) = \min_{w\in\Delta} \{M(w,p) + c(w)\} \qquad \forall p \in \mathbb{R}^n \tag{208}$$

where $M : \Delta \times \mathbb{R}^n \to \mathbb{R}$ is defined by $M(w,p) = \ln\left(\sum_{j=1}^n e^{p_j} w_j\right)$ for all $(w,p) \in \Delta \times \mathbb{R}^n$ and $c : \Delta \to [0,\infty]$ is defined by $c(w) = -\ln G(w)$ for all $w \in \Delta$. Since $K_G \in \mathcal{K}_{\text{nor}}$, we have that κ_G is normalized, monotone, and translation invariant. We define $R(\cdot||\cdot) : \Delta \times \Delta \to [0,\infty]$ by

$$R(w||\tilde{w}) = \begin{cases} \sum_{j=1}^n \ln\left(\frac{w_j}{\tilde{w}_j}\right) w_j & w \ll \tilde{w} \\ \infty & \text{otherwise} \end{cases} \qquad \forall (w,\tilde{w}) \in \Delta \times \Delta.$$

We define the absolute entropy $\text{Ent} : \Delta \to \mathbb{R}$ by $\text{Ent}(w) = -\sum_{j=1}^n w_j \ln w_j$ for all $w \in \Delta$. Recall that Ent is continuous and strictly concave.[25] Given a convex subset C of Δ, we will say that a function $f : C \to \mathbb{R} \cup \{\infty\}$ is *entropy convex* if and only if $f + \text{Ent}$ is convex. By subtracting Ent, it is immediate to see that if f is entropy convex, then f is strictly convex on its effective domain and convex in general. Given a function $A : C \to [1,\infty]$, we will say it is *entropy log-convex* if and only if $d : C \to [0,\infty]$, defined by $d(w) = \ln A(w)$ for all $w \in C$, is entropy convex.

[25] Define $\phi : [0,\infty) \to \mathbb{R}$ by $\phi(0) = 0$ and $\phi(t) = t\ln(t)$ for all $t > 0$. Observe that ϕ is continuous and strictly convex.

Lemma 22. *Let $G \in \mathcal{G}_{\text{nor}}$. If $\kappa_G : \mathbb{R}^n \to \mathbb{R}$ is convex, then there exists a grounded, lower semicontinuous, and convex $d : \Delta \to [0, \infty]$ such that*

$$\kappa_G(p) = \max_{w \in \Delta} \{\langle w, p\rangle - d(w)\} \qquad \forall p \in \mathbb{R}^n \tag{209}$$

and

$$d(w) = \sup_{\tilde{w} \in \operatorname{dom} c} \{R(w||\tilde{w}) - c(\tilde{w})\} \qquad \forall w \in \Delta. \tag{210}$$

Moreover, if $\tilde{w} \in \operatorname{int} \Delta$, then $w \mapsto d(w) - R(w||\tilde{w})$ is lower semicontinuous and convex on Δ. In particular, d is lower semicontinuous and entropy convex.

Proof. Define $\bar{\kappa}_G : \mathbb{R}^n \to \mathbb{R}$ by $\bar{\kappa}_G(p) = -\kappa_G(-p)$ for all $p \in \mathbb{R}^n$. We have that $\bar{\kappa}_G$ is normalized, monotone, translation invariant, concave, and such that

$$\bar{\kappa}_G(p) = \max_{w \in \Delta} \{m(w, p) - c(w)\} \qquad \forall p \in \mathbb{R}^n \tag{211}$$

where $m : \Delta \times \mathbb{R}^n \to \mathbb{R}$ is defined by $m(w, p) = -\ln\left(\sum_{j=1}^n e^{-p_j} w_j\right)$ for all $(w, p) \in \Delta \times \mathbb{R}^n$. By Cerreia-Vioglio et al. (2014) and since $\bar{\kappa}_G$ is normalized, monotone, translation invariant, and concave, we have that $d : \Delta \to [0, \infty]$, defined by $d(w) = \sup_{p \in \mathbb{R}^n} \{\bar{\kappa}_G(p) - \langle w, p\rangle\}$ for all $w \in \Delta$, is such that

$$\bar{\kappa}_G(p) = \min_{w \in \Delta} \{\langle w, p\rangle + d(w)\} \qquad \forall p \in \mathbb{R}^n.$$

By definition of d and since $\bar{\kappa}_G$ is normalized, we have that d is grounded, lower semicontinuous, and convex. By definition of d and (211) and Donsker-Varadhan's variational formula, we also have that

$$\begin{aligned}
d(w) &= \sup_{p \in \mathbb{R}^n} \left\{\max_{\tilde{w} \in \Delta} \{m(\tilde{w}, p) - c(\tilde{w})\} - \langle w, p\rangle\right\} = \sup_{p \in \mathbb{R}^n} \sup_{\tilde{w} \in \Delta} \{m(\tilde{w}, p) - c(\tilde{w}) - \langle w, p\rangle\} \\
&= \sup_{p \in \mathbb{R}^n} \sup_{\tilde{w} \in \operatorname{dom} c} \{m(\tilde{w}, p) - c(\tilde{w}) - \langle w, p\rangle\} = \sup_{\tilde{w} \in \operatorname{dom} c} \sup_{p \in \mathbb{R}^n} \{m(\tilde{w}, p) - \langle w, p\rangle - c(\tilde{w})\} \\
&= \sup_{\tilde{w} \in \operatorname{dom} c} \left\{\sup_{p \in \mathbb{R}^n} \{m(\tilde{w}, p) - \langle w, p\rangle\} - c(\tilde{w})\right\} = \sup_{\tilde{w} \in \operatorname{dom} c} \{R(w||\tilde{w}) - c(\tilde{w})\} \quad \forall w \in \Delta.
\end{aligned}$$

Since $\kappa_G(p) = -\bar{\kappa}_G(-p)$ for all $p \in \mathbb{R}^n$, the main statement and (210) follow. Fix $\breve{w} \in \operatorname{int} \Delta$. Since $\breve{w} \in \operatorname{int} \Delta$, we have that $R(w||\breve{w}) < \infty$ for all $w \in \Delta$. For each $\tilde{w} \in \operatorname{dom} c$ define $f_{\tilde{w}} : \Delta \to \mathbb{R} \cup \{\infty\}$ by $f_{\tilde{w}}(w) = R(w||\tilde{w}) - c(\tilde{w}) - R(w||\breve{w})$ for all $w \in \Delta$. Fix $\tilde{w} \in \operatorname{dom} c$. Since $R(w||\breve{w}) \in \mathbb{R}$ for all $w \in \Delta$ and $c(\tilde{w}) \in \mathbb{R}$, note that $w \in \operatorname{dom} f_{\tilde{w}}$ if and only if $f_{\tilde{w}}(w) < \infty$ if and only if $R(w||\tilde{w}) < \infty$, proving that $\operatorname{dom} f_{\tilde{w}} = \operatorname{dom} R(\cdot||\tilde{w})$ is convex. Since $R(w||\breve{w}) \in \mathbb{R}$

for all $w \in \Delta$ and $R(w||\tilde{w}) \in \mathbb{R}$ for all $w \in \operatorname{dom} f_{\tilde{w}}$, we have that $w_j = 0$ whenever $\tilde{w}_j = 0$ and

$$f_{\tilde{w}}(w) = R(w||\tilde{w}) - c(\tilde{w}) - R(w||\breve{w}) = \sum_{j=1}^{n} \ln\left(\frac{w_j}{\tilde{w}_j}\right) w_j - c(\tilde{w}) - \sum_{j=1}^{n} \ln\left(\frac{w_j}{\breve{w}_j}\right) w_j$$
$$= \sum_{j=1}^{n} \ln\left(\frac{\breve{w}_j}{\tilde{w}_j}\right) w_j - c(\tilde{w}) \qquad \forall w \in \operatorname{dom} f_{\tilde{w}}.$$

This implies that $f_{\tilde{w}}$ is affine over $\operatorname{dom} f_{\tilde{w}}$ and, in particular, convex on Δ. By definition of d and since $\tilde{w}$ was arbitrarily chosen, we have that

$$d(w) - R(w||\breve{w}) = \sup_{\tilde{w}\in\operatorname{dom} c} \{R(w||\tilde{w}) - c(\tilde{w})\} - R(w||\breve{w})$$
$$= \sup_{\tilde{w}\in\operatorname{dom} c} \{R(w||\tilde{w}) - c(\tilde{w}) - R(w||\breve{w})\} = \sup_{\tilde{w}\in\operatorname{dom} c} f_{\tilde{w}}(w) \quad \forall w \in \Delta,$$

proving that $w \mapsto d(w) - R(w||\breve{w})$ is convex. Since $\breve{w} \in \operatorname{int}\Delta$ and d is lower semicontinuous, we have that $w \mapsto R(w||\breve{w})$ is continuous and $w \mapsto d(w) - R(w||\breve{w})$ is lower semicontinuous. Fix $\breve{w} \in \operatorname{int}\Delta$. Since $w \mapsto \sum_{j=1}^{n} \ln(\breve{w}_j) w_j$ is affine and $w \mapsto d(w) - R(w||\breve{w})$ is convex, we have that $w \mapsto d(w) + \operatorname{Ent}(w) = d(w) - R(w||\breve{w}) - \sum_{j=1}^{n} \ln(\breve{w}_j) w_j$, proving that $d + \operatorname{Ent}$ is convex. □

Proposition 13. *Let $K : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$. The following statements are equivalent:*

(i) $K \in \mathcal{K}_{\text{nor}}$ and κ is convex;

(ii) There exists $G \in \mathcal{G}_{\text{nor}}$ such that $K = K_G$ and $\kappa = \kappa_G$ is convex;

(iii) There exists a unique $G^\star \in \mathcal{G}_{\text{nor}-\text{con}}$ such that $K = K_{G^\star}$, $\kappa = \kappa_{G^\star}$, and

$$G^\star(Q) = \min_{w\in C} A(w)\, \Pi_{j=1}^{n} \left(\frac{Q_j}{w_j}\right)^{w_j} \qquad \forall Q \in \mathbb{R}^n_+$$

where C is a nonempty, compact, and convex subset of Δ and $A : C \to [1,\infty]$ is lower semicontinuous, entropy log-convex, and such that $\min_{w\in C} A(w) = 1$.

Proof. (i) implies (ii). By Theorem 3, the implication follows.

(ii) implies (iii). Define $L : \mathbb{R}^n_{++} \to \mathbb{R}^n$ by $L(P)$ being the vector whose j-th component is $\ln(P_j)$ for all $j \in [n]$ and for all $P \in \mathbb{R}^n_{++}$. By Lemma 22 and definition of κ_G, this implies that

$$K(P) = \max_{w\in\Delta} e^{\langle w, L(P)\rangle - d(w)} = \max_{w\in\Delta} \frac{e^{\langle w, L(P)\rangle}}{e^{d(w)}} = \max_{w\in\Delta} \frac{\Pi_{j=1}^{n} P_j^{w_j}}{e^{d(w)}} \quad \forall P \in \mathbb{R}^n_{++} \tag{212}$$

where $d : \Delta \to [0,\infty]$ is grounded, lower semicontinuous, and convex. For each $w \in \Delta$ define $K_w : \mathbb{R}^n_{++} \to \mathbb{R}_+$ by $K_w(P) = \Pi_{j=1}^{n} P_j^{w_j} / e^{d(w)}$ for all $P \in \mathbb{R}^n_{++}$. If $d(w) = \infty$, note that $K_w = 0$.

Given $w \in \Delta$, define $H, H_w : \Delta \to [0, \infty]$ by

$$H(\tilde{w}) = \sup \left\{ K(P) : P \in \mathbb{R}^n_{++} \text{ s.t. } \langle \tilde{w}, P \rangle \leq 1 \right\} \qquad \forall \tilde{w} \in \Delta$$

and

$$H_w(\tilde{w}) = \sup \left\{ K_w(P) : P \in \mathbb{R}^n_{++} \text{ s.t. } \langle \tilde{w}, P \rangle \leq 1 \right\} \qquad \forall \tilde{w} \in \Delta.$$

Note that

$$H_w(\tilde{w}) = \frac{e^{R(w||\tilde{w})}}{e^{d(w)}} = \frac{1}{e^{d(w)-R(w||\tilde{w})}} = \frac{1}{e^{d(w)} \Pi_{j=1}^n \left(\frac{\tilde{w}_j}{w_j} \right)^{w_j}} > 0 \quad \forall (\tilde{w}, w) \in \Delta \times \operatorname{dom} d$$

while $H_w(\tilde{w}) = 0$ for all $(\tilde{w}, w) \in \Delta \times (\operatorname{dom} d)^c$. By (212), note that $H(\tilde{w}) = \sup_{w \in \Delta} H_w(\tilde{w}) = \sup_{w \in \operatorname{dom} d} H_w(\tilde{w})$ for all $\tilde{w} \in \Delta$ and

$$H(\tilde{w}) = \sup_{w \in \operatorname{dom} d} H_w(\tilde{w}) = \sup_{w \in \operatorname{dom} d} \left\{ \frac{1}{e^{d(w)} \Pi_{j=1}^n \left(\frac{\tilde{w}_j}{w_j} \right)^{w_j}} \right\} = \frac{1}{\inf_{w \in \operatorname{dom} d} e^{d(w)} \Pi_{j=1}^n \left(\frac{\tilde{w}_j}{w_j} \right)^{w_j}} \quad \forall \tilde{w} \in \Delta.$$

By Lemma 17 and Theorem 3 and points 1–3 and since $K = K_G \in \mathcal{K}_{\text{nor}}$, we can conclude that $K = K_{G^\star}$ where $\Gamma : \Delta \to [0, \infty)$ and $G^\star : \mathbb{R}^n_+ \to \mathbb{R}_+$ are such that

$$\Gamma(\tilde{w}) = \frac{1}{H(\tilde{w})} = \inf_{w \in \operatorname{dom} d} e^{d(w)} \Pi_{j=1}^n \left(\frac{\tilde{w}_j}{w_j} \right)^{w_j} \qquad \forall \tilde{w} \in \Delta$$

and

$$G^\star(Q) = \inf_{w \in \operatorname{dom} d} e^{d(w)} \Pi_{j=1}^n \left(\frac{Q_j}{w_j} \right)^{w_j} \qquad \forall Q \in \mathbb{R}^n_+.$$

By Lemma 22, we have that $C = \operatorname{cl}(\operatorname{dom} d)$ is a nonempty, compact, and convex subset of Δ and $A = e^d$ from C to $[1, \infty]$ is lower semicontinuous, entropy log-convex, and such that $\min_{w \in C} A(w) = 1$. We next show that the inf can be extended from $\operatorname{dom} d$ to C and it is a min. Consider $Q \in \mathbb{R}^n_+$. We have two cases:

1. $A(w) \Pi_{j=1}^n \left(\frac{Q_j}{w_j} \right)^{w_j} > 0$ for all $w \in \operatorname{dom} d$. In this case, since $A(w) \in [1, \infty)$, we have that for each $w \in \operatorname{dom} d$

$$w_j > 0 \implies Q_j > 0.$$

Define $\Delta(Q) = \{w \in \Delta : Q_j = 0 \implies w_j = 0\}$. Clearly, $\Delta(Q)$ is closed and $\operatorname{dom} d \subseteq \Delta(Q)$ and, in particular, $C \subseteq \Delta(Q)$. Note also that $\Pi_{j=1}^n \left(\frac{Q_j}{w_j} \right)^{w_j} = \exp\left(\sum_{j=1}^n \ln \frac{Q_j}{w_j} w_j \right) = \exp\left(\sum_{j=1}^n \ln Q_j w_j + \operatorname{Ent}(w) \right)$ for all $w \in \Delta(Q)$. Note that the function $w \mapsto \sum_{j=1}^n \ln Q_j w_j + \operatorname{Ent}(w)$ is real-valued and continuous on $\Delta(Q)$ and, in particular, on C. This implies that the function

$$w \mapsto A(w)\,\Pi_{j=1}^{n}\left(\frac{Q_j}{w_j}\right)^{w_j} = \exp\left(d(w) + \sum_{j=1}^{n} \ln Q_j w_j + \mathrm{Ent}(w)\right)$$

is lower semicontinuous on C, proving that the inf over C is a min. Moreover, since

$$A(w)\,\Pi_{j=1}^{n}\left(\frac{Q_j}{w_j}\right)^{w_j} = \exp\left(d(w) + \sum_{j=1}^{n} \ln Q_j w_j + \mathrm{Ent}(w)\right) = \infty \quad \forall w \in C \setminus \mathrm{dom}\, d,$$

we have that extending the inf to C does not change the value of the minimization.

2. $A(\bar{w})\,\Pi_{j=1}^{n}\left(\frac{Q_j}{\bar{w}_j}\right)^{\bar{w}_j} = 0$ for some $\bar{w} \in \mathrm{dom}\, d$. Since $w \mapsto A(w)\,\Pi_{j=1}^{n}\left(\frac{Q_j}{w_j}\right)^{w_j} \geq 0$ for all $w \in C$, we have that

$$\begin{aligned} G^{\star}(Q) &= \inf_{w \in \mathrm{dom}\, d} e^{d(w)}\Pi_{j=1}^{n}\left(\frac{Q_j}{w_j}\right)^{w_j} = \inf_{w \in C} e^{d(w)}\Pi_{j=1}^{n}\left(\frac{Q_j}{w_j}\right)^{w_j} \\ &= A(\bar{w})\,\Pi_{j=1}^{n}\left(\frac{Q_j}{\bar{w}_j}\right)^{\bar{w}_j} = 0, \end{aligned}$$

proving that inf can be extended from $\mathrm{dom}\, d$ to C and it is a min.

Points 1 and 2 prove the implication.

(iii) implies (i). Define $\tilde{C} = \{w \in C : A(w) < \infty\}$. Since $\tilde{C}$ is a nonempty subset of a compact set and A is such that $\min_{w \in C} A(w) = 1$ and entropy log-convex and, in particular, quasiconvex,we have that $\tilde{C}$ is nonempty, precompact, and convex. Define $\mathfrak{G} : \tilde{C} \times \mathbb{R}_{+}^{n} \to \mathbb{R}_{+}$ by

$$\mathfrak{G}(w, Q) = A(w)\,\Pi_{j=1}^{n}\left(\frac{Q_j}{w_j}\right)^{w_j} \qquad \forall (w, Q) \in \tilde{C} \times \mathbb{R}_{+}^{n}.$$

Clearly, $\mathfrak{G}(w, \cdot) \in \mathcal{G}_{\mathrm{con}}$ for all $w \in \mathrm{cl}\,\tilde{C}$. Define $d : \mathrm{cl}\,\tilde{C} \to [0, \infty]$ by $d(w) = \ln A(w)$ for all $w \in C$. Given $Q \in \mathbb{R}_{++}^{n}$, note that $\mathfrak{G}(w, Q) = \|Q\|_1 e^{d(w) + \mathrm{Ent}(w) + \sum_{j=1}^{n} \ln(\breve{w}_j) w_j}$ where $\breve{w} = Q / \|Q\|_1$. Since A is lower semicontinuous on C and entropy log-convex, it follows that $w \mapsto d(w) + \mathrm{Ent}(w) + \sum_{j=1}^{n} \ln(\breve{w}_j)\, w_j$ is lower semicontinuous and convex on $\mathrm{cl}\,\tilde{C}$ and, in particular, the lower contour sets of this function are closed and convex subsets of $\mathrm{cl}\,\tilde{C}$, that is, they are compact subsets. At the same time, since $d(w) = \infty$ for all $w \in \mathrm{cl}\,\tilde{C} \setminus \tilde{C}$, they are subsets of $\tilde{C}$, proving that $w \mapsto d(w) + \mathrm{Ent}(w) + \sum_{j=1}^{n} \ln(\breve{w}_j)\, w_j$ is lower coercive and quasiconvex on $\tilde{C}$ and so is $w \mapsto \mathfrak{G}(w, Q)$. By Lemma 20 and since $G^{\star} : \mathbb{R}_{+}^{n} \to \mathbb{R}_{+}$ is such that

$$G^{\star}(Q) = \min_{w \in \tilde{C}} \mathfrak{G}(w, Q) \qquad \forall Q \in \mathbb{R}_{+}^{n},$$

then

$$K_{G^{\star}}(P)=\max_{w\in\tilde{C}}K_{w}(P)=\max_{w\in\tilde{C}}\frac{\Pi_{j=1}^{n}P_{j}^{w_{j}}}{e^{d(w)}}\qquad\forall P\in\mathbb{R}_{++}^{n}.$$

Since $\kappa(p)=\kappa_{G^{\star}}(p)=\ln(K_{G^{\star}}(\exp(p)))$ for all $p\in\mathbb{R}^{n}$, it follows that

$$\kappa(p)=\ln(K_{G^{\star}}(\exp(p)))=\ln\left(\max_{w\in\tilde{C}}\frac{\Pi_{j=1}^{n}e^{w_{j}p_{j}}}{e^{d(w)}}\right)=\max_{w\in\tilde{C}}\left\{\sum_{j=1}^{n}w_{j}p_{j}-d(w)\right\}\quad\forall p\in\mathbb{R}^{n},$$

proving the convexity of κ. □

Smoothness and density. In this section, we show that smooth unit log-cost functions are dense. In light of Theorem 3, we can therefore think that for any $G\in\mathcal{G}_{\text{nor}}$ there exists a sequence $\{G_{m}\}_{m\in\mathbb{N}}\subseteq\mathcal{G}_{\text{nor}}$ such that $\kappa_{G_{m}}\in C^{\infty}(\mathbb{R}^{n})$ for all $m\in\mathbb{N}$ and $\lim_{m}\kappa_{G_{m}}(p)=\kappa_{G}(p)$ for all $p\in\mathbb{R}^{n}$. By Arzela-Ascoli's Theorem and since the sequence $\{\kappa_{G_{m}}\}_{m\in\mathbb{N}}$ is uniformly bounded over compacta and Lipschitz of order 1, this implies that $\kappa_{G_{m}}\overset{d}{\to}\kappa_{G}$ where d is the distance on $C^{\infty}(\mathbb{R}^{n})$ metrizing convergence over compacta. This is our density result (cf. Proposition 14).

As one might suspect, we carry out these approximations by convolution with mollifiers. To this extent, we need an intermediate approach. This intermediate approach brings us to consider the following new objects. Given a function $K\in\mathcal{K}_{\text{nor}}$, we define $\mathfrak{K}:\mathbb{R}^{n}\to\mathbb{R}$ by

$$\mathfrak{K}(p)=K(\exp(p))\qquad\forall p\in\mathbb{R}^{n}$$

where as usual $\exp(p)$ denotes the vector whose i-th component is $e^{p_{i}}$. Recall that $\mathbb{R}^{n}$ is endowed with the supnorm $\|\ \|_{\infty}$ and we denote by B the unit ball. Consider a function $\varphi\in C_{0}^{\infty}(\mathbb{R}^{n})$ such that $\varphi\geq0$, $\int_{\mathbb{R}^{n}}\varphi(x)\,dx=1$, and $\operatorname{supp}\varphi\subseteq B$. For each $\varepsilon>0$ define $\varphi_{\varepsilon}(x)=\varepsilon^{-n}\varphi\left(\frac{x}{\varepsilon}\right)$ for all $x\in\mathbb{R}^{n}$. By the Change of Variable Formula, note that $\varphi_{\varepsilon}\geq0$, $\int_{\mathbb{R}^{n}}\varphi_{\varepsilon}(x)\,dx=1$, and $\operatorname{supp}\varphi_{\varepsilon}\subseteq\varepsilon B$ for all $\varepsilon>0$. Given a function $K\in\mathcal{K}_{\text{nor}}$ and $\varepsilon>0$, we define $\mathfrak{K}_{\varepsilon}:\mathbb{R}^{n}\to\mathbb{R}$ by

$$\mathfrak{K}_{\varepsilon}(\bar{p})=(\varphi_{\varepsilon}\star\mathfrak{K})(\bar{p})=\int_{\mathbb{R}^{n}}\varphi_{\varepsilon}(\bar{p}-p)\,\mathfrak{K}(p)\,dp\quad\forall\bar{p}\in\mathbb{R}^{n}.$$

We define $K_{\varepsilon}:\mathbb{R}_{++}^{n}\to\mathbb{R}_{++}$ by $K_{\varepsilon}(P)=\frac{1}{\mathfrak{K}_{\varepsilon}(0)}\mathfrak{K}_{\varepsilon}(\ln P)$ where $\ln P$ is the vector whose i-th component is $\ln P_{i}$ for all $P\in\mathbb{R}_{++}^{n}$. As before, we define $\kappa_{\varepsilon}:\mathbb{R}^{n}\to\mathbb{R}$ by $\kappa_{\varepsilon}(p)=\ln(K_{\varepsilon}(\exp(p)))$ for all $p\in\mathbb{R}^{n}$.

Proposition 14. *If $K\in\mathcal{K}_{\text{nor}}$ and $\{\varepsilon_{m}\}_{m\in\mathbb{N}}\subseteq(0,1)$ such that $\lim_{m}\varepsilon_{m}=0$, then $K_{\varepsilon_{m}}\in\mathcal{K}_{\text{nor}}$ and $\kappa_{\varepsilon_{m}}\in C^{\infty}(\mathbb{R}^{n})$ for all $m\in\mathbb{N}$ and $\kappa_{\varepsilon_{m}}\overset{d}{\to}\kappa$.*

Proof. Since K is continuous, so is $\mathfrak{K}$ and , in particular, $\mathfrak{K}\in L_{\text{loc}}^{1}(\mathbb{R}^{n})$. By (Brezis, 2011, Propositions 4.20 and 4.21), it follows that $\mathfrak{K}_{\varepsilon_{m}}\in C^{\infty}(\mathbb{R}^{n})$ for all $m\in\mathbb{N}$ and $\mathfrak{K}_{\varepsilon_{m}}$ converges uniformly to $\mathfrak{K}$ over compacta. By the Change of Variable Formula, recall that convolution is commutative. It follows that

$$\mathfrak{K}_{\varepsilon_m}(\bar{p}) = (\varphi_{\varepsilon_m} \star \mathfrak{K})(\bar{p}) = (\mathfrak{K} \star \varphi_{\varepsilon_m})(\bar{p}) = \int_{\mathbb{R}^n} \mathfrak{K}(\bar{p} - p)\, \varphi_{\varepsilon_m}(p)\, dp \quad \forall \bar{p} \in \mathbb{R}^n, \forall m \in \mathbb{N}.$$

Since $K > 0$ and $\varphi_{\varepsilon_m} \geq 0$ and $\int_{\mathbb{R}^n} \varphi_{\varepsilon_m}(x)\, dx = 1$ for all $m \in \mathbb{N}$, we have that $\mathfrak{K} > 0$ and, in particular, $\mathfrak{K}_{\varepsilon_m} > 0$ for all $m \in \mathbb{N}$. It follows that K_{ε_m} is well defined and continuous for all $m \in \mathbb{N}$. Moreover, by construction, we have that $K_{\varepsilon_m}(e) = 1$ for all $m \in \mathbb{N}$. Since K is monotone, so is $\mathfrak{K}$. Since $\mathfrak{K}$ is monotone and $\varphi_{\varepsilon_m} \geq 0$ and $\int_{\mathbb{R}^n} \varphi_{\varepsilon_m}(p)\, dp = 1$ for all $m \in \mathbb{N}$, it follows that

$$\hat{p} \geq \tilde{p} \implies \mathfrak{K}_{\varepsilon_m}(\hat{p}) = \int_{\mathbb{R}^n} \mathfrak{K}(\hat{p} - p)\, \varphi_{\varepsilon_m}(p)\, dp \geq \int_{\mathbb{R}^n} \mathfrak{K}(\tilde{p} - p)\, \varphi_{\varepsilon_m}(p)\, dp = \mathfrak{K}_{\varepsilon_m}(\tilde{p}) \quad \forall m \in \mathbb{N},$$

proving monotonicity of $\mathfrak{K}_{\varepsilon_m}$ and K_{ε_m} for all $m \in \mathbb{N}$. Since K is positively homogeneous and $\varphi_{\varepsilon_m} \geq 0$ and $\int_{\mathbb{R}^n} \varphi_{\varepsilon_m}(p)\, dp = 1$ for all $m \in \mathbb{N}$, it follows that

$$\begin{aligned}
\mathfrak{K}_{\varepsilon_m}(\bar{p} + he) &= \int_{\mathbb{R}^n} \mathfrak{K}(\bar{p} + he - p)\, \varphi_{\varepsilon_m}(p)\, dp = \int_{\mathbb{R}^n} K(\exp(\bar{p} + he - p))\, \varphi_{\varepsilon_m}(p)\, dp \\
&= e^h \int_{\mathbb{R}^n} K(\exp(\bar{p} - p))\, \varphi_{\varepsilon_m}(p)\, dp = e^h \int_{\mathbb{R}^n} \mathfrak{K}(\bar{p} - p)\, \varphi_{\varepsilon_m}(p)\, dp \\
&= e^h \mathfrak{K}_{\varepsilon_m}(\bar{p}) \quad \forall \bar{p} \in \mathbb{R}^n, \forall h \in \mathbb{R}, \forall m \in \mathbb{N}.
\end{aligned}$$

It follows that

$$\begin{aligned}
K_{\varepsilon_m}(\lambda P) &= \frac{1}{\mathfrak{K}_{\varepsilon_m}(0)} \mathfrak{K}_{\varepsilon_m}(\ln(\lambda P)) = \frac{1}{\mathfrak{K}_{\varepsilon_m}(0)} \mathfrak{K}_{\varepsilon_m}(\ln(P) + \ln(\lambda)\, e) \\
&= \frac{1}{\mathfrak{K}_{\varepsilon_m}(0)} e^{\ln(\lambda)} \mathfrak{K}_{\varepsilon_m}(\ln(P)) = \lambda K_{\varepsilon_m}(P), \qquad \forall P \in \mathbb{R}^n_{++}, \forall \lambda > 0, \forall m \in \mathbb{N},
\end{aligned}$$

proving that K_{ε_m} is positively homogeneous for all $m \in \mathbb{N}$. Consider $P, \tilde{P} \in \mathbb{R}^n_{++}$ and $\lambda \in (0, 1)$. Given a vector $p \in \mathbb{R}^n$ define P_p (resp. $\tilde{P}_p$) as the vector whose i-th component is $P_i \backslash \exp(p_i)$

(resp. $\tilde{P}_i \backslash \exp(p_i)$). Since K is concave, we have that

$$
\begin{aligned}
&\mathfrak{K}_{\varepsilon_m}\left(\ln\left(\lambda P + (1-\lambda)\tilde{P}\right)\right)\\
&= \int_{\mathbb{R}^n} \mathfrak{K}\left(\ln\left(\lambda P + (1-\lambda)\tilde{P}\right) - p\right)\varphi_{\varepsilon_m}(p)\,dp\\
&= \int_{\mathbb{R}^n} \mathfrak{K}\left(\ln\left(\lambda P_p + (1-\lambda)\tilde{P}_p\right)\right)\varphi_{\varepsilon_m}(p)\,dp\\
&= \int_{\mathbb{R}^n} K\left(\lambda P_p + (1-\lambda)\tilde{P}_p\right)\varphi_{\varepsilon_m}(p)\,dp\\
&\geq \lambda\int_{\mathbb{R}^n} K(P_p)\,\varphi_{\varepsilon_m}(p)\,dp + (1-\lambda)\int_{\mathbb{R}^n} K\left(\tilde{P}_p\right)\varphi_{\varepsilon_m}(p)\,dp\\
&= \lambda\int_{\mathbb{R}^n} \mathfrak{K}(\ln(P)-p)\,\varphi_{\varepsilon_m}(p)\,dp + (1-\lambda)\int_{\mathbb{R}^n} \mathfrak{K}\left(\ln\left(\tilde{P}\right)-p\right)\varphi_{\varepsilon_m}(p)\,dp\\
&= \lambda\mathfrak{K}_{\varepsilon_m}(\ln(P)) + (1-\lambda)\,\mathfrak{K}_{\varepsilon_m}\left(\ln\left(\tilde{P}\right)\right) \quad \forall m\in\mathbb{N},
\end{aligned}
$$

proving concavity of K_{ε_m} for all $m \in \mathbb{N}$. We can conclude that $K_{\varepsilon_m} \in \mathcal{K}_{\text{nor}}$ for all $m \in \mathbb{N}$. Since $\mathfrak{K}_{\varepsilon_m} \in C^\infty(\mathbb{R}^n)$ and $\mathfrak{K}_{\varepsilon_m} > 0$ for all $m \in \mathbb{N}$ and ln is infinitely many times differentiable on $(0,\infty)$, we can conclude that $\kappa_{\varepsilon_m} \in C^\infty(\mathbb{R}^n)$ for all $m \in \mathbb{N}$. Since $\lim_m \mathfrak{K}_{\varepsilon_m}(p) = \mathfrak{K}(p)$ for all $p \in \mathbb{R}^n$ and ln is continuous on $(0,\infty)$, we have that $\lim_m \mathfrak{K}_{\varepsilon_m}(0) = \mathfrak{K}(0) = 1$ and, in particular,

$$
\begin{aligned}
\lim_m \kappa_{\varepsilon_m}(p) &= \lim_m\left[\ln\mathfrak{K}_{\varepsilon_m}(p) - \ln\mathfrak{K}_{\varepsilon_m}(0)\right] = \lim_m \ln\mathfrak{K}_{\varepsilon_m}(p) - \lim_m \ln\mathfrak{K}_{\varepsilon_m}(0)\\
&= \ln\mathfrak{K}(p) = \kappa(p) \quad \forall p\in\mathbb{R}^n,
\end{aligned}
$$

proving that κ_{ε_m} converges pointwise to κ. By Arzela-Ascoli's Theorem and since the sequence $\{\kappa_{\varepsilon_m}\}_{m\in\mathbb{N}}$ is uniformly bounded over compacta and Lipschitz of order 1, this implies that $\kappa_{\varepsilon_m} \stackrel{d}{\to} \kappa$.

Proposition 15. *If T is a production operator, then there exists a sequence $\{T_m\}_{m\in\mathbb{N}}$ of production operators such that $T_m \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ for all $m \in \mathbb{N}$ and*

$$
\lim_m T_m(x) = T(x) \ \textit{ and } \lim_m \Pi_m(x) = \Pi(x) \quad \forall x\in\mathbb{R}^n.
$$

Proof. By Proposition 14, we have that for each $i \in [n]$ there exists a sequence $\{K_{i,m}\}_{m\in\mathbb{N}} \subseteq \mathcal{K}_{\text{nor}}$ such that $\{\kappa_{i,m}\}_{m\in\mathbb{N}} \subseteq C^\infty(\mathbb{R}^n)$ and $\kappa_{i,m} \stackrel{d}{\to} T_i$ where d is the distance on $C^\infty(\mathbb{R}^n)$ metrizing convergence over compacta. Define $T_m : \mathbb{R}^n \to \mathbb{R}^n$ by $T_{m,i}(x) = \kappa_{i,m}(x)$ for all $x \in \mathbb{R}^n$. It follows that each T_m is a production operator and belongs to $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Moreover, $\{T_m\}_{m\in\mathbb{N}}$ converges uniformly over compacta to T and, in particular, $\lim_m T_m(x) = T(x)$ for all $x \in \mathbb{R}^n$. Fix $x \in \mathbb{R}^n$. Since each T_m and Π are normalized and nonexpansive, we have that

$$\begin{aligned}\|\Pi_m(x) - \Pi(x)\|_\infty &= \|\beta T_m(\Pi_m(x)) - \beta T(\Pi(x))\|_\infty \\ &\leq \|\beta T_m(\Pi_m(x)) - \beta T_m(\Pi(x))\|_\infty + \|\beta T_m(\Pi(x)) - \beta T(\Pi(x))\|_\infty \\ &\leq \beta \|\Pi_m(x) - \Pi(x)\|_\infty + \beta \max_{z \in \mathbb{R}^n : \|z\|_\infty \leq \|x\|_\infty} \|T_m(z) - T(z)\|_\infty,\end{aligned}$$

proving that

$$\lim_m \|\Pi_m(x) - \Pi(x)\|_\infty \leq \frac{\beta}{1-\beta} \lim_m \max_{z \in \mathbb{R}^n : \|z\|_\infty \leq \|x\|_\infty} \|T_m(z) - T(z)\|_\infty = 0.$$

Since x was arbitrarily chosen, the statement follows. □

Semiconcavity. In this section, we prove first that if $K \in \mathcal{K}_{\text{nor}}$ and κ is twice continuously differentiable, then $\kappa - \| \|_2^2 \backslash 2$ is a concave function where $\| \|_2$ is the Euclidean norm over $\mathbb{R}^n$. In this section, we show that if $K \in \mathcal{K}_{\text{nor}}$, then κ is λ-semiconcave, that is, $\kappa - \frac{\lambda}{2} \| \|_2^2$ is concave. In particular, λ can be chosen to be 1. We first start by proving the result when κ is twice continuously differentiable. In this section, given two symmetric $n \times n$ matrices A and B, we write $A \succeq B$ if and only if $\langle Ax, x\rangle \geq \langle Bx, x\rangle$ for all $x \in \mathbb{R}^n$. With a small abuse of notation, we denote by I both the $n \times n$ identity matrix and the identity operator.

Lemma 23. *If $K \in \mathcal{K}_{\text{nor}}$ and κ is twice continuously differentiable, then*

$$\max_{h \in B_\infty} \left\langle \nabla^2 \kappa(p) h, h \right\rangle \leq 1$$

and, in particular, $I \succeq \nabla^2 \kappa(p)$ for all $p \in \mathbb{R}^n$. Moreover, $\kappa - \| \|_2^2 \backslash 2$ is concave.

Proof. Fix $p \in \mathbb{R}^n$ and $w \in \sigma_\kappa(p)$. By Corollary 4 and since $M(w, \cdot)$ and κ are twice continuously differentiable, we have that $\nabla \kappa(p) = \nabla M(w, p)$,

$$\kappa(p+h) = \kappa(p) + \langle \nabla \kappa(p), h\rangle + \frac{1}{2} \left\langle \nabla^2 \kappa(p) h, h \right\rangle + R_\kappa(h) \quad \forall h \in \mathbb{R}^n,$$

and

$$M(w, p+h) = M(w, p) + \langle \nabla M(w, p), h\rangle + \frac{1}{2} \left\langle \nabla^2 M(w, p) h, h \right\rangle + R_w(h) \quad \forall h \in \mathbb{R}^n$$

where $R_w(\cdot) = o\left(\| \|_2^2\right)$ and $R_\kappa(\cdot) = o\left(\| \|_2^2\right)$. Fix $h \in \mathbb{R}^n$. It follows that

$$
\begin{aligned}
&\langle \nabla \kappa (p), th \rangle + \frac{1}{2} \left\langle \nabla^2 \kappa (p)\, th, th \right\rangle + R_\kappa (th) \\
&= \kappa (p + th) - \kappa (p) \\
&\leq M (w, p + th) - M (w, p) \\
&= \langle \nabla M (w, p), th \rangle + \frac{1}{2} \left\langle \nabla^2 M (w, p)\, th, th \right\rangle + R_w (th) \quad \forall t > 0,
\end{aligned}
$$

proving that

$$
\frac{1}{2} \left\langle \nabla^2 \kappa (p)\, h, h \right\rangle + \frac{R_\kappa (th)}{t^2} \leq \frac{1}{2} \left\langle \nabla^2 M (w, p)\, h, h \right\rangle + \frac{R_w (th)}{t^2} \quad \forall t > 0.
$$

By passing to the limit and since h was arbitrarily chosen, we have that $\langle \nabla^2 \kappa (p)\, h, h \rangle \leq \langle \nabla^2 M (w, p)\, h, h \rangle$ for all $h \in \mathbb{R}^n$ and, in particular, $\nabla^2 M (w, p) \succeq \nabla^2 \kappa (p)$. Denote by S_2 the unit sphere induced by $\| \ \|_2$ and by B_∞ the unit ball induced by $\| \ \|_\infty$. Since $M (w, \cdot)$ is convex, we have that $h \mapsto \langle \nabla^2 M (w, p)\, h, h \rangle$ is a convex function on $\mathbb{R}^n$. Since $\| \ \|_\infty \leq \| \ \|_2$, this implies that

$$
\begin{aligned}
\max_{h \in S_2} \left\langle \nabla^2 \kappa (p)\, h, h \right\rangle \leq \max_{h \in B_\infty} \left\langle \nabla^2 \kappa (p)\, h, h \right\rangle &\leq \max_{h \in B_\infty} \left\langle \nabla^2 M (w, p)\, h, h \right\rangle \\
&= \max_{h \in \operatorname{ext}(B_\infty)} \left\langle \nabla^2 M (w, p)\, h, h \right\rangle \leq 1,
\end{aligned}
$$

proving that $\langle \nabla^2 \kappa (p)\, h, h \rangle \leq \|h\|_2^2 = \langle I h, h \rangle$ for all $h \in \mathbb{R}^n$, that is, $I \succeq \nabla^2 \kappa (p)$. Since p was arbitrarily chosen, the statement follows. $\square$

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